Algebraic Geometry
Problem Sheet 3 – Solutions

Question 1.

Sub-question 1(a). *Let $\varphi\colon \mathbb{P}^n \to \mathbb{P}^m$ be a regular map. Prove that there exist homogeneous polynomials $f_0, \dots, f_m \in k[X_0, \dots, X_n]$ such that the expression*

$$[f_0(x_0, \dots, x_n) : \dots : f_m(x_0, \dots, x_n)]$$

defines φ at every point of $\mathbb{P}^n$. (Use the fact that $k[X_0, \dots, X_n]$ is a UFD.)

Pick an open set $U \subseteq \mathbb{P}^n$ and a list of polynomials $f_0, \dots, f_m \in k[X_0, \dots, X_n]$ which define φ on U. If any polynomial h divides all of $f_0, \dots, f_m$, then $[f_0/h : \dots : f_m/h]$ still defines φ on U. So we can assume that $f_0, \dots, f_m$ have no common factor.

Consider a point $x \in \mathbb{P}^n \setminus U$. Because φ is regular at x, there exist polynomials $g_0, \dots, g_m \in k[X_0, \dots, X_n]$ such that the g_i do not all vanish at x, and such that

$$[f_0 : \dots : f_m] = [g_0 : \dots : g_m]$$

on some open set U' where both expressions make sense. This implies that

$$f_i g_j = g_i f_j \text{ on } U' \text{ for all } i, j.$$

But $\mathbb{P}^n$ is irreducible, so U' is dense in $\mathbb{P}^n$. Hence we can conclude that

$$f_i g_j = g_i f_j \text{ as polynomials in } k[X_0, \dots, X_n].$$

Let h_i denote the highest common factor of f_i and g_i. Then $f_i' = f_i/h_i$ has no common factors with $g_i' = g_i/h_i$, and

$$f_i'.g_j = g_i'.f_j.$$

Hence the fact that $k[X_0, \dots, X_n]$ is a unique factorisation domain implies that f_i' divides f_j (for every i and j).

For any single value of i, we have shown that f_i' divides f_j for all j. But the f_j have no common factor, so f_i' is a constant. Hence f_i divides g_i. This holds for all i.

Since there is some g_i which does not vanish at x, and f_i divides g_i, we conclude that there is some f_i which does not vanish at x. Hence the expression $[f_0 : \dots : f_m]$ is defined at x.

Finally, because U is dense in $\mathbb{P}^n$, we must have $\varphi = [f_0 : \dots : f_m]$ everywhere (Lemma 14.1).

Sub-question 1(b). *Prove that, if $\varphi\colon \mathbb{P}^n \to \mathbb{P}^1$ is a non-constant regular map, then the image of φ is not contained in $\mathbb{A}^1$.*

Using completeness, we have seen in the lectures (Corollary 17.3) that this holds not only for $\mathbb{P}^n$, but for all irreducible projective varieties. However the point of this question was to give a simpler proof for the case of $\mathbb{P}^n$, using part (a).

The question is vague about how we embed $\mathbb{A}^1$ in $\mathbb{P}^1$. It does not really matter, let's use $x \mapsto [1 : x]$.

By part (a), if $\varphi\colon \mathbb{P}^n \to \mathbb{P}^1$ is a regular map, then there exist homogeneous polynomials $f, g \in k[X_0, \dots, X_n]$ of the same degree such that $\varphi(x) = [f(x) : g(x)]$ at every point $x \in \mathbb{P}^n$.

Since φ is non-constant, at least one of f and g is non-constant. Since f and g have the same degree, they are both non-constant.

In particular, f is a non-constant homogeneous polynomial, so there exist points $x \in \mathbb{P}^n$ where $f(x) = 0$. At such a point,

$$\varphi(x) = [0 : g(x)] \notin \mathbb{A}^1.$$

Sub-question 1(c). *Prove that every rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is regular.*

Let $\varphi\colon \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ be a rational map, determined by homogeneous polynomials $f_0, \dots, f_m \in k[X,Y]$. As in part (a), we may assume that $f_0, \dots, f_m$ have no common factor.

