Algebraic Geometry

Problem Sheet 5 – Solutions

Question 1. *Let $V \subseteq \mathbb{P}^n$ be a projective algebraic set in which all components have dimension $n-1$. Prove that V is a hypersurface.*

We may assume that V is irreducible. (This is because a finite union of hypersurfaces is still a hypersurface, as you can take the product of the defining polynomials.)

Since $\dim V \neq n$, $V \neq \mathbb{P}^n$. So we can choose a non-zero homogeneous polynomial f which vanishes on V. Let the irreducible factors of f be $f_1, \dots, f_m$. Now

$$V = \bigcup_{i=1}^{n} \{x \in V : f_i(x) = 0\}$$

so the irreducibility of V forces one of the f_i to vanish on V.

Let W be the hypersurface defined by f_i. Then $\dim W = n - 1 = \dim V$ because W is a hypersurface and so $\dim W = \dim V$. Furthermore W is irreducible because f_i is irreducible, and V is a closed subset of W. Hence by Lemma 21.2, $W = V$.

Question 2. *Let $H \subseteq \mathbb{P}^n$ be a hyperplane. Let $V \subseteq H$ be an irreducible projective algebraic set (note: $V \subseteq H$, rather than $V \subseteq \mathbb{P}^n$).*

*Let x be a point of $\mathbb{P}^n \setminus H$. Let C be the union of all lines joining V to x (this is called the **cone** over V with vertex x).*

Prove that C is a projective algebraic set, that it is irreducible and that $\dim C = \dim V + 1$.

Make a linear change of coordinates so that H is the hyperplane $X_0 = 0$ and x is the point $[1 : 0 : \cdots : 0]$. (In other words, H is the hyperplane at infinity for the standard embedding $\mathbb{A}^n \to \mathbb{P}^n$, while x is the origin.)

For any point $y = [0 : y_1 : \cdots : y_n] \in H$, the line L_y joining y to x is

$$\{[a : by_1 : \cdots : by_n] : a, b \in k, \text{ not both zero}\}.$$

We can write this as

$$L_y = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n : [z_1 : \cdots : z_n] = [0 : \cdots : 0] \text{ or } [y_1 : \cdots : y_n]\}.$$

Thus

$$C = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n : [z_1 : \cdots : z_n] = [0 : \cdots : 0] \text{ or } [z_1 : \cdots : z_n] \in V\}.$$

Let $f_1, \dots, f_m \in k[X_1, \dots, X_n]$ be homogeneous polynomials defining V (together with the equation $X_0 = 0$). Then C is the zero set of $f_1, \dots, f_m$ in $\mathbb{P}^n$ – observe that $f_1, \dots, f_m$ are non-constant homogeneous polynomials, so they do indeed vanish at $(0, \dots, 0)$.

Now we show that C is irreducible. If $C = S \cup T$ where S and T are closed subsets, then the irreducibility of the line L_y implies that for each $y \in V$ we have either $L_y \subseteq S$ or $L_y \subseteq T$. The sets

$$\{y \in V : L_y \subseteq S\}, \quad \{y \in V : L_y \subseteq T\}$$

are closed subsets of V (by similar reasoning to sheet 1, question 10). So the irreducibility of V forces one of these sets to be all of V, and hence $C = S$ or $C = T$. Thus C is irreducible.

Finally we calculate $\dim C$.

We know that $V = C \cap H$ so we will apply Proposition 20.4 from lectures. Since $x \in C$ and $x \notin H$, we conclude that $C \not\subseteq H$. Since C is irreducible, this tells us that H does not contain any irreducible component of C. Therefore by Proposition 20.4, $\dim V = \dim(C \cap H) = \dim C - 1$.

Question 3. *Let $V_{n,d}$ denote the vector space of homogeneous polynomials of degree d in $k[X_0, \dots, X_n]$ and let $P_{n,d}$ denote the projective space attached to $V_{n,d}$*

Sub-question 3(a). *Show that, for each e such that $0 \le e \le d$, multiplication of polynomials $V_{n,e} \times V_{n,d-e} \to V_{n,d}$ induces a regular map*

$$\mu_{n,d,e} \colon P_{n,e} \times P_{n,d-e} \to P_{n,d}.$$

If $f, g \in k[X_0, \dots, X_n]$ are homogeneous polynomials of degrees e and $d - e$ respectively, then fg is a homogeneous polynomial of degree d. Each coefficient of fg is given by a bihomogeneous polynomial of degree $(1, 1)$ in the coefficients of f and g. Hence $([f], [g]) \mapsto [fg]$ is a regular map of projective algebraic sets $P_{n,e} \times P_{n,d-e} \to P_{n,d}$.

