Algebraic Geometry
Problem Sheet 4 – Solutions

Question 1.

Sub-question 1(a). *Prove that the rational map $\varphi\colon [x:y:z] \mapsto [xy:yz:zx]$ is dominant and gives a birational equivalence $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Write down a formula for a rational inverse ψ of φ.*

The image of φ contains $\{[u:v:w] \in \mathbb{P}^2 : uvw \neq 0\}$ (because if u, v, w are all non-zero, then

$$\varphi([uw:uv:vw]) = [u^2vw : uw^2v : uvw^2] = [u:v:w].$$

The rational inverse is given by

$$\psi([u:v:w]) = [uw:uv:vw].$$

To check this, the same proof a for φ shows that ψ is dominant, while the calculation above shows that $\varphi \circ \psi = \mathrm{id}$ on the open subset of $\{[u:v:w] \in \mathbb{P}^2 : uvw \neq 0\}$. A similar calculation shows that $\varphi \circ \psi = \mathrm{id}$.

Sub-question 1(b). *What are the domains of definition of φ and ψ?*

The representative $[xy:yz::zx$ for φ is defined at those $[x:y:z] \in \mathbb{P}^2$ where at least one of xy, yz, zx is non-zero, in other words whenever at most one out of x, y, z is zero. If two out of x, y, z are zero, then because we are using homogeneous coordinates, we may as well assume that the other coordinate is equal to 1. Hence

$$\operatorname{dom}\varphi \supseteq \mathbb{P}^2 \setminus \{[0:0:1], [0:1:0], [1:0:0]\}.$$

Now we show that $[0:0:1] \notin \operatorname{dom}\varphi$. Suppose that there was some other representative $[f:g:h]$ for φ which is well-defined at $[0:0:1]$. Then $XYg = YZf$ as polynomials, so X divides f. Therefore $f(0,0,1) = 0$. Similarly, $ZXg = YZh$ so $Y \mid g$ and $X \mid h$, so $g(0,0,1) = h(0,0,1) = 0$. Thus $[f:g:h]$ cannot be well-defined at $[0:0:1]$ for any representative of φ.

Reordering the coordinates, similar arguments show that $[0:1:0] \notin \operatorname{dom}\varphi$ and $[1:0:0] \notin \operatorname{dom}\varphi$. Therefore

$$\operatorname{dom}\varphi = \mathbb{P}^2 \setminus \{[0:0:1], [0:1:0], [1:0:0]\}.$$

Because ψ has the same formula as φ except for changing the order of the coordinates, $\operatorname{dom}\psi = \operatorname{dom}\varphi$.

Sub-question 1(c). *Write down open subsets $A, B \subseteq \mathbb{P}^2$ such that φ induces an isomorphism $A \to B$.*

Following the proof of Lemma 15.2 from the lectures, we need to take

$$A = \varphi^{-1}(\operatorname{dom}\psi), \quad B = \psi^{-1}(\operatorname{dom}\varphi).$$

Now

$$\begin{aligned}
\varphi([x:y:z]) = [0:0:1] &\Leftrightarrow xy = yz = 0 \\
&\Leftrightarrow y = 0 \text{ or } (x = 0 \text{ and } z = 0), \\
\varphi([x:y:z]) = [0:1:0] &\Leftrightarrow x = 0 \text{ or } (y = 0 \text{ and } z = 0), \\
\varphi([x:y:z]) = [1:0:0] &\Leftrightarrow z = 0 \text{ or } (x = 0 \text{ and } y = 0).
\end{aligned}$$

Combining this, we see that $\varphi([x:y:z]) \in \{[0:0:1],[0:1:0],[1:0:0]\}$ if and only if at least one of x, y, z is zero. Hence

$$A = \{[x:y:z] \in \mathbb{P}^2 : x, y, z \text{ are all non-zero}\}.$$

Likewise,

$$B = \{[u:v:w] \in \mathbb{P}^2 : u, v, w \text{ are all non-zero}\}.$$

Question 2.

