Topological definitions for Algebraic Geometry

The aim of these notes is to briefly introduce the definitions from topology which we will require in the Algebraic Geometry course.

In algebraic geometry at the level of this course, mostly we use topology just as a convenient (and suggestive) language. Hence why these notes consist mainly of definitions, and a few exercises to check you understand the definitions. Higher-level algebraic geometry (e.g. schemes) uses topology a bit more seriously.

These notes are written with the assumption that you have already seen metric spaces. Most of the concepts defined should be familiar from metric spaces, but the definition for a general topological space may look different (you should check that each definition here is equivalent to the definition you already know in the case of metric spaces).

In addition to the definitions in these notes, there will be other topological concepts which I define in the lectures – these will be concepts which are not important for metric spaces, and I therefore assume most students will not have seen them at all before.

Here is one fuller introduction to topology that seems to contain all the concepts I need (see list at the end of these notes) without too much of a bias towards metric spaces: `https://www.math.cornell.edu/~hatcher/Top/TopNotes.pdf`

1. Definition of topological space, open and closed sets

Let X be a set. A *topology* on X is defined by specifying a collection of subsets of X, called the *open sets*. These have to satisfy certain axioms, as follows.

Definition. A **topology** on X is a collection $\mathcal{T}$ of subsets of X such that:

(i) $\emptyset$ and X are in $\mathcal{T}$;
(ii) for any finite subcollection $\{U_1, \dots, U_n\} \subseteq \mathcal{T}$, the intersection $\bigcup_{i=1}^n U_i$ is in $\mathcal{T}$ ("finite intersections of open sets are open");
(iii) for any subcollection (finite or infinite) $\{U_i : i \in I\} \subseteq \mathcal{T}$, the union $\bigcup_{i \in I} U_i$ is in $\mathcal{T}$ ("arbitrary unions of open sets are open").

The sets in $\mathcal{T}$ are called **open sets** of the topology.

A **topological space** is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on X.

A lot of the time, we just say "the topological space X," and it is obvious from context what $\mathcal{T}$ is. However, there may be more than one topology on the same set so sometimes it is necessary to make it clear which one we are using. For example, in algebraic geometry there are two topologies on $\mathbb{C}^n$ which are useful: the "Euclidean" (or "usual" or "complex") topology and the "Zariski" topology.

Examples.

(1) Any metric space, with the usual notion of open sets in a metric space, that is

$$U \in \mathcal{T} \quad \Leftrightarrow \quad \forall x \in U,\ \exists \epsilon > 0 \text{ s.t. } B(x, \epsilon) \subseteq U.$$

(2) In particular this includes $\mathbb{R}^n$ and $\mathbb{C}^n$ with the topology induced by the standard metric. We will call this the **Euclidean topology** in this course.

(3) The **discrete topology** on any set X, in which $\mathcal{T}$ consists of all subsets of X.
(4) The **indiscrete topology** on any set X, where $\mathcal{T} = \{\emptyset, X\}$.
(5) The **cofinite topology** on any set X, where

$$\mathcal{T} = \{U \subseteq X : X \setminus U \text{ is finite}\}.$$

Exercise 1.1.

(1) Verify that each of these examples is a topology.
(2) The axioms for a topology do *not* include the condition "arbitrary intersections of open sets are open." Find a counter-example to this condition for the Euclidean topology on $\mathbb{R}^n$.
(3) Check that the cofinite topology is the same as the discrete topology if and only if X is a finite set.

Closed sets.

Definition. If $(X, \mathcal{T})$ is a topological space, we define a **closed set** to be the complement of an open set in X.

This agrees with the usual definition of closed sets in metric spaces.

Exercise 1.2.

(1) Check that, for the discrete and indiscrete topologies, a subset of X is closed if and only if it is open.
(2) Check that, for the cofinite topology, a subset of X is closed if and only if it is finite.

Note that closed sets satisfy the reverse of the axioms for open sets, i.e.

(i) $\emptyset$ and X are closed;
(ii) finite unions of closed sets are closed;
(iii) arbitrary intersections of closed sets are closed.

If you know the clsoed sets of a topological space, you have the information required to determine the open sets.

2. The subspace topology

Let $(X, \mathcal{T})$ be a topological space, and let Y be any subset of X. The **subspace topology** on Y is defined by the condition:

(*) $V \subseteq Y$ is open in the subspace topology on Y if and only if $V = U \cap Y$ for some U which is open in the topology on X.

Note that a set may be open in Y without being open in X. For example, let $X = \mathbb{R}$ (with the Euclidean topology) and $Y = [0, 1]$. The half-open interval $[0, 1/2)$ is open in the subspace topology on Y but is not open in the Euclidean topology on X.

