Algebraic Geometry

Problem Sheet 6 – Solutions

Question 1. *Suppose that k does not have characteristic 2 or 3.*

For $a \in k$, let V_a denote the surface in $\mathbb{A}^3$ defined by the equation

$$x^3 + y^3 + z^3 - 3a(x^2 + y^2 + z^2) - a^2 = 0.$$

You may assume that this polynomial generates the ideal $\mathbb{I}(V_a)$.

For which values of a does V_a have singular points? For each a, find all the singular points of V_a.

Let $f_a(x, y, z) = x^3 + y^3 + z^3 - 3a(x^2 + y^2 + z^2) - a^2$. We calculate

$$\frac{\partial f_a}{\partial x} = 3x^2 - 6ax$$

so $\partial f_a/\partial x = 0$ if and only if $x = 0$ or $x = 2a$. Similarly $\partial f_a/\partial y = 0$ if and only if $y = 0$ or $y = 2a$, and $\partial f_a/\partial z = 0$ if and only if $z = 0$ or $z = 2a$.

Because V_a is a hypersurface, $(x, y, z) \in V_a$ is a singular point if and only if $df_{a,(x,y,z)}$ is the zero map. Thus (x, y, z) is a singular point of V_a if and only if $f_a(x, y, z) = 0$ and each of x, y, z is in $\{0, 2a\}$.

Hence if $a = 0$, then $(0, 0, 0)$ is the unique singular point of V_0 (and this point is indeed in V_0).

Otherwise, suppose that $a \neq 0$ (so that $2a \neq 0$). Suppose that $x, y, z \in \{0, 2a\}$ and let $\delta \in \{0, 1, 2, 3\}$ be the number of occurrences of $2a$ among x, y, z. Then

$$f_a(x, y, z) = 8\delta a^3 - 12\delta a^3 - a^2 = -4\delta a^3 - a^2.$$

So (x, y, z) is a singular point of V_a if and only if $a^2(4\delta a + 1) = 0$.

Since we are assuming that $a \neq 0$, we conclude that if V_a has a singular point, then $a = -1/4\delta$ for some $\delta \in \{1, 2, 3\}$. We find the following singular points:

$$\begin{array}{lll}
\delta = 1 & a = -\frac{1}{4} & (x,y,z) = \left(-\frac{1}{2}, 0, 0\right) \text{ or permutations thereof} \\
\delta = 2 & a = -\frac{1}{8} & (x,y,z) = \left(-\frac{1}{4}, -\frac{1}{4}, 0\right) \text{ or permutations thereof} \\
\delta = 3 & a = -\frac{1}{12} & (x,y,z) = \left(-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}\right)
\end{array}$$

as well as the case we already dealt with:

$$a = 0 \qquad (x, y, z) = (0, 0, 0).$$

Question 2. *Let V, W be affine varieties. Let $v \in V$ and $w \in W$*

Prove that $V \times W$ is non-singular at (v, w) if and only if V is non-singular at v and W is non-singular at w.

Suppose that $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$. Write $X_1, \dots, X_n$ for the coordinates on $\mathbb{A}^n$ and $Y_1, \dots, Y_m$ for the coordinates on $\mathbb{A}^m$.

In order to determine whether $V \times W$ is non-singular at (v, w), we have to work out the tangent space $T_{(v,w)}(V \times W)$. To do this, we have to work out $\ker df_{(v,w)}$ for some polynomials f which generate $\mathbb{I}(V \times W)$.

Let $f_1, \dots, f_r \in k[X_1, \dots, X_n]$ generate $\mathbb{I}(V)$ and let $g_1, \dots, g_s \in k[Y_1, \dots, Y_m]$ generate $\mathbb{I}(W)$. Now $\mathbb{I}(V \times W)$ is generated by $f_1, \dots, f_r, g_1, \dots, g_s$ considered as elements of $k[X_1, \dots, X_n, Y_1, \dots, Y_m]$.