Sub-question 2(a). *Prove that $\mathbb{P}^1 \times \mathbb{A}^1$ is not isomorphic to either an affine or a projective algebraic set.*

There are many ways to do this, almost all relying in some way on the completeness of projective varieties. This is probably the quickest:

We use the fact that every regular map from an irreducible projective variety to an affine variety is constant.

Observe that $\mathbb{P}^1 \times \mathbb{A}^1$ is irreducible because it is a product of irreducible varieties (by sheet 1, question 10 – the argument works for quasi-projective varieties as well as affine ones).

There is a non-constant regular map $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ (projection onto the second factor) so $\mathbb{P}^1 \times \mathbb{A}^1$ is not projective.

There is a non-constant regular map $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{A}^1$ ($x \mapsto (x, y)$ for any fixed $y \in \mathbb{A}^1$) so $\mathbb{P}^1 \times \mathbb{A}^1$ is not affine.

Sub-question 2(b). *Write down homogeneous polynomials defining closed subsets $V, Z \subseteq \mathbb{P}^3$ such that $\mathbb{P}^1 \times \mathbb{A}^1$ is isomorphic to $V \cap (\mathbb{P}^3 \setminus Z)$. (Use the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$.)*

$\mathbb{P}^1 \times \mathbb{A}^1$ is the complement of the closed set $\mathbb{P}^1 \times \{[0:1]\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Since the Segre embedding $\sigma_{1,1}$ is an isomorphism onto its image $\Sigma_{1,1} \subseteq \mathbb{P}^3$, it induces an isomorphism between $\mathbb{P}^1 \times \mathbb{A}^1$ and

$$\Sigma_{1,1} \setminus \sigma_{1,1}(\mathbb{P}^1 \times \{[0:1]\}).$$

Thus we can take $V = \Sigma_{1,1}$ and $Z = \sigma_{1,1}(\mathbb{P}^1 \times \{[0:1]\})$ (which is closed by completeness).

Now $V = \Sigma_{1,1}$ is defined by the polynomial $WZ - XY$.

The Segre embedding is given by

$$([x_0 : x_1], [y_0 : y_1] \mapsto [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1]$$

so

$$\sigma_{1,1}(\mathbb{P}^1 \times \{[0:1]\}) = \{[0:0:x_0:x_1] \in \mathbb{P}^3 : [x_0:x_1] \in \mathbb{P}^1\}.$$

In other words, we can take $Z = \sigma_{1,1}(\mathbb{P}^1 \times \{[0:1]\})$ defined by the polynomials $W = Y = 0$.

Question 3. *Let $H \subseteq \mathbb{P}^n$ be a hypersurface, defined by a homogeneous polynomial f of degree d. Let $V \subseteq \mathbb{P}^n$ be an irreducible projective algebraic set such that $V \cap H$ is empty.*

Sub-question 3(a). *Prove that the functions $X_i X_j^{d-1}/f$ are constant on V for every i, j. (Make clear how you are using the condition that $V \cap H$ is empty.)*

We can define a regular map $V \to \mathbb{P}^1$ by $[f : X_i X_j^{d-1}]$ (note that this is well-defined because $\deg(X_i X_j^{d-1}) = d = \deg f$). Because $V \cap H$ is empty, f is never zero on V and so this map has image inside $\mathbb{A}^1$ (this is just a way of writing the regular function $X_i X_j^{d-1}/f$). Thus we have a regular function on an irreducible projective algebraic set, so it is constant by Lemma 17.3 from lectures.

Sub-question 3(b). *Deduce that V is a point.*

Pick some j such that X_j is non-zero at some point of V. (This is possible because $[0 : \cdots : 0]$ is not a valid list of homogeneous coordinates.) Taking $i = j$ in (a), X_j^d is constant on V so X_j is non-zero everywhere on V. Hence, for every point $[x_0 : \cdots : x_n] \in V$, we can say

$$[x_0 : \cdots : x_n] = [x_0 x_j^{d-1} : \cdots : x_n x_j^{d-1}].$$

But all the functions $X_i X_j^{d-1}$ are constant on V, so there is only a single point in V.

