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INTRODUCTION TO UNLIKELY INTERSECTIONS

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ABSTRACT. Statement of conjectures such as Manin-Mumford, André-Oort, Zilber-Pink. History and motivation. Overview of known results, and methods used. High-level outline of the steps in the Pila-Zannier strategy. If time permits: Definition of O-minimal structure, idea of 'tame topology'. Examples: $\mathbb{R}_{\text{alg}}, \mathbb{R}_{\text{exp}}, \mathbb{R}_{\text{an}}, \mathbb{R}_{\text{an,exp}}$. Programme of the study group.

CONTENTS

1. STATEMENTS

1.1. Mordell's Conjecture.

Theorem 1.1 (Mordell's Conjecture)**.** *Let C be a smooth projective curve defined over a number field k of genus bigger or equal than 2, then for every L/k finite, the set of L-rational points $C(L)$ is finite.*

First proven by Faltings in 1983 (via the Tate conjecture). Later by Vojta and Faltings (1993) and Masser-Wustholtz (1994). The three proofs are quite different.

Idea: Let C/k, if $C(k)$ is empty we are done, if not we can enlarge k and suppose that there exists $P \in C(k)$. This point gives rise to an embedding of C into its Jacobian:

$$C \to \text{Jac}(C),\ Q \mapsto (P) - (Q).$$

$\text{Jac}(C)$ is an abelian variety defined over k, and its set of rational point, $\Gamma := \text{Jac}(C)(k)$, is a finitely generated abelian group. If we knew that if a curve C in an abelian variety intersects a given finitely generated abelian group Γ at infinitely many points, then $g(C) < 2$, then this would imply Mordell!

More than this is known, but such results use Mordell (Faltings theorem).

Theorem 1.2 (Mordell-Lang)**.** *Let A be an abelian variety, $\Gamma \subset A(\mathbb{C})$ be a subgroup of finite rank (i.e. there exits $\Gamma_0 \subset A$ free subgroup s.t. for every $\gamma \in \Gamma, \exists m \in \mathbb{Z}$ s.t. $m\gamma \in \Gamma_0$). Then if $Z \subset A$ is an algebraic subvariety such that $\Gamma \cap Z$ is Zariski dense in Z, then Z can be written as a finite union of translates of abelian subvarieties of A.*

Manin-Mumford is a special case of Mordell-Lang.

1.2. Manin-Mumford.

Theorem 1.3. *Let A be a complex abelian variety and $Z \subset A$ an irreducible subvariety containing a Zariski dense set of torsion points, then Z is a translate of an abelian subvariety of A by a torsion point, i.e.*

$$Z = B + x, \text{ where } B \text{ is an abelian subvariety of } A \text{ and } x \text{ is a torsion point}.$$

It was first proven by Raynaud in 1983, but we will follow the proof given by Pila and Zannier (2008) which uses O-minimality. A simpler case is the multiplicative Manin-Mumford: consider the complex algebraic group $\mathbb{C}^{*n}$ and the followign definitions

- special point = n-tuple of roots of unity,
- special subvarieties = translates of subgroups of $\mathbb{C}^{*n}$ by a special point.

Manin-Mumford here reads as follows: an irreducible subvariety of $\mathbb{C}^{*n}$ containing a dense set of special points is special.

2. Interlude about Shimura varieties

When S is a Shimura variety we still have a notion of *special point*s and *special subvarieties*. The statement generalising Manin-Mumford is the André-Oort conjecture. It is known in the most interesting cases

There is also a notion of *mixed Shimura varieties* which allows to speak simultaneously about Shimura varieties and abelian varieties. Idea: mixed Shimura variety= a combination of a torus, an abelian variety and a Shimura variety.

We can go beyond André-Oort: Zilber-Pink conjectures on unlikely intersections. Only a few examples are known.

Conjecture 2.1 (Zilber-Pink conjecture, around 2004)**.** *Let S be a mixed Shimura variety of dimension d, $Z \subset S$ an irreducible subvariety of dimension n. Oonsider the set*

$$Z^{ul} := \{Z \cap S' \text{ such that } S' \text{ is special }, \dim S' < \dim S - \dim Z\}.$$

If Z is not contained in a proper special subvariety of S, then Z^{ul} is not Zariski dense in Z.