Sub-question 3(b). *By applying completeness to $\mu_{n,d,e}$, show that the set*

$$P_{n,d,\text{irr}} = \{[f] \in P_{n,d} : f \text{ is irreducible}\}$$

is an open subset of $P_{n,d}$.

A polynomial $f \in V_{n,d}$ is *reducible* if and only if it can be written as a produt of two polynomials of smaller degree, i.e. if and only if $[f]$ is in the image of $\mu_{n,d,e} \colon P_{n,e} \times P_{n,d-e} \to P_{n,d}$ for some e such that $1 \le e \le d - 1$. Thus

$$P_{n,d,\text{irr}} = P_{n,d} \setminus \bigcup_{e=1}^{d-1} \operatorname{im} \mu_{n,d,e}.$$

Because $P_{n,e} \times P_{n,d-e}$ is a projective variety, completeness tells us that $\operatorname{im} \mu_{n,d,e}$ is a closed subset of $P_{n,d}$ for each e. Hence the finite union $\bigcup_{e=1}^{d-1} \operatorname{im} \mu_{n,d,e}$ is also a closed subset of $P_{n,d}$, and therefore its complement is an open subset.

Sub-question 3(c). *Let $n \ge 2$. Show that $P_{n,d,\text{irr}}$ is non-empty. (Easy approach: write down an irreducible homogeneous polynomial of degree d. Harder approach, making use of dimension: Calculate the dimension of $P_{n,e} \times P_{n,d-e}$ for each e and compare with* $\dim P_{n,d}$. *What goes wrong when $n = 1$ and $d \ge 2$?)*

Easy approach: the polynomial $f = X_0 X_1^{d-1} - X_2^d$ is homogeneous of degree d and irreducible. (It is irreducible because it has degree 1 in X_0, so any factor must divide both coefficients if we consider f as a polynomial in X_0 i.e. the factor must divide X_1^{d-1} and X_2^d but these have no common factor.) This fails when $n = 1$ because then we only have the variables X_0 and X_1 in our polynomial ring, but this example requires three variables.

Note that when $n = 1$ and $d \ge 2$ there are actually no irreducible polynomials: given a homogeneous polynomial $f \in V_{1,d}$, we can substitute in $X_0 = 1$ to get a one-variable polynomial $f(1, X_1)$. Either every term of f is divisible by X_0, in which case X is reducible, or else $f(1, X_1)$ still has degree d. In the second case, because k is algebraically closed, $f(1, X_1)$ has a root a and so $f(1, X_1)$ has a factor $X_1 - a$. Then $f(X_0, X_1)$ has the factor $X_1 - aX_0$ (which is a non-trivial factor when $d \ge 2$).

Harder approach: we have

$$\dim P_{n,d} = \dim V_{n,d} - 1 = \binom{n+d}{n} - 1$$

and

$$\dim(P_{n,e} \times P_{n,d-e}) = \binom{n+e}{n} + \binom{n+d-e}{n} - 2.$$

I think the easiest way to compare these is combinatorially: $\binom{n+d}{n}$ counts the number of ways to choose n elements from a set of size $n+d$. Let's label the elements of our set of size $n+d$ as $a_1, \ldots, a_n, b_1, \ldots, b_e, c_1, \ldots, c_{d-e}$.

Now $\binom{n+e}{n}$ counts the ways to choose n elements from $\{a_1, \ldots, a_n, b_1, \ldots, b_e\}$ while $\binom{n+d-e}{n}$ counts the ways to choose n elements from $\{a_1, \ldots, a_n, c_1, \ldots, c_{d-e}\}$. Combining these two together, we count $\{a_1, \ldots, a_n\}$ twice but that is the only thing counted twice. Hence $\binom{n+e}{n} + \binom{n+d-e}{n} - 1$ counts the ways to choose n elements from $\{a_1, \ldots, a_n, b_1, \ldots, b_e, c_1, \ldots, c_{d-e}\}$, subject to the condition that we can choose only bs or cs but not both (while we can choose as many as as we want). When $n \geq 2$ and $1 \leq e \leq d-1$, this condition gives fewer options than just choosing n elements from $a_1, \ldots, a_n, b_1, \ldots, b_e, c_1, \ldots, c_{d-e}$ (for example, we cannot choose $\{a_1, \ldots, a_{n-2}, b_1, c_1\}$). Therefore

$$\binom{n+e}{n} + \binom{n+d-e}{n} - 1 < \binom{n+d}{n}$$

and so $\dim(P_{n,e} \times P_{n,d-e}) < \dim P_{n,d}$.

This implies that $\dim \operatorname{im} \mu_{n,d,e} < \dim P_{n,d}$ and therefore $\operatorname{im} \mu_{n,d,e}$ is a proper closed subset of $P_{n,d}$. Because $P_{n,d}$ is irreducible (it is isomorphic to a projective space), we deduce that

$$P_{n,d} \neq \bigcup_{e=1}^{d-1} \operatorname{im} \mu_{n,d,e}$$

and so $P_{n,d,\mathrm{irr}}$ is non-empty.