Question 4. *By considering the rank of the matrix*

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ y_0 & y_1 & \cdots & y_n \\ z_0 & z_1 & \cdots & z_n \end{pmatrix}$$

or otherwise, show that the set

$$\{(x, y, z) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n : \textit{there exists a line in } \mathbb{P}^n \textit{ containing all of } x, y, z\}$$

is a closed subset of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$. (You may assume without proof that closed subsets of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ are described by "trihomogeneous" polynomials, i.e. polynomials in $k[X_0, \ldots, X_n, Y_0, \ldots, Y_n, Z_0, \ldots, Z_n]$ which are separately homogeneous in the X-variables, in the Y-variables and in the Z-variables.)

The points $x, y, z \in \mathbb{P}^n$ are contained in the line L if and only if the vectors $(x_0, \ldots, x_n)$, $(y_0, \ldots, y_n)$, $(z_0, \ldots, z_n) \in k^{n+1}$ (given by the homogeneous coordinates of x, y, z) are contained in $C(L)$, which is a vector subspace of dimension 2. In other words, x, y, z are contained in a line if and only if the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ y_0 & y_1 & \cdots & y_n \\ z_0 & z_1 & \cdots & z_n \end{pmatrix}$$

has rank less than or equal to 2.

By linear algebra, a matrix has rank less than or equal to 2 if and only if the determinants of all its 3×3 submatrices are zero. Because the determinant of a 3×3 matrix is a trihomogeneous polynomial in the entries of the matrix (homogeneous of degree 1 in the entries of each row), this shows that the set

$$\{(x, y, z) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n : \text{there exists a line in } \mathbb{P}^n \text{ containing all of } x, y, z\}$$

is a closed subset of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. More explicitly, this set is equal to

$$\left\{ (x, y, z) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n : \det \begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix} = 0 \text{ for all } i, j, k \right\}.$$

Question 5. *Let* $\Sigma_{1,2}$ *denote the image of the Segre embedding* $\sigma_{1,2}\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$. *Consider two distinct points in* $\mathbb{P}^1 \times \mathbb{P}^2$*:*

$$(a, x) = ([a_0 : a_1], [x_0 : x_1 : x_2]) \text{ and } (b, y) = ([b_0 : b_1], [y_0 : y_1 : y_2]).$$

Let $p = \sigma_{1,2}(a, x)$ *and* $q = \sigma_{1,2}(b, y)$*.*

Sub-question 5(a). *Write down three homogeneous polynomials of degree 2 which define* $\Sigma_{1,2}$ *as a subset of* $\mathbb{P}^5$*.*

If we label homogeneous coordinates of points in $\mathbb{P}^5$ as $[z_{00} : z_{01} : z_{02} : z_{10} : z_{11} : z_{12}]$, then $\Sigma_{1,2}$ is defined by the homogeneous polynomials

$$\det\begin{pmatrix} z_{00} & z_{01} \\ z_{10} & z_{11} \end{pmatrix} = \det\begin{pmatrix} z_{00} & z_{02} \\ z_{10} & z_{12} \end{pmatrix} = \det\begin{pmatrix} z_{01} & z_{02} \\ z_{11} & z_{12} \end{pmatrix} = 0$$

or, expanding these out,

$$z_{00}z_{11} - z_{10}z_{01} = z_{00}z_{12} - z_{10}z_{02} = z_{01}z_{12} - z_{11}z_{02} = 0.$$

Sub-question 5(b). *Write down a regular map* $\mathbb{P}^1 \to \mathbb{P}^5$ *whose image is the line* $L_{pq} \subseteq \mathbb{P}^5$ *which passes through* p *and* q *(no proofs are required).*

We have

$$p = \sigma_{1,2}(a, x) = [a_0x_0 : a_0x_1 : a_0x_2 : a_1x_0 : a_1x_1 : a_1x_2],$$
$$q = \sigma_{1,2}(b, y) = [b_0y_0 : b_0y_1 : b_0y_2 : b_1y_0 : b_1y_1 : b_1y_2].$$