Because the base field k is algebraically closed, every homogeneous polynomial $f \in k[X,Y]$ factors as a product of linear homogeneous polynomials. (To prove this, consider the one-variable non-homogeneous polynomial $f(X,1)$. Because k is algebraically closed, this factors as a product of linear polynomials in X. We can then homogenise each factor to get a factorisation of the original f.)

If $f_j(X,Y) = \prod_{j=1}^d (a_j X + b_j Y)$, then

$$\{[x : y] : f_j(x,y) = 0\} = \{[b_j : -a_j] : j = 1, \dots, d\}.$$

If there is some point $[u : v]$ where all the f_j vanish, then $vX - uY$ divides all the f_j. This contradicts the fact that $f_0, \dots, f_m$ have no common factor.

Hence at every point of $\mathbb{P}^1$, at least one of the f_j is non-zero, and so φ is regular.

Question 2.

Sub-question 2(a). *Prove that, if $\varphi\colon \mathbb{A}^2 \dashrightarrow k$ is a rational function, then there exist polynomials $f, g \in k[X,Y]$ such that $\varphi = f/g$ and such that $g(x) \neq 0$ on all of* $\operatorname{dom}\varphi$.

Choose polynomials $f, g \in k[X,Y]$ such that $\varphi = f/g$. We may assume that f and g have no common factor.

If f'/g' is any other fraction of polynomials representing φ, then

$$fg' = f'g.$$

Because f, g have no common factor, and because $k[X,Y]$ is a unique factorisation domain, g divides g'. But then $g'(x,y) \neq 0$ implies that $g(x,y) \neq 0$, so g is non-zero everywhere on $\operatorname{dom}\varphi$.

Sub-question 2(b). *Deduce that, if $\varphi\colon \mathbb{A}^2 \dashrightarrow k$ is regular on $\mathbb{A}^2 \setminus \{(0,0)\}$, then it is regular at $(0,0)$ too.*

Write $\varphi = f/g$ as in (a): we have $g(x,y) \neq 0$ for all $(x,y) \neq (0,0)$.

We claim that g is a constant polynomial. To prove this, write

$$g(X,Y) = \sum_{i=0}^d a_i(X) Y.$$

For each value of x other than 0, the one-variable polynomial $\sum_{i=0}^d a_i(x) Y$ has no roots, and therefore must be a constant polynomial. Thus

$$a_i(x) = 0 \text{ for all } x \neq 0 \text{ for } i > 0.$$

But $a_i(X)$ is itself a polynomial, so this implies that $a_i(X)$ is identically zero for $i > 0$. In other words, g is a constant.

The constant value of g must be non-zero, so $g(0,0) \neq 0$ and hence φ is regular at $(0,0)$.

Sub-question 2(c). *Consider the quasi-projective algebraic set $V = \mathbb{A}^2 \setminus \{(0,0)\}$. Prove that the ring of regular functions on V is the polynomial ring $k[X,Y]$.*

If f is a regular function on V, then f is also a rational function on V. Since V is open in $\mathbb{A}^2$, f extends to a rational function on $\mathbb{A}^2$. By (b), since this function is regular on V, it is regular on all of $\mathbb{A}^2$.

Hence the natural restriction map $k[\mathbb{A}^2] \to k[V]$ is surjective. This map is injective because V is dense in $\mathbb{A}^2$. So $k[V] \cong k[\mathbb{A}^2] = k[X,Y]$.

Sub-question 2(d). *Prove that V is not isomorphic to any affine algebraic set. (Use the fact that, in an affine algebraic set W, every proper ideal in $k[W]$ defines a non-empty subset.)*

Consider the maximal ideal $(X,Y) \subseteq k[X,Y] \cong k[V]$. The equations $x=0$, $y=0$ have no common solution in V. Hence the vanishing set of this ideal in V is empty.

If V were affine, then the ideal $(X,Y) \subsetneq k[V]$ would define a non-empty subset of V, contradicting this example.

