Algebraic Geometry
Problem Sheet 2 – Solutions

Throughout, we let k be an algebraically closed field whose characteristic is not 2.

Question 1. *Let $\varphi\colon V \to W$ be a regular map between affine algebraic sets. Prove that $\varphi^*\colon k[W] \to k[V]$ is injective if and only if the image of φ is dense in W.*

If the image of φ is dense in W: Let f be a regular function on W such that $\varphi^* f = 0$. Then $\varphi(V)$ is contained in the Zariski closed set

$$\{x \in W : f(x) = 0\}.$$

Since $\varphi(V)$ is dense in W, this set must be equal to all of W and therefore $f = 0$ (as an element of $k[W]$). Or you could quote Lemma 6.2 from lectures.

If the image of φ is not dense in W: Let A be the closure of the image of φ. Because A is an affine algebraic set strictly contained in W, there exists some regular function $f \in k[W]$ which vanishes on A but not on all of W. Thus $\varphi^* f = 0$ but $f \neq 0$.

Question 2. *Prove that a regular function $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism if and only if it is given by a polynomial of degree 1. (Do not use differentiation because it does not behave nicely when the characteristic of the base field is not zero.)*

A regular function $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^1$ is simply given by a polynomial in $k[X]$.

If φ has degree 1: $\varphi(x) = ax + b$ and the inverse is given by $y \mapsto a^{-1}y - a^{-1}b$, which is also a regular map. So φ is an isomorphism.

If φ has degree 0: φ is a constant function, which is not injective (or surjective) so not an isomorphism.

If φ has degree $d > 1$: Suppose that φ has inverse $\psi\colon \mathbb{A}^1 \to \mathbb{A}^1$. Then ψ is also a polynomial, say of degree e. In order for $\varphi \circ \psi$ to be non-constant, we must have $e \neq 0$. If we expand out $\varphi(\psi(X))$, then the leading term will be a non-zero constant times X^{de}. Thus $\deg(\varphi \circ \psi) = de > 1$ so $\varphi(\psi(X))$ cannot be equal to the degree-1 polynomial X. This contradicts the assumption that $\varphi \circ \psi = \mathrm{id}_{\mathbb{A}^1}$.

You cannot simply say that if $\deg \varphi = d > 1$, then φ has d roots and so is injective, because all its roots might be repeated. In characteristic zero, it is possible to deal with this by looking at a point where the derivative $\varphi'(t)$ is non-zero, but this does not work in positive characteristic (because, over a field of characteristic p, the derivative of X^p is $pX^{p-1} = 0$). Note that, over an algebraically closed field of characteristic p, the regular map $\mathbb{A}^1 \to \mathbb{A}^1$ given by X^p is a bijection but not an isomorphism.

Question 3. *Let C denote the circle $\mathbb{V}(X^2 + Y^2 - 1) \subseteq \mathbb{A}^2$. Let φ be the rational function $(1 - Y)/X$ on C.*

Find a point $\underline{a} \in C$ where φ is not *regular (you need to prove that there is no fraction f/g which is equal to φ in $k(C)$ and satisfies $g(\underline{a}) \neq 0$).*

Prove that φ is regular at all points of C other than $\underline{a}$.

Answer: $\underline{a} = \{(0, -1)\}$.

Using the expression $\varphi = (1 - Y)/X$, we see that φ is regular whenever $x \neq 0$.

There are two points on C where $x = 0$: $(0, 1)$ and $(0, -1)$.

At $(0,1)$: Observe that $X^2 = 1 - Y^2 = (1-Y)(1+Y)$ in the ring of regular functions $k[C]$. Hence in the field of fractions of $k[C]$, we can compute:

$$\frac{1-Y}{X} = \frac{(1-Y)(1+Y)}{X(1+Y)} = \frac{X^2}{X(1+Y)} = \frac{X}{1+Y}.$$

This shows that φ is regular whenever $1+y \neq 0$, including at the point $(0,1)$.

At $(0,-1)$: Let f/g be any fraction between elements of $k[C]$ which is equal to φ. Multiplying up to clear denominators in $\varphi = f/g$, we get

$$Xf = (1-Y)g.$$

Now X is zero at $(0,-1)$, so this equation implies that $(1-Y)g$ is zero at $(0,-1)$. But $1-Y$ has value 2 at $(0,-1)$, so this forces $g(0,-1)$ to be zero. Hence the expression f/g does not give a value for φ at $(0,-1)$. Since this argument applies to all possible expressions for φ, we conclude that φ is not regular at $(0,-1)$.