Let us introduce some notation: write $\bar{f}_i(X_1, \dots, X_n, Y_1, \dots, Y_m)$ for $f_i(X_1, \dots, X_n)$ considered as a polynomial in $k[X_1, \dots, X_n, Y_1, \dots, Y_m]$ (which happens to not involve the variables $Y_1, \dots, Y_m$). Similarly write $\bar{g}_j$ for $g_j(Y_1, \dots, Y_m)$ considered as a polynomial in $k[X_1, \dots, X_n, Y_1, \dots, Y_m]$. The purpose of this notation is to distinguish between the linear maps

$$d\bar{f}_{i,(v,w)} \colon k^{n+m} \to k,$$

and

$$df_{i,v} \colon k^n \to k.$$

Because $(\partial \bar{f}_i/\partial X_j)|_{(v,w)} = (\partial f_i/\partial X_j)|_v$ while $(\partial \bar{f}_i/\partial Y_j)|_{(v,w)} = 0$, the definition gives us that

$$d\bar{f}_{i,(v,w)}(a_1, \dots, a_n, b_1, \dots, b_m) = df_{i,v}(a_1, \dots, a_n).$$

Therefore

$$\ker d\bar{f}_{i,(v,w)} = \ker df_{i,v} \oplus k^m$$

and so

$$\bigcap_{i=1}^{r} \ker d\bar{f}_{i,(v,w)} = T_v V \oplus k^m.$$

Similarly

$$d\bar{g}_{j,(v,w)}(a_1, \dots, a_n, b_1, \dots, b_m) = dg_{j,w}(b_1, \dots, b_m)$$

and so

$$\ker d\bar{g}_{j,(v,w)} = k^n \oplus \ker dg_{j,w},$$

giving

$$\bigcap_{j=1}^{s} \ker d\bar{g}_{j,(v,w)} = k^n \oplus T_w W.$$

By definition,

$$T_{(v,w)}(V \times W) = \bigcap_{i=1}^{r} \ker d\bar{f}_{i,(v,w)} \cap \bigcap_{j=1}^{s} \ker d\bar{g}_{j,(v,w)}$$

therefore

$$\begin{aligned} T_{(v,w)}(V \times W) &= (T_v V \oplus k^m) \cap (k^n \oplus T_w W) \\ &= T_v V \oplus T_w W. \end{aligned}$$

Thus $\dim T_{(v,w)}(V \times W) = \dim T_v V + \dim T_w W$.

If $V = \bigcup_i V_i$ and $W = \bigcup_j W_j$ are the decompositions into irreducible components, then the irreducible components of $V \times W$ are $V_i \times W_j$. By definition, $\dim_{(v,w)}(V \times W)$ is the maximum dimension of the irreducible components of $V \times W$ which contain (v, w). The components of $V \times W$ containing (v, w) of maximum dimension have the form $V_i \times W_j$, where V_i is a component of V containing v of maximum dimension and W_j is a component of W_j containing w of maximum dimension. Hence

$$\dim_{(v,w)}(V \times W) = \dim(V_i \times W_j) = \dim V_i + \dim W_j = \dim_v V + \dim_w W.$$

By Lemma A.10 from the mastery mateiral,

$$\dim T_v V \geq \dim_v V \text{ and } \dim T_w W \geq \dim_w W.(*)$$

It follows that

$$\dim T_v V + \dim T_w W \geq \dim_v V + \dim_w W$$

with equality if and only if equality holds in both parts of (*). Given our earlier calculations, this implies that

$$\dim T_{(v,w)}(V \times W) = \dim_{(v,w)}(V \times W)$$
$$\Leftrightarrow$$
$$\dim T_v V = \dim_v V \text{ and } \dim T_w W = \dim_w W.$$

By definition, $V \times W$ is non-singular at (v, w) if and only if $\dim T_{(v,w)}(V \times W) = \dim_{(v,w)}(V \times W)$, and similarly for V and W. So we have proved that

$$V \times W \text{ is non-singular at } (v, w)$$
$$\Leftrightarrow$$
$$V \text{ is non-singular at } v \text{ and } W \text{ is non-singular at } w.$$

Question 3. *Let $V \subseteq \mathbb{A}^n$ be an affine algebraic set with irreducible components V_1 and V_2. Let $x \in V_1 \cap V_2$. Prove that*

$$T_x V_1 + T_x V_2 \subseteq T_x V.$$

Is $T_x V_1 + T_x V_2$ always equal to $T_x V$?

(Here, $T_x V_1 + T_x V_2$ means the vector space spanned by $T_x V_1$ and $T_x V_2$.)

We have $V_1 \subseteq V$ so $\mathbb{I}(V_1) \supseteq \mathbb{I}(V)$. Hence

$$T_x V_1 = \bigcap_{f \in \mathbb{I}(V_1)} \ker df_x$$

is contained in $T_x V = \bigcap_{f \in \mathbb{I}(V)} \ker df_x$.

Similarly $V_2 \subseteq V$ so $T_x V_2 \subseteq T_x V$.

$T_x V$ is a vector space, so as it contains both $T_x V_1$ and $T_x V_2$ it contains the space generated by them, namely $T_x V_1 + T_x V_2$.