Question 3. *Consider the example from lecture 15:*

$$\overline{C} = \{[w : x : y] \in \mathbb{P}^2 : w^2 y = x^3\}$$

and $\varphi\colon \overline{C} \dashrightarrow \mathbb{P}^1$ is the rational map represented by $[w:x:y] \mapsto [w:x]$. Prove that φ is not regular at $[0:0:1]$.

Following the outline given on the problem sheet: There is a regular map $\psi\colon \mathbb{A}^1 \to C = \mathbb{V}(Y - X^3)$ given by $t \mapsto (t, t^3)$. Homogenising, we get the regular map $\overline{\psi}\colon \mathbb{P}^1 \to \overline{C}$ defined by

$$\overline{\psi}([s:t]) = [s^3 : s^2 t : t^3]$$

(where we embed $\mathbb{A}^2$ into $\mathbb{P}^2$ by $(x,y) \mapsto [1:x:y]$).

Suppose we can represent $\varphi\colon \overline{C} \dashrightarrow \mathbb{P}^1$ by some homogeneous polynomials $f, g \in k[W,X,Y]$ which are not both zero at $[0:0:1]$. The equivalence relation between representations of a rational map tells us that

$$Xf(W,X,Y) = Wg(W,X,Y)$$

on $\overline{C}$. Substituting $\overline{\psi}([s:t])$ into this equation, we get

$$s^2 t f(s^3, s^2 t, t^3) = s^3 g(s^3, s^2 t, t^3). \tag{1}$$

Since $\overline{\psi}([s:t]) \in \overline{C}$ for all $(s,t) \neq (0,0)$, (1) holds for all such s,t.

Since $\mathbb{A}^2 \setminus \{(0,0)\}$ is Zariski dense in $\mathbb{A}^2$, we conclude that (1) holds as an equation in the polynomial ring $k[S,T]$. Now we use the same trick as in sheet 2, question 6: since $k[S,T]$ is an integral domain, (1) implies that

$$Tf(S^3, S^2T, T^3) = Sg(S^3, S^2T, T^3) \tag{2}$$

and hence that S divides $f(S^3, S^2T, T^3)$. But each term of $f(S^3, S^2T, T^3)$ either contains no power of S or is divisible by S^2, so this implies that

$$S^2 \mid f(S^3, S^2T, T^3).$$

Therefore (2) implies that

$$S \mid g(S^3, S^2T, T^3). \tag{3}$$

Substituting $S = 0$, $T = 1$ in (3), we see that $g(0,0,1) = 0$.

Now substituting $S = 0$, $T = 1$ in (2), we see that $f(0,0,1) = 0$.

Thus, for every representative $[f : g]$ of φ, $f(0,0,1) = g(0,0,1) = 0$ and so φ is not regular at $[0 : 0 : 1]$.

Alternative solution. Consider a different embedding of $\mathbb{A}^2$ into $\mathbb{P}^2$, given by $(w, x) \mapsto [w : x : 1]$. Then $\overline{C} \cup \mathbb{A}^2 = \mathbb{V}(Y^2 - X^3)$, and $[0 : 0 : 1] \in \mathbb{P}^2$ corresponds to the point $(0, 0) \in \mathbb{A}^2$.

The rational function $\varphi \colon \overline{C} \dashrightarrow \mathbb{P}^1$ restricts to the rational function $\overline{C} \dashrightarrow \mathbb{A}^1$ given by

$$(w, x) \mapsto w/x.$$

We showed in sheet 2, question 6 that this rational function is not regular at $(0, 0)$.

Question 4. *Let*

$$f(X_0, \dots, X_n) = \sum_{0 \leq i \leq j \leq n} a_{ij} X_i X_j \in k[X_0, \dots, X_n]$$

be a homogeneous polynomial of degree 2 *(where a_{ij} are not all zero).*

The projective algebraic set $V \subseteq \mathbb{P}^n$ defined by the equation $f = 0$ is called a ***quadric****. We aim to show that, if V is irreducible, then it is birational to $\mathbb{P}^{n-1}$.*

This question contained several typos and a mathematical error in the statement of (a). I apologise for that.