Question 4. *Determine (with proof) the irreducible components of the affine algebraic set*

$$\mathbb{V}(Y^2 - XZ,\, X^2Y - Z) \subseteq \mathbb{A}^3.$$

Let $V = \mathbb{V}(Y^2 - XZ,\, X^2Y - Z) \subseteq \mathbb{A}^3$.

We have

$$f - Xg = Y^2 - XZ - X^3Y + XZ = Y^2 - X^3Y = Y(Y - X^3).$$

Hence at every point $(x,y,z) \in V$, either $y = 0$ or $y = x^3$.

Let $V_1 = V \cap \mathbb{V}(Y)$. Then V_1 is described by the polynomials

$$f(X,0,Z) = -XZ, \quad g(X,0,Z) = -Z$$

(as well as Y). Thus $V_1 = \mathbb{V}(Y,\, Z)$.

Now let $V_2 = V \cap \mathbb{V}(Y - X^3)$. This is described by the polynomials

$$f(X,X^3,Z) = X^6 - XZ = X(X^5 - Z), \quad g(X,X^3,Z) = X^5 - Z$$

(as well as $Y - X^3$). Thus we see that $V_2 = \mathbb{V}(X^5 - Z,\, Y - X^3)$. These equations for V_2 show us how to write y and z as functions of x, so we have

$$V_2 = \{(t, t^3, t^5) \in \mathbb{A}^3 : t \in k\}.$$

Thus V_1 and V_2 are closed subsets of V such that $V = V_1 \cup V_2$. It is clear that neither $V_1 \subseteq V_2$ nor $V_2 \subseteq V_1$. It remains to check that V_1 and V_2 are irreducible.

V_1 is irreducible because it is a line.

To show that V_2 is irreducible, we will prove that every proper Zariski closed subset of V_2 is finite. This is because, if $h \in k[X,Y,Z]$ is any polynomial, then $h(X,X^3,X^5)$ is a polynomial in one variable so either has finitely many roots or is identically zero. Since each value of x corresponds to exactly one point $(x,x^3,x^5) \in V_2$, it follows that $\{(x,y,z) \in V_2 : h(x,y,z) = 0\}$ is either finite or all of V_2. Consequently V_2 is irreducible, by the same reasoning as for $\mathbb{A}^1$.

Thus the irreducible components of V are

$$V_1 = \mathbb{V}(Y,\, Z) = \{(t,0,0)\}, \quad V_2 = \mathbb{V}(X^5 - Z,\, Y - X^3) = \{(t,t^3,t^5)\}.$$

Question 5. *Let $V = \mathbb{V}(XY)$ and $W = \mathbb{V}(Y(Y - X^2))$.*

Sub-question 5(a). *Describe the irreducible components of W. (A proof is not required.)*

The irreducible components of W are the line $W_1 = \mathbb{V}(Y = 0)$ and the parabola $W_2 = \mathbb{V}(Y - X^2)$.

Sub-question 5(b). *Which points lie in more than one irreducible component of W?*

$W_1 \cap W_2 = \{(0, 0)\}$.

Sub-question 5(c). *Consider the function $f\colon W \to k$ defined by*

$$f(x, 0) = x^2, \quad f(x, y) = 0 \text{ if } y \neq 0.$$

Prove that f is a regular function on W by writing down a polynomial $F \in k[X, Y]$ such that $F_{|W} = f$.

$F(X, Y) = X^2 - Y$. (On W_1, F restricts to X^2. On W_2, $Y = X^2$ and so F restricts to zero.)

Sub-question 5(d). *Prove that the function $g\colon W \to k$ defined by*

$$g(x, 0) = x, \quad g(x, y) = 0 \text{ if } y \neq 0$$

is not regular.

Assume for contradiction that there exists some polynomial $G \in k[X, Y]$ such that $G_{|W} = g$.

For every $x \neq 0$, we have $(x, x^2) \in W$ with $y = x^2 \neq 0$. Therefore $g(x, x^2) = 0$ for all $x \neq 0$, and so the one-variable polynomial $g(X, X^2)$ is the zero polynomial.