It implies both Manin-Mumford and André-Oort, where you only look at case where S' are special points, i.e. $\dim S' = 0$.

Pink's motivation was to have a new approach to the Mordell-Lang conjecture, which implies Mordell. Zilber's motivations has to do with transcendence theory (?).

3. Pila-Zannier strategy

Let us assume O-minimality as a black box, and prove (multiplicative) Manin-Mumford for $\mathbb{C}^{*2}$, where all the main ideas appear.

Question 3.1 (From Lang's book on Algebra, originally asked by Tate)**.** *Let $C = V(f) \subset \mathbb{C}^* \times \mathbb{C}^*$. Suppose C contains infinitely many points (ξ_1, ξ_2) where ξ_is are roots of 1. What is C?*

The answer is: C is *special*, i.e. it is a translate of a subgroup by a root of 1, i.e. C is given by an equation of the form $x^m y^n = \xi$.

3.1. **Step one: Bi-algebraic geometry.** In our three cases (tori, abelian varieties and Shimura varieties) we have a transcendental map between two complex varieties, and special subvarieties can be characterised in therms of this. This is the step where functional transcendence appears.

In the $\mathbb{C}^* \times \mathbb{C}^*$ case, it is as follows. Consider

$$\pi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*,\ (z_1, z_2) \mapsto (e^{2\pi i z_1}, e^{2\pi i z_2}).$$

An algebraic subvariety $C \subset \mathbb{C}^* \times \mathbb{C}^*$ is special if and only if analytic component of $\pi^{-1}(C)$ are (complex) algebraic varieties. This is a baby case of the Ax-Lindemann-Wietrass theorem.

3.2. **Step two: O-minimality.** We need to find algebraic subvarieties in $\pi^{-1}(C)$. In fact showing that $\pi^{-1}(C)$ contains a positive dimensional semi-algebraic (like a polidisc) subset of $\pi^{-1}(C)$ will be enough. This s done using O-minimality. To do so you need to fix a fundamental domain for the action of $\mathbb{Z} \times \mathbb{Z}$ in $\mathbb{C} \times \mathbb{C}$. So call $\mathcal{F}$ the band given by $|\Re(z)| < 1$ and consider

$$\pi^{-1}(Z) \cap (\mathcal{F} \times \mathcal{F}) =: \widetilde{Z},$$

where Z is a definable set in $\mathbb{C}^* \times \mathbb{C}^*$. Then Pila-Wilkie theorem says that if there are more than T^c points in $\widetilde{Z} \cap \mathbb{Q} \times \mathbb{Q}$ of height $\leq T$, then $\widetilde{Z}$ contains a positive dimensional semi-algebraic subset.

To check the assumption of Pila-Wilkie, we need to understand the Galois action on roots of unity.

3.3. **Step three: Galois and height.** Let $x = (\xi_1, \xi_2) = (e^{2\pi i \frac{a}{b}}, e^{2\pi i \frac{a'}{b'}}) \in C$, and assume that

$$x = \pi(\frac{a}{b}, \frac{a'}{b'}), \text{ where } (\frac{a}{b}, \frac{a'}{b'}) \in \mathcal{F} \times \mathcal{F} \text{ and } a < b,\ a' < b'.$$

Let n_x be the order of x, we have that n_x tends to infinity and

$$H(\tilde{x}) = \max(b, b') \leq n_x^2.$$

In particular the $\mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$ orbit of x grows at least as $\sqrt{n_x}$. So we have at least $\sqrt{n_x}$ points in $\widetilde{Z}$ of height $\leq n_x^2$.

4. Next talks

- Manin-Mumford (Michele)
- AO for $Y(1)^n$ (Domenico)
- Ax-Lindemann for $\mathcal{A}_g$ (Andrei)
- Heights of abelian varieties (Netan)
- AO for $\mathcal{A}_g$ (Greg)
- Masser-Wustholz (Martin)
- Families of abelian varieties (Laura)