Sub-question 4(a). *Prove that V is irreducible if and only if f is irreducible.*

This is not quite true: it should say V is irreducible if and only if f is irreducible or $f = g^2$ where g is a linear homogeneous polynomial.

If g is a linear homogeneous polynomial, then V is a hyperplane which is certainly irreducible.

Otherwise, assume that f is not of the form g^2. Then the ideal (f) is radical, so by the Projective Nullstellensatz, the homogeneous ideal I of polynomials which vanish on V is equal to (f).

In one direction: if V is reducible, say $V = V_1 \cup V_2$, then we can find homogeneous polynomials $f_1, f_2 \notin I$ such that f_1 vanishes on V_1 and f_2 vanishes on V_2. Then $f_1 f_2$ vanishes on V, so $f_1 f_2 \in I$. By the above argument, this implies that $f \mid f_1 f_2$. But $f \nmid f_1$ and $f \nmid f_2$, so the fact that $k[X_0, \dots, X_n]$ is a UFD implies that $f = g_1 g_2$ for some non-constant polynomials g_1, g_2 such that $g_1 \mid f_1$ and $g_2 \mid f_2$. Thus the polynomial f is reducible.

In the other direction: if f is reducible, then $f = f_1 f_2$ where f_1 and f_2 are both homogeneous linear polynomials. Since f is not of the form g^2, f_1 is not a scalar multiple of f_2. Hence neither of the hyperplanes $V_1 : f_1 = 0$ and $V_2 : f_2 = 0$ is contained in the other. But $V = V_1 \cup V_2$ so V is reducible.

Sub-question 4(b). *Let $p = [0 : \cdots : 0 : 1] \in \mathbb{P}^n$. What condition on the coefficients a_{ij} is equivalent to: $p \in V$?*

$a_{nn} = 0$.

Sub-question 4(c). *From now on, we assume that V is irreducible and that $p \in V$. (Actually, we should assume that f is irreducible, which we saw is slightly stronger than V being irreducible.)*

Let $H \subseteq \mathbb{P}^n$ be the hyperplane defined by the equation $X_n = 0$. For each point $x \in H$, let L_x denote the line through x and p. Prove that either $L_x \subseteq V$ or $L_x \cap V$ contains 1 or 2 points.

Let $x = [x_0 : \cdots : x_{n-1} : 0] \in H$. The line through x and p is

$$L_x = \{[sx_0 : sx_1 : \cdots : sx_{n-1} : t] \in \mathbb{P}^n : [s : t] \in \mathbb{P}^1\}.$$

Substituting this into the equation defining V gives

$$f(sx_0, \ldots, sx_{n-1}, t) = \sum_{0 \le i \le j \le n-1} a_{ij} x_i x_j s^2 + \sum_{i=0}^{n-1} a_{in} x_i st = 0$$

which is a homogeneous equation of degree 2 in s and t.

Since $a_{nn} = 0$, this equation factorises as

$$s.h_x(s,t) = 0 \tag{4}$$

where $h_x(s,t)$ is a linear homogeneous polynomial in s and t, depending on x. Whenever (*) is satisfied, either $s = 0$ (corresponding to the point $p \in V$) or $h_x(s,t) = 0$.

Since h_x is homogeneous linear, either h_x is identically zero or h_x has a unique solution $[s : t] \in \mathbb{P}^1$.

(i) If $h_x \equiv 0$, then $L_x \subseteq V$.
(ii) If $h_x(s,t) = 0$ has unique solution $[0 : 1]$, then $L_x \cap V = \{p\}$ i.e. just one point.
(iii) If $h_x(s,t) = 0$ has a unique solution other than $[0 : 1]$, then $L_x \cap V$ consists of two points, p and the point coming from the solution of $h_x(s,t) = 0$.

Sub-question 4(d). *Let $U = \{x \in H : \#(L_x \cap V) = 2\}$. By finding equations for $H \setminus U$, prove that U is a Zariski open subset of H.*

$x \in H \setminus U$ if and only if x leads to either case (i) or (ii) above. We see that this happens if and only if $[0 : 1]$ is a solution to $h_x(s,t) = 0$.