Write

$$g(X, Y) = \sum_{i,j=0}^{n} a_{ij} X^i Y^j$$

(for some $n \in \mathbb{N}$ and $a_{ij} \in k$). The only term in $g(X, X^2)$ which has degree 1 is $a_1 X$. Thus the fact that $g(X, X^2) = 0$ implies that $a_1 = 0$.

But $g(X, 0) = \sum_{i=0}^{n} a_{i0} X^i$, so if $g(X, 0) = X$ then we must have $a_{i0} = 1$. This gives a contradiction.

Sub-question 5(e). *Write down polynomials in $k[X, Y]$ which define a surjective regular map $V \to W$.*

$\varphi(X, Y) = (X + Y, X^2)$ (other answers are possible).

You were not required to prove that φ surjects onto W, but here is a proof: V is the union of the lines $X = 0$ and $Y = 0$. On the line $Y = 0$, we have $\varphi(X, 0) = (X, 0)$ so this line surjects onto W_1. On the line $X = 0$, we have $\varphi(0, Y) = (Y, Y^2)$ so this line surjects onto W_2.

Sub-question 5(f). *Is W isomorphic to V? Give a proof.*

You may use any facts or examples from the lecture notes without proof.

V and W are not isomorphic.

Proof 1. Label the irreducible components of V as $V_1 = \mathbb{V}(X)$, $V_2 = \mathbb{V}(Y)$.

Suppose that there exists some isomorphism $\varphi \colon W \to V$. This isomorphism must map each irreducible component of W to an irreducible component of W. There exists an isomorphism of V with itself swapping V_1 and V_2 (just swap the coordinates X and Y), so we may assume WLOG that $\varphi(W_1) = V_1$ and $\varphi(W_2) = V_2$.

Now $\varphi_{|W_1}$ is an isomorphism between $\mathbb{A}^1$ and itself, so by question 2 it is given by a polynomial of degree 1:

$$\varphi(x, 0) = (ax + b, 0) \text{ for some } a, b \in k, \text{ where } a \neq 0.$$

The point $(0, 0) \in W_1 \cap W_2$ must get mapped to $(0, 0) \in V_1 \cap V_2$, so $b = 0$.

Because points of W_2 get mapped into V_2, the x-coordinate of $\varphi(x, y)$ must be zero for every point of W_2. Hence the x-coordinate of φ looks like

$$\varphi_1(x, y) = ax \text{ if } y = 0, \quad \varphi_1(x, y) = 0 \text{ if } y \neq 0.$$

In other words, $\varphi_1 = ag$ where g is the function from part (d).

By the definition of regular map, φ_1 must be a regular function on W. Since $a \neq 0$, g is also a regular function but this contradicts part (d).

Proof 2. Suppose that there is an isomorphism $\varphi \colon W \to V$. Then $\varphi^* \colon k[V] \to k[W]$ is an isomorphism of k-algebras.

By the Nullstellensatz, $\mathbb{I}(V) = k[X, Y]/(XY)$ and $\mathbb{I}(W) = k[X, Y]/(Y(Y - X^2))$. Note that ψ^* does not have to map X to X and Y to Y, so we can't just say that $XY = 0$ in $k[V]$ but $XY \neq 0$ in $k[W]$ (or something like that).

The isomorphism φ^* must map zero divisors to zero divisors. The zero divisors in $k[V]$ are of the form $Yp(X, Y)$ or $(Y - X^2)q(X, Y)$ (this follows from unique factorisation in $k[X, Y]$). Hence $\varphi^*(X) = Yp(X, Y)$ and $\varphi^*(Y) = (Y - X^2)q(X, Y)$ (or the other way round, but we can just swap the coordinates X and Y on V, so we can assume WLOG that it is this way round).

Since φ^* is an isomorphism, it is surjective so there exists $f \in k[W]$ such that $\varphi^*(f) = X$. Writing $f = \sum_{i,j} a_{ij} X^i Y^j$, we get

$$X = \sum_{i,j} a_{ij} (Yp)^i ((Y - X^2)q)^j \in k[W]$$

and so

$$X = \sum_{i,j} a_{ij} (Yp)^i ((Y - X^2)q)^j + Y(Y - X^2)r \in k[X, Y]$$

for some $r \in k[X, Y]$. Every non-constant term of the RHS has either a Y in it or at least X^2 in it, so X cannot be in the image of φ^*.