It is not true that $T_x V$ is always equal to $T_x V_1 + T_x V_2$. Here is a counter-example (many others are possible):

Let V_1 and V_2 be the affine plane curves $Y = 0$ and $Y - X^2 = 0$, and let V be their union. The origin $(0, 0)$ is in $V_1 \cap V_2$.

Now

$$\left.\frac{\partial Y}{\partial X}\right|_{(0,0)} = 0 \text{ and } \left.\frac{\partial Y}{\partial Y}\right|_{(0,0)} = 1$$

while

$$\left.\frac{\partial Y - X^2}{\partial X}\right|_{(0,0)} = -2X|_{(0,0)} = 0 \text{ and } \left.\frac{\partial Y - X^2}{\partial Y}\right|_{(0,0)} = 1.$$

Hence

$$T_{(0,0)} V_1 = T_{(0,0)} V_2 = \{(a, b) : b = 0\}.$$

This is geometrically clear: V_2 is tangent to the x-axis V_1. It follows that

$$T_{(0,0)}V_1 + T_{(0,0)}V_2 = \{(a,b) : b = 0\}.$$

Now V is defined by the polynomial $f(X,Y) = Y(Y - X^2)$. (This polynomial has no repeated factors, so it generates a radical ideal. By the Nullstellensatz, f generates $\mathbb{I}(V)$.)

Now

$$\left.\frac{\partial f}{\partial X}\right|_{(0,0)} = -2XY|_{(0,0)} = 0 \text{ and } \left.\frac{\partial f}{\partial Y}\right|_{(0,0)} = (2Y - X^2)\big|_{(0,0)} = 0.$$

Hence $df_{(0,0)} = 0$ and so

$$T_{(0,0)}V = \ker df_{(0,0)} = k^2 \neq T_{(0,0)}V_1 + T_{(0,0)}V_2.$$

This illustrates how intuition about tangent spaces can break down at a singular point.

Question 4. *Let f be a non-zero polynomial in $k[X_1, \dots, X_n]$ with no repeated factors. Let $V \subseteq \mathbb{A}^n$ be the hypersurface defined by f.*

Sub-question 4(a). *Let x be a point in V and let u be a vector in k^n.*

Prove that the polynomial

$$f(x + Tu) \in k[T]$$

has a repeated root at $T = 0$ if and only if $u \in T_xV$.

Since f has no repeated factors, f generates $\mathbb{I}(V)$. Hence T_xV is the kernel of the linear map

$$df_x \colon (a_1, \dots, a_n) \mapsto \sum_{i=1}^{n} \left.\frac{\partial f}{\partial X_i}\right|_x a_i.$$

Since $x \in V$, $f(x + Tu)$ has a root at $T = 0$. So it has a repeated root at $T = 0$ if and only if

$$\left.\frac{d}{dT} f(x + Tu)\right|_{T=0} = 0.$$

By the chain rule for partial derivatives,

$$\left.\frac{d}{dT} f(x + Tu)\right|_{T=0} = \sum_{i=1}^{n} \left.\frac{\partial f}{\partial X_i}\right|_x u_i.$$

Hence $u \in T_xV = \ker df_x$ if and only if $f(x + Tu)$ has a repeated root at $T = 0$.

Sub-question 4(b). *Suppose that* $\deg f = 2$. *Prove that if x is a singular point of V and $L \subseteq \mathbb{A}^n$ is a line through x, then either $L \subseteq V$ or $L \cap V = \{x\}$.*

Because x is a singular point of V, $\dim T_xV > \dim V$. Because V is a hypersurface in $\mathbb{A}^n$, $\dim V = n - 1$. But $T_xV \subseteq k^n$. So we must have $\dim T_xV = n$ and $T_xV = k^n$.

Let u be a vector in k^n such that $L = \{x + tu : t \in k\}$. Since $T_xV = k^n$, we get $u \in T_xV$.

By (c), the polynomial $f(x+Tu)$ has a repeated root at $T = 0$. But f has degree 2, so $f(x+Tu)$ has degree ≤ 2. So in order to have a repeated root at 0, we must have

$$f(x + Tu) = aT^2$$

for some $a \in k$ (a may be zero or non-zero).

Now $L \cap V = \{x + tv : t \in k, f(x + tu) = 0\}$. So if $f(x + Tu) = aT^2$ with $a \neq 0$, then $L \cap V = \{x\}$. On the other hand if $f(x + Tu) = 0$ then $L \subseteq V$.