If we compute $h_x(s,t)$ explicitly, we get

$$h_x(s,t) = \sum_{0 \le i \le j \le n-1} a_{ij} x_i x_j s + \sum_{i=0}^{n-1} a_{in} x_i t. \tag{5}$$

This has $[0 : 1]$ as a solution if and only if the coefficient of t is zero. In other words,

$$H \setminus U = \left\{ [x_0 : \cdots : x_{n-1} : 0] \in H : \sum_{i=0}^{n-1} a_{in} x_i = 0 \right\}.$$

This shows that $H \setminus U$ is Zariski closed, and so U is Zariski open in H.

Sub-question 4(e). *What condition on the coefficients a_{ij} is equivalent to: $U \neq \emptyset$?*

$U \neq \emptyset$ if and only if the polynomial defining $H \setminus U$ is non-zero, in other words if and only if $a_{in} \neq 0$ for at least one $i \in \{0, \ldots, n-1\}$.

Sub-question 4(f). *From now on, we assume that $U \neq \emptyset$.*

For $x \in U$, write $L_x \cap V = \{p, \psi(x)\}$. Find the coordinates of the point $\psi(x)$. Conclude that ψ is a rational map $H \dashrightarrow V$.

To find $\psi(x)$, we need solve $h_x(s,t) = 0$. We can read off from (5) that this has solution

$$[s:t] = \Big[-\sum_{i=0}^{n-1} a_{in}x_i : \sum_{0\le i\le j\le n-1} a_{ij}x_ix_j\Big].$$

The coordinates of $\psi(x)$ are given by substituting this into $[sx_0 : sx_1 : \cdots : sx_{n-1} : t]$. In other words,

$$\psi(x) = \Big[-\sum_{i=0}^{n-1} a_{in}x_i\, x_0 : \cdots : -\sum_{i=0}^{n-1} a_{in}x_i\, x_{n-1} : \sum_{0\le i\le j\le n-1} a_{ij}x_ix_j\Big]. \tag{6}$$

Each term is given by a homogeneous polynomial of degree 2 in $x_0, \dots, x_{n-1}$ and these polynomials are not all identically zero (since at least one of the a_{in} is non-zero). So this defines a rational map $\psi \dashrightarrow \mathbb{P}^n$.

The image of this rational map is contained in V because we selected $\psi(x)$ in V for all $x \in U$, and U is a dense subset of H.

Sub-question 4(g). *By considering the projection from p to H, deduce that V is birational to H.*

The projection π from p to H is a rational map $\pi\colon V \dashrightarrow H$. By construction, ψ and π induce are an inverse pair of bijections between the open subsets $U \subseteq H$ and $V \setminus \{p\} \subseteq V$. Thus they are dominant and rational inverses to each other, so V is birational to H.

Sub-question 4(h). *Determine the domain of definition of ψ. What is $\psi(x)$ for $x \in \operatorname{dom}\psi \setminus U$?*

By the representation for ψ which we obtained in part (f), we see that $x \in \operatorname{dom}\psi$ if either g or h is non-zero at x, where

$$g(x) = \sum_{i=0}^{n-1} a_{in}x_i, \quad h(x) = \sum_{0\le i\le j\le n-1} a_{ij}x_ix_j. \tag{7}$$

We shall show that ψ is not regular at any other points (that is: $x \notin \operatorname{dom}\psi$ if $g(x) = h(x) = 0$).

Suppose we have some representation $[f_0 : \cdots : f_n]$ for ψ. From (6), we get that

$$hf_0 = -gX_0f_n, \ \dots, \ hf_{n-1} = -gX_{n-1}f_n \tag{8}$$

in $k[X_0, \dots, X_{n-1}]$.

Note that $g \nmid h$ because otherwise g divides $f = h + gX_n$, contradicting our assumption that f is irreducible. Furthermore, g is linear and therefore irreducible. Since $k[X_0, \dots, X_{n-1}]$ is a UFD, we conclude from (8) that g divides each of $f_0, \dots, f_{n-1}$.