Thus every element of the image of φ^* must have the form

$$Ya(X, Y) + (Y - X^2)b(X, Y) + c$$

for some $a(X, Y), b(X, Y) \in k[X, Y]$ and $c \in k$. But every non-constant term of such a polynomial has either a Y in it, or X to the power of at least 2. In particular, X is not in the image of φ^*.

Note the similarity between proof 1 and proof 2: the argument about zero divisors in proof 2 is the algebraic version of the argument about irreducible components in *proof 1. We were able to avoid lots more algebra in proof 1 because we already did it in questions 2 and 5(d). The argument in Proof 2 that $X \notin \operatorname{im} \varphi^*$ is closely related to 5(d).*

Remark. We showed in lecture 6 (example (iv)) that

$$k[V] = \{(h_1, h_2) \in k[V_1] \times k[V_2] = k[X] \times k[Y] : h_1(0) = h_2(0)\}$$

What is $k[W]$? It is

$$\{(h_1, h_2) \in k[W_1] \times k[W_2] \cong k[S] \times k[T] : h_1(0) = h_2(0) \text{ and } h_1'(0) = h_2'(0)\}.$$

(The derivatives h_1', h_2' can be defined formally even in positive characteristic. Saying $h_1'(0) = h_2'(0)$ means that the X-terms of h_1 and h_2 are the same.)

Geometrically, this is explained by the fact that W_1 and W_2 are tangent to each other at $(0, 0)$.

The proof of the above description of $k[W]$ is similar to parts (c) and (d). Once you have proved this, showing that $k[W]$ is not isomorphic to $k[V]$ doesn't seem particularly easy. Overall this gives a solution to 5(f) which is at least as long as either of the proofs above.

Question 6. *Let $C = \mathbb{V}(Y^2 - X^3) \subseteq \mathbb{A}^2$. (You may use without proof the fact that C is irreducible.) Let $\varphi \in k(C)$ denote the rational function Y/X.*

Prove that φ is not regular at $(0, 0)$.

Suggested method (you are free to use a different method if you wish):

Suppose that we can write $\varphi = f/g$ for some regular functions $f, g \in k[C]$. Observe that for every $t \in k$, we have $(t^2, t^3) \in C$ and use this to prove that $f(T^2, T^3) = Tg(T^2, T^3)$ in the ring of polynomials $k[T]$. Deduce that T^2 divides $f(T^2, T^3)$ and then that $g(0, 0) = 0$.

Following the suggested method, write $\varphi = f/g$, where $f, g \in k[C]$. Then we have the equation $Y/X = f/g$ in $k(C)$. Since C is irreducible, $k[C]$ is an integral domain and so we can cross-multiply to get

$$Y\, g(X, Y) = X\, f(X, Y) \text{ in } k[C]. \tag{1}$$

For every $t \in k$, we have $(t^2, t^3) \in C$ and so we can substitute $x = t^2, y = t^3$ into (1). We get that

$$t^3\, g(t^2, t^3) = t^2\, f(t^2, t^3) \tag{2}$$

for all $t \in k$. Hence (2) holds as an equation in the polynomial ring $k[T]$.

Since $k[T]$ is an integral domain, we can cancel factors of T^2 and get

$$T\, g(T^2, T^3) = f(T^2, T^3).$$

Thus T divides $f(T^2, T^3)$. In particular $f(T^2, T^3)$ has no constant term, and so the smallest-degree term which can appear in $f(T^2, T^3)$ is T^2. Since $k[X, Y]$ is a unique factorisation domain, we conclude that T divides $g(T^2, T^3)$.

But this implies that $g(0, 0) = 0$. Since this holds for all possible fractions f/g representing φ, we conclude that φ is not regular at $(0, 0)$.

Alternative method. The rational map φ is regular at all points of C other than $(0,0)$, because they must have $y \neq 0$. So by Lemma 8.2 from lectures, if φ is regular at $(0,0)$, then φ is a regular map $C \to \mathbb{A}^1$. Then there is some polynomial $F(X,Y) \in k[X,Y]$ such that $F_{|C} = Y/X \in k(C)$ and so $XF_{|C} = Y \in k[C]$.

The ideal $(Y^2 - X^3)$ is radical, so $k[C] = k[X,Y]/(Y^2 - X^3)$. Therefore

$$XF(X,Y) - Y = G(X,Y)(Y^2 - X^3)$$

for some polynomial $G(X,Y) \in k[X,Y]$. But every term of $G(X,Y)(Y^2 - X^3)$ has degree at least 2, so the term $-Y$ can never appear.