So a map $\lambda\colon \mathbb{P}^1 \to \mathbb{P}^5$ with image L_{pq} is given by

$$[s : t] \mapsto [sa_0x_0+tb_0y_0 : sa_0x_1+tb_0y_1 : sa_0x_2+tb_0y_2 : sa_1x_0+tb_1y_0 : sa_1x_1+tb_1y_1 : sa_1x_2+tb_1y_2].$$

Sub-question 5(c). *Find (with proof) three polynomial equations, homogeneous of degree 1 with respect to each of* a, x, b, y*, which must be satisfied by* (a, x) *and* (b, y) *if* $L_{pq} \subseteq \Sigma_{1,2}$*.*

$L_{pq} \subseteq \Sigma_{1,2}$ if and only if $\lambda([s : t]) \in \Sigma_{1,2}$ for all $[s : t] \in \mathbb{P}^1$, where λ denotes the map from part (b).

Substituting the formula for λ into the equations from (a), we see that $\lambda([s : t]) \in \Sigma_{1,2}$ if and only if

$$\begin{aligned}
(sa_0x_0 + tb_0y_0)(sa_1x_1 + tb_1y_1) - (sa_1x_0 + tb_1y_0)(sa_0x_1 + tb_0y_1) &= 0,\\
(sa_0x_0 + tb_0y_0)(sa_1x_2 + tb_1y_2) - (sa_1x_0 + tb_1y_0)(sa_0x_2 + tb_0y_2) &= 0,\\
(sa_0x_1 + tb_0y_1)(sa_1x_2 + tb_1y_2) - (sa_1x_1 + tb_1y_1)(sa_0x_2 + tb_0y_2) &= 0.
\end{aligned} \tag{1}$$

Multiplying out the first of these simplifying, we get

$$sta_0b_1x_0y_1 + sta_1b_0x_1y_0 - sta_1b_0x_0y_1 - sta_0b_1x_1y_0 = 0.$$

This equation has a factor of st so if it holds for all $[s : t] \in \mathbb{P}^1$, then it must still hold after dividing out that factor. In other words, if $L_{pq} \subseteq \Sigma_{1,2}$ then

$$a_0b_1x_0y_1 + a_1b_0x_1y_0 - a_1b_0x_0y_1 - a_0b_1x_1y_0 = 0.$$

Applying the same argument to the second and third lines from (1), we get

$$a_0b_1x_0y_2 + a_1b_0x_2y_0 - a_1b_0x_0y_2 - a_0b_1x_2y_0 = 0,$$
$$a_0b_1x_1y_2 + a_1b_0x_2y_1 - a_1b_0x_1y_2 - a_0b_1x_2y_1 = 0.$$

Sub-question 5(d). *By factorising the equations from (c), conclude that if* $L_{pq} \subseteq \Sigma_{1,2}$*, then either* $a = b$ *or* $x = y$*.*

The equations from part (c) factorise as

$$(a_0b_1 - a_1b_0)(x_0y_1 - x_1y_0) = 0,$$
$$(a_0b_1 - a_1b_0)(x_0y_2 - x_2y_0) = 0,$$
$$(a_0b_1 - a_1b_0)(x_1y_2 - x_2y_1) = 0.$$

Hence if $L_{pq} \subseteq \Sigma_{1,2}$, then either

$$a_0b_1 - a_1b_0 = 0 \tag{2}$$

or

$$x_0y_1 - x_1y_0 = x_0y_2 - x_2y_0 = x_1y_2 - x_2y_1 = 0. \tag{3}$$

If (2) holds, then $[a_0 : a_1] = [b_0 : b_1]$ as homogeneous coordinates, so $a = b$ in $\mathbb{P}^1$.

If (3) holds, then $[x_0 : x_1 : x_2] = [y_0 : y_1 : y_2]$ as homogeneous coordinates, so $x = y$ in $\mathbb{P}^2$.