(*) is equivalent to the same thing for closed sets:

(*') $B \subseteq Y$ is closed in the subspace topology on Y if and only if $B = A \cap Y$ for some A which is closed in the topology on X.

In Algebraic Geometry, we will be particularly interested in the case where Y is itself a closed subset of X. Then we have:

Lemma 2.1. If Y is a closed subset of X, then a subset $B \subseteq Y$ is closed in the subspace topology on Y if and only if B is closed in X.

The example above shows that, even when Y closed in X, open subsets of Y need not be open in X.

3. Closure and dense sets

Definition. Let X be a topological space and let A be any subset of X. The **closure** of A, written $\overline{A}$, is the smallest closed subset of X which contains A. This means:

(i) $A \subseteq \overline{A}$;
(ii) $\overline{A}$ is closed; and
(iii) if B is a closed set such that $A \subseteq B$ then $\overline{A} \subseteq B$.

To prove that such a set exists, note that the intersection of all closed sets containing A satisfies the conditions. Check that this is the only set satisfying the conditions.

Note that A is closed if and only if $\overline{A} = A$.

Let Y be a subset of X. If $A \subseteq Y$, then the closure of A might be different depending on whether we use the topology on X or the subspace topology on Y.

Exercise 3.1. We let $\overline{A}$ denote the closure of A in the topology on X. Show that the closure of A in the topology on Y is $\overline{A} \cap Y$.

If Y is closed in X, then the closure of A in the subspace topology on Y is the same as its closure in the topology on X.

Definition. We say that a subset $A \subseteq X$ is **dense** if $\overline{A} = X$.

If $A \subseteq Y \subseteq X$, then we say that A is **dense in** Y if it is dense for the subspace topology on Y. Again letting $\overline{A}$ denote the closure in the topology on X, this is equivalent to $Y \subseteq \overline{A}$ but not necessarily $Y = \overline{A}$.

4. Connected spaces and connected components

Definition. A topological space $(X, \mathcal{T})$ is **connected** if it is not possible to write X as the union of two disjoint non-empty open sets.

This is equivalent to each of the following conditions (check this if you are not familiar with the definition!):

(1) It is not possible to write X as the union of two disjoint non-empty closed sets.
(2) X is non-empty and the only subsets of X which are both open and closed are X and $\emptyset$.

Definitions vary as to whether the empty set is connected or not. According to the above definition, the empty set is not connected. I think this is the most convenient convention, but it doesn't matter very much.

If $(X, \mathcal{T})$ is a topological space and $Y \subseteq X$, then we say that Y is **connected** if the subspace topology on Y is connected.

Definition. Let $(X, \mathcal{T})$ be a topological space. A **connected component** of X is a maximal connected subset. In other words, $C \subseteq X$ is a connected component if C is connected and, if B is a subset of X which properly contains C, then B is not connected.

Exercise 4.1. Check that, for any topological space X, the connected components of X are disjoint and their union is all of X. Furthermore each connected component is a closed subset of X (but connected components need not be open subsets of X).

The notion of connectedness is sometimes useful in algebraic geometry, but we will define a slightly different notion which is more often important: irreducibility.

5. Hausdorff spaces

The fundamental difference between metric spaces and the Zariski topology used in algebraic geometry is that metric spaces are always Hausdorff while the Zariski topology is usually not Hausdorff. "Hausdorff spaces" are also called "separated spaces," especially in languages other than English.

We don't need the definition of Hausdorff spaces except to observe that the Zariski topology is not Hausdorff, so don't worry too much about it.

Definition. A topological space X is **Hausdorff** if, for every pair of points $x, y \in X$, where $x \neq y$, there exist disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $y \in V$.

Lemma 5.1. Every metric space is Hausdorff.

The Hausdorff property is essential to make limits of sequences behave sensibly. Hence most methods of analysis work only in Hausdorff spaces, and thus are not applicable to the Zariski topology.

6. Continuous functions

The natural type of functions between topological spaces are continuous functions. We can't define these with limits or with epsilons and deltas as in metric spaces: we can only use open and closed sets. The following definition turns out to be equivalent to the $\epsilon - \delta$ definition in the case of metric spaces.

Definition. Let X, Y be topological spaces. A function $f \colon X \to Y$ is **continuous** if, for every open set $U \subseteq Y$, the preimage $f^{-1}(U)$ is open in X.

We can equivalently state this in terms of closed sets:

Fact. A function $f \colon X \to Y$ is continuous if and only if, for every closed set $V \subseteq Y$, the preimage $f^{-1}(V)$ is closed in X.

We will also mention the following concepts in the course, but like Hausdorff spaces this will only be in order to explain why they are not useful in algebraic geometry:

(1) product topology;
(2) compact spaces.