Letting $f_0 = gf_0'$, (8) tells us that $f_n = -hX_0f_0'$. Thus $h \mid f_n$.

Since $g \mid f_0, \dots, f_{n-1}$ and $h \mid f_n$, if $g(x) = h(x) = 0$, then $f_0, \dots, f_n$ are all zero at x. So there is no representation of ψ which is defined at x.

Thus $\operatorname{dom}\psi = \{x \in H : g(x) \neq 0 \text{ or } h(x) \neq 0\}$.

Now look at $\operatorname{dom}\psi \setminus U = \{x \in \mathbb{P}^n : g(x) = 0 \text{ and } h(x) \neq 0\}$. For $x \in \operatorname{dom}\psi \setminus U$, (6) tells us that

$$\psi(x) = [0 : \cdots : 0 : h(x)] = p.$$

Sub-question 4(i). *Apply this to the quadric defined by the equation $XY = WZ$ in $\mathbb{P}^3$: write down the rational map $\psi\colon H \dashrightarrow V$ and its domain of definition for this case.*

Interpreting $[W : X : Y : Z]$ as $[X_0 : X_1 : X_2 : X_3]$, the coefficients of f are $a_{12} = 1$, $a_{03} = -1$ and the remaining a_{ij} are zero. Thus (in the notation of (7)),

$$g(w, x, y) = -w, \quad h(w, x, y) = xy.$$

Hence

$$\psi([w : x : y : 0]) = [-gw : -gx : -gy : h] = [w^2 : wx : wy : xy]$$

while

$$\begin{aligned} \operatorname{dom}\psi &= \{[w : x : y : 0] \in H : w \neq 0 \text{ or } xy \neq 0\} \\ &= H \setminus \{[0 : 1 : 0 : 0], [0 : 0 : 1 : 0]\}. \end{aligned}$$

Sub-question 4(j). *Let $\pi\colon V \dashrightarrow H$ denote the projection from p. What is* $\operatorname{dom}\pi$*?*

The answer is different for $n = 2$ and $n \geq 3$. For $n = 2$, give an algebraic proof and an informal geometric explanation of what is going on. For $n \geq 3$, give either a geometric or an algebraic proof.

As a rational map $\mathbb{P}^n \dashrightarrow H$, the projection is regular on $\mathbb{P}^n \setminus \{p\}$. Therefore it is still regular on this set when we restrict to V. The only question to settle is whether $\pi\colon V \dashrightarrow H$ is regular at p. (The rational map $V \dashrightarrow H$ may be regular at p even though the rational map $\mathbb{P}^n \dashrightarrow H$ is not.)

Lemma. If $n = 2$, then $\pi\colon V \dashrightarrow H$ is regular at p, and so $\operatorname{dom}\pi = V$.

Proof. We have

$$f(w, x, y) = a_{00}w^2 + a_{11}x^2 + a_{01}wx + a_{02}wy + a_{12}xy$$

(recall that since $p \in V$, there is no y^2 term). Therefore we can obtain new representations of the rational map π by

$$\begin{aligned} \pi([w : x : y]) = [w : x : 0] &= [a_{00}w^2 + a_{01}wx + a_{02}wy : a_{00}wx + a_{01}x^2 + a_{02}xy : 0] \\ &= [-a_{11}x - a_{12}xy : a_{00}wx + a_{01}x^2 + a_{02}xy : 0] \\ &= [-a_{11}x - a_{12}y : a_{00}x + a_{01}x + a_{02}y : 0] \end{aligned}$$

(the first and last equality are because we are using homogeneous coordinates, the middle one is because $f = 0$ on V).

Substituting $p = [0 : 0 : 1]$ into the last expression, we get

$$\pi([0 : 0 : 1]) = [-a_{12} : a_{02} : 0].$$

We want to show that at least one of a_{12} and a_{02} is non-zero. Assume not: then $f(w, x, y)$ is a homogeneous polynomial of degree 2 in w and x only. Dehomogenising to get a polynomial of degree 2 in one variable (over an algebraically closed field), we see that it must factorise as a product of linear polynomials. This contradicts our hypothesis that f is irreducible.