When $n = 1$ this argument goes wrong because if we have to choose one element from $\{a_1, b_1, \ldots, b_d, c_1, \ldots, c_{d-e}\}$, it will not be possible to choose both bs and cs because we have just one choice.

Sub-question 3(d). *By considering maps of the form*

$$P_{n,a} \times P_{n,b} \to P_{n,d} \;:\; ([f],[g]) \mapsto [f^r g]$$

for all triples a, b, r such that $ar + b = d$ and $r \geq 2$, show that the set

$$P_{n,d,\mathrm{rad}} = \{[f] \in P_{n,d} : f \text{ generates a radical ideal}\}$$

is an open subset of $P_{n,d}$.

If $ar+b = d$, then by similar reasoning to part (a), the map $\nu_{n,r,a,b} \colon P_{n,a} \times P_{n,b} \to P_{n,d}$ given by $([f],[g]) \mapsto [f^r g]$ is a regular map $P_{n,a} \times P_{n,b} \to P_{n,d}$. (If f, g are homogeneous polynomials of degrees a and b respectively, then $f^r g$ is a homogeneous polynomial of degree d, and the coefficients of $f^r g$ are bihomogeneous polynomials of degree $(r,1)$ in the coefficients of f and g.) Hence by completeness, the image of $\nu_{n,r,a,b}$ is a closed subset of $P_{n,a}$.

Now $f \in V_{n,d}$ generates a radical ideal if and only if $[f]$ is not in the image of any $\nu_{n,r,a,d}$ (for $ar + b = d$ and $r \geq 2$). To prove this: if $[h]$ is in the image of $\nu_{n,r,a,b}$ for some $r \geq 2$, then $h = f^r g$ and so $fg \in \operatorname{rad}(h)$ but $fg \notin (h)$. Conversely, if h generates a radical ideal, then the irreducible factorisation of h must be $h_1 \cdots h_m$ where the factors are distinct, and so h is not divisible by f^r for any non-constant polynomial f and any $r \geq 2$.

Thus

$$P_{n,d,\mathrm{rad}} = P_{n,d} \setminus \bigcup_{\substack{ar+b=d \\ r \geq 2}} \operatorname{im} \nu_{n,r,a,b}.$$

Thus $P_{n,d,\mathrm{rad}}$ is the complement fo a finite union of closed sets, so it is an open set in $P_{n,d}$.

Note that $P_{n,d,\mathrm{irr}} \subseteq P_{n,d,\mathrm{rad}}$ so part (c) implies that $P_{n,d,\mathrm{rad}}$ is non-empty.

Sub-question 3(e). *Give an example (for some n, d) of two polynomials f, g such that $[f], [g] \in P_{n,d} \setminus P_{n,d,\mathrm{rad}}$ and f and g define the same hypersurface in $\mathbb{P}^n$, but f is not a scalar multiple of g.*

(Note: it follows from the Nullstellensatz that, if $[f], [g] \in P_{n,d,\mathrm{rad}}$ and f and g define the same hypersurface, then f must be a scalar multiple of g.)

Pick two coprime homogeneous polynomials u, v of the same degree e. Let $f = u^2 v$ and $g = uv^2$. Thus $f, g \in V_{n,3e}$. Now $\operatorname{rad}(f) = (uv) = \operatorname{rad}(g)$ so f and g define the same hypersurface.

Question 4. *Let $V \subseteq \mathbb{P}^n$ be an irreducible projective algebraic set of dimension d.*

Sub-question 4(a). *Let*

$$\Sigma = \{(p, q, r) \in V \times V \times \mathbb{P}^n : p \neq q \text{ and } r \in L_{pq}\}$$

and let $\overline{\Sigma}$ denote the Zariski closure of Σ. You may assume that $\overline{\Sigma}$ is irreducible, and that

$$\overline{\Sigma} \cap \{(p, q, r) \in V \times V \times \mathbb{P}^n : p \neq q\} = \Sigma.$$

Prove that the projection $\pi_{1,2} \colon \overline{\Sigma} \to V \times V$ (onto the first two factors) is surjective.

Consider some pair $(p, q) \in V \times V$ with $p \neq q$. Then $q \in L_{pq}$ so $(p, q, q) \in \Sigma$. Thus $(p, q) \in \pi_{1,2}(\Sigma) \subseteq \pi_{1,2}(\overline{\Sigma})$.