Thus $\pi([0 : 0 : 1]) = [-a_{12} : a_{02} : 0]$ is well-defined, so π is regular at p. ☐

Informal geometrical explanation. When $x \in V$ becomes equal to p, "the line through x and p" becomes tangent to V at p. When $n = 2$, V is a curve and we can check that it is smooth at p (we have not defined this yet so I am speaking informally for now). Hence there is a unique tangent line to V at p, and $\pi(p)$ is the point where this tangent intersects H.

Lemma. If $n \geq 3$, then $\pi \colon V \dashrightarrow H$ is not regular at p, and so $\operatorname{dom} \pi = V \setminus \{p\}$.

Algebraic proof. Suppose that there exists some representation $[g_0 : \cdots : g_{n-1} : 0]$ for π such that $g_0, \ldots, g_{n-1}$ are not all zero at p. Assume without loss of generality that $g_0(p) \neq 0$.

The polynomials $g_0, \ldots, g_{n-1}$ are all homogeneous of the same degree, say d.

Since π can be represented by $[X_0 : \cdots : X_{n-1} : 0]$, we have

$$X_i g_0(X_0, \ldots, X_n) - X_0 g_i(X_0, \ldots, X_n) = 0 \text{ on } V$$

for each $i = 1, \ldots, n-1$. In other words,

$$X_i g_0(X_0, \ldots, X_n) - X_0 g_i(X_0, \ldots, X_n) = f h_i \text{ in } k[X_0, \ldots, X_n]$$

for some homogeneous polynomial h_i of degree $d + 1 - 2 = d - 1$.

Since $g_0(0, \ldots, 0, 1) \neq 0$, the X_n^d term in g_0 is non-zero. Therefore $X_i g_0 - X_0 g_i$ has a non-zero $X_i X_n^d$ term. Because f has no X_n^2 term, in order to get a non-zero $X_i X_n^d$ term in $f h_i$ we must have (i) $a_{in} \neq 0$ and (ii) h_i has non-zero X_n^{d-1} term.

This holds for every $i = 1, \ldots, n-1$. In particular, (picking different indices for (i) and (ii) – this is where we use $n \geq 3$) $a_{1n} \neq 0$ and h_2 has non-zero X_n^{d-1} term. This implies that $h_2 f$ has non-zero $X_1 X_n^d$ term.

But the only multiples of X_n^d which can appear in $X_2 g_0 - X_0 g_2$ are $X_2 X_n^d$ and $X_0 X_n^d$, giving a contradiction. □

Geometric proof. The idea is that the "informal geometric explanation" from the case $n = 2$ breaks down when $n \geq 3$, because instead of a tangent line at p, V now has a tangent hyperplane so there are many different lines to choose from and $\pi(p)$ should lie on all of these lines.

We can make this precise as follows: Assume for contradiction that π is regular at p (and thus is a regular map $V \to H$.

Consider any hyperplane $Z \subseteq \mathbb{P}^n$ which passes through p. Then π restricts to a regular map $V \cap Z \to H$. For each $x \in V \cap Z \setminus \{p\}$, $\pi(x)$ lies on the line L_x through x and p (by construction). This line lies entirely in the hyperplane Z and so $\pi(x) \in Z$ for all $x \in V \cap Z \setminus \{p\}$.

Claim. $V \cap Z \setminus \{p\}$ is dense in $V \cap Z$.

Proof of claim. $V \cap Z$ is defined by a homogeneous polynomial of degree 2 in the homogeneous coordinates on $Z \cong \mathbb{P}^{n-1}$. This new polynomial might not be irreducible. But from part (a), we see that either $V \cap Z$ is irreducible or it is a union of two hyperplanes in Z.

If $V \cap Z$ is irreducible, then the fact that $V \cap Z \setminus \{p\}$ is a non-empty open subset of $V \cap Z$ implies that it is dense.