What about the diagonal in $V \times V$? $\overline{\Sigma}$ is a closed subset of $V \times V \times \mathbb{P}^n$ (by construction) and therefore a projective algebraic set. Hence by completeness, $\pi_{1,2}(\overline{\Sigma})$ is a closed subset of $V \times V$. But the complement of the diagonal is a non-empty open subset of $V \times V$, which is irreducible, so it is dense in $V \times V$. Thus we have shown that $\pi_{1,2}(\overline{\Sigma})$ is a closed subset of $V \times V$ is a closed subset of $V \times V$ which contains a dense subset, so it must be all of $V \times V$.

Sub-question 4(b). *By applying the fibre dimension theorem to the projection $\pi_{1,2} \colon \overline{\Sigma} \to V \times V$, show that* $\dim \overline{\Sigma} \leq 2d + 1$.

By part (a), $\pi_{1,2}\colon \overline{\Sigma} \to V \times V$ is surjective. According to the question, we may assume that $\overline{\Sigma}$ is irreducible. Hence we can apply the fibre dimension theorem: for every $(p,q) \in V \times V$, we have

$$\dim \pi_{1,2}^{-1}(p,q) \geq \dim \overline{\Sigma} - \dim(V \times V).$$

Now $\dim(V \times V) = 2d$ so this becomes

$$\dim \overline{\Sigma} \leq 2d + \dim \pi_{1,2}^{-1}(p,q).$$

For any $(p,q) \in V \times V$ such that $p \neq q$, the fibre $\pi_{1,2}^{-1}(p,q)$ is

$$(\{p\} \times \{q\} \times \mathbb{P}^n) \cap \overline{\Sigma} = (\{p\} \times \{q\} \times \mathbb{P}^n) \cap \Sigma = \{p\} \times \{q\} \times L_{pq}$$

(where the first equality is an assumption from the question). Hence $\dim \pi_{1,2}^{-1}(p,q) = \dim L_{pq} = 1$.

Substituting this in (*), along with $\dim(V \times V) = 2d$, we get

$$\dim \overline{\Sigma} \leq 2d + 1.$$

(In fact, we have shown that $\dim \pi_{1,2}^{-1}(p,q) = 1$ for all (p,q) in a dense subset of $V \times V$, so we could use the second part of the fibre dimension theorem to conclude that $\dim \overline{\Sigma} = 2d + 1$.)

Sub-question 4(c). *Let*

$$S = \bigcup_{\substack{(p,q) \in V \times V \\ p \neq q}} L_{pq}$$

and let π_3 denote the projection onto the last factor $\overline{\Sigma} \to \mathbb{P}^n$. Prove that $\pi_3(\Sigma) = S$ and deduce that S is contained in a closed subset of $\mathbb{P}^n$ of dimension at most $2d+1$.

If $(p,q,r) \in \Sigma$, then $r \in L_{pq}$ with $p, q \in V \times V$ and $p \neq q$, so $r = \pi_3(p,q,r) \in S$. Conversely, if $r \in S$, then there exists $(p,q) \in V \times V$ such that $p \neq q$ and $r = L_{pq}$. Thus $(p,q,r) \in \Sigma$ so $r \in \pi_3(\Sigma)$.

Thus $S = \pi_3(\Sigma) \subseteq \pi_3(\overline{\Sigma})$, where $\pi_3(\overline{\Sigma})$ is a closed subset of $\mathbb{P}^n$ by completeness.

By fact (1) from Lecture 20, $\dim \pi_3(\overline{\Sigma}) \leq \dim \Sigma$. Using part (b) gives $\dim \pi_3(\overline{\Sigma}) \leq 2d + 1$.

Alternatively, we could use the fibre dimension theorem: According to the question, we are assuming that $\overline{\Sigma}$ is irreducible. Hence by the fibre dimension theorem, for r in some non-empty open subset of $\pi_3(\overline{\Sigma})$, we have

$$\dim \pi_3^{-1}(r) = \dim \overline{\Sigma} - \dim \pi_3(\overline{\Sigma}).$$

But $\dim \pi_3^{-1}(r) \geq 0$, so this implies that $\dim \Sigma - \dim \pi_3(\overline{\Sigma}) \geq 0$. Again we conclude using part (b).

Sub-question 4(d). *Deduce that if $2d+1 < n$, then there exists a point $r \in \mathbb{P}^n$ such that every line through r intersects V in at most one point.*

By part (c), if $2d+1 < n$, then S is contained in a closed subset of $\mathbb{P}^n$. In particular, $S \neq \mathbb{P}^n$ so we can pick $r \in \mathbb{P}^n \setminus S$.

Consider any line L through r. If $L \cap V$ has more than one point, we can pick distinct $p, q \in V$. Because two points suffice to determine a line, $L = L_{pq}$ and so $r \in L_{pq}$. This contradicts the fact that $r \notin S$. So $L \cap V$ contains at most one point.