If $V \cap Z$ is a union of two hyperplanes, then we either p is in just one of these hyperplanes, or it is in their intersection. In either case, one can check by hand that $V \cap Z \setminus \{p\}$ is dense in $V \cap Z$. (This is where we use the hypothesis that $n \geq 3$: if

$n = 2$, then $Z \cong \mathbb{P}^1$ so "the union of two hyperplanes in Z" means just two points, and $\{q\}$ is not dense in $\{p, q\}$.)

This completes the proof of the claim □

Using the claim, we deduce that $\pi(x) \in Z$ for all $x \in V \cap Z$. In particular $\pi(p) \in Z$.

For each point $y \in H$, it is possible to choose a hyperplane $Z_y \subseteq \mathbb{P}^n$ which contains p but not y. The above argument shows that $\pi(p) \in Z_y$, so $\pi(p) \neq y$. Thus $\pi(p)$ cannot be any point of H, giving a contradiction. □

Question 5. *A quadric in $\mathbb{P}^2$ is called a **conic**. Let*

$$f(X, Y, Z) = aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ \in k[X, Y, Z]$$

(where a, b, c, d, e, f are not all zero). Let $V \subseteq \mathbb{P}^2$ be the conic defined by f.

Sub-question 5(a). *Prove that two quadratics f and g define the same conic if and only if $f = \lambda g$ for some $\lambda \in k \setminus \{0\}$.*

Hence we can identify the set of conics with the projective space $\mathbb{P}^5$.

As a consequence of the Nullstellensatz, the quadratics f and g define the same conic if and only if $\mathrm{rad}(f) = \mathrm{rad}(g) \subseteq k[X, Y, Z]$. Since f is homogeneous of degree 2, either $\mathrm{rad}(f) = (f)$ or $f = \ell^2$ for some linear homogeneous polynomial ℓ (and similarly for g).

If $\mathrm{rad}(f) = (f)$, then in order to have $g \in \mathrm{rad}(f)$, it must be a constant multiple of f (since both f and g have degree 2). The same argument applies if $\mathrm{rad}(g) = (g)$: then f must be a constant multiple of g.

That leaves us with the case where $f = \ell^2$ and $g = m^2$ for smoe linear homogeneous polynomials $\ell, m \in k[X, Y, Z]$. Then $\mathrm{rad}(f) = (\ell)$ and $\mathrm{rad}(g) = (m)$. Since both ℓ and m have degree 1, if $(\ell) = (m)$ then m is a constant multiple of ℓ and so g is a constant multiple of f.

We can thus identify the set of conics with $\mathbb{P}^5$ by associating the conic $f = 0$ with the point whose homogeneous coordinates are given by the coefficients of f.

Sub-question 5(b). *For any point $p \in \mathbb{P}^2$, prove that the set of conics which contain p corresponds to a hyperplane in $\mathbb{P}^5$.*

Let $p = [x : y : z]$. The condition that p is in a conic is

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz.$$

Considering x, y, z as fixed and a, b, c, d, e, f as variables, this is a homogeneous linear polynomial and so defines a hyperplane in $\mathbb{P}^5$.

Sub-question 5(c). *Prove that, given any 5 points in $\mathbb{P}^2$, there exists a conic passing through all 5 points.*

For each of the points p_i ($i = 1, \ldots, 5$) there is a hyperplane $H_i \subseteq \mathbb{P}^5$ corresponding to the conics which contain p_i.

Five hyperplanes in $\mathbb{P}^5$ always have non-empty intersection (looking at the associated affine cones in k^6, we are imposing five linear equations in k^6, so the intersection will be a vector space of dimension at least $6 - 5 = 1$ and thus its image in $\mathbb{P}^5$ is non-empty).

Take a point in the intersection $H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5$: it corresponds to a conic which contains all of p_1, p_2, p_3, p_4, p_5.

Sub-question 5(d). *Find conditions on* 5 *points in* $\mathbb{P}^2$ *which are equivalent to:*

(i) There exists an irreducible conic passing through the 5 *points.*
(ii) There exists a unique conic passing through the 5 *points.*

I am not going to give proofs for this question (you can continue to think about them yourself if you like), but the answers are:

(i) No three of the points are collinear.
(ii) No four of the points are collinear.