Algebraic Geometry
Problem Sheet 1 – Solutions

Question 1. *Let A be any subset of $\mathbb{A}^n$. Prove that $\mathbb{V}(\mathbb{I}(A))$ is the Zariski closure of A.*

Every set of the form $\mathbb{V}(S)$ is Zariski closed, for any $S \subseteq k[X_1, \dots, X_n]$. Hence in particular $\mathbb{V}(\mathbb{I}(A))$ is Zariski closed.

Let $B \subseteq \mathbb{A}^n$ be a Zariski closed set such that $A \subseteq B$. Since $\mathbb{V}$ and $\mathbb{I}$ both reverse the direction of inclusions, $\mathbb{I}(B) \subseteq \mathbb{I}(A)$ and so $\mathbb{V}(\mathbb{I}(A)) \subseteq \mathbb{V}(\mathbb{I}(B))$. Because B is an affine algebraic set, $B = \mathbb{V}(\mathbb{I}(B))$. Hence $\mathbb{V}(\mathbb{I}(A)) \subseteq B$.

We conclude that $\mathbb{V}(\mathbb{I}(A))$ is the smallest Zariski closed subset of $\mathbb{A}^n$ which contains A.

Question 2. *Determine all Zariski closed subsets of the union of two lines $\mathbb{V}(XY) \subseteq \mathbb{A}^2$. Deduce that $\mathbb{V}(XY)$ is connected in the Zariski topology.*

Let $V_1 = \mathbb{V}(X)$ and $V_2 = \mathbb{V}(Y)$. The Zariski closed subsets of $V = \mathbb{V}(XY) = V_1 \cup V_2$ are of the following forms:

(i) a finite subset of V.
(ii) $A \cup V_2$, where A is a finite subset of V_1.
(iii) $V_1 \cup B$, where B is a finite subset of V_2.
(iv) V itself.

To prove that these are the only Zariski closed subsets: let $S \subseteq V$ be a Zariski closed subset. Then $S \cap V_1$ is a Zariski closed subset of V_1 but V_1 is a line, so its only Zariski closed subsets are finite or V_1 itself. Similarly, $S \cap V_2$ is a Zariski closed subset of V_2, so is either finite or V_2 itself.

When $S \cap V_1$ and $S \cap V_2$ are both finite, we get (i). When $S \cap V_1$ is finite and $S \cap V_2 = V_2$, we get (ii). When $S \cap V_1 = V_1$ and $S \cap V_2$ is finite, we get (ii). When $S \cap V_1 = V_1$ and $S \cap V_2 = V_2$, we get (iv).

To prove that each of (i)–(iv) are indeed Zariski closed in V: each is of the form $A \cup B$, where A is a Zariski closed subset of V_1 and B is a Zariski closed subset of V_2. Since V_1 is a closed subset of V, it follows that A is closed as a subset of V. Similarly B is closed as a subset of V. A finite union of closed subsets is closed, so $A \cup B$ isa closed subset of V.

To prove that V is connected: try to write $V = S \cup T$ where S and T are disjoint proper subsets. We cannot have both $S \cap V_1$ and $T \cap V_1$ being finite, otherwise V_1 would not be in $S \cup T$. Therefore either $S \cap V_1 = V_1$ or $T \cap V_1 = V_1$.

Similarly either $S \cap V_2 = V_2$ or $T \cap V_2 = V_2$.

Since S and T are both proper subsets of V, we conclude that one of them contains V_1 while the other contains V_2 (say $V_1 \subseteq S$ and $V_2 \subseteq T$). But then the intersection $S \cap T$ contains $(0, 0)$ so is non-empty. Thus V cannot be written as a union of disjoint proper Zariski closed subsets, so it is connected.

Question 3. *Let V be the subset of $\mathbb{A}^2$ defined by the polynomials*

$$f = X^2 + Y^2 - 1, \quad g = X - 1.$$

Find $\mathbb{I}(V)$. Is $\mathbb{I}(V) = (f, g)$?

Observe that

$$(f-(X+1)g,\ g)=(f,g).$$

We have

$$f-(X+1)g=Y^2$$

and so

$$(f,g)=(Y^2,X-1).$$

It follows that V consists of the single point $(1,0)$ and

$$\mathbb{I}(V)=\mathbb{I}(\{(1,0)\})=(X-1,Y).$$

Thus $Y\in\mathbb{I}(V)$ but $Y\notin(Y^2,X-1)$ so

$$\mathbb{I}(V)\neq(f,g).$$

Note that we could also work out that $V=\{(1,0)\}$ by substituting $x=1$ (from $g(x,y)=0$) into $f(x,y)=0$. It is still necessary to work out that $(f,g)=(Y^2,X-1)$ in order to prove that $\mathbb{I}(V)\neq(f,g)$.

Geometrically, V is the intersection between the circle $X^2+Y^2=1$ and the tangent line $X=1$.

Question 4. *Prove that a hypersurface $\{\underline{x}\in\mathbb{A}^n : f(\underline{x})=0\}$ is irreducible if and only if f is a power of an irreducible polynomial. (You will need to use the Nullstellensatz and the fact that $k[X_1,\dots,X_n]$ is a unique factorisation domain.)*

Let

$$V=\{\underline{x}\in\mathbb{A}^n : f(\underline{x})=0\}.$$

Let I be the ideal generated by f. By Hilbert's Nullstellensatz, we have

$$\mathbb{I}(V)=\operatorname{rad}I.$$

Let the factorisation of f into irreducible polynomials be

$$f=f_1^{a_1}f_2^{a_2}\cdots f_m^{a_m}$$

and let $a=\max(a_1,\dots,a_m)$.

Consider any polynomial $g\in k[X_1,\dots,X_m]$. If $f_1f_2\cdots f_m$ divides g, then f divides g^a so $g\in\operatorname{rad}I$.

On the other hand, if $f_1f_2\cdots f_m$ does not divide g, then (by unique factorisation in $k[X_1,\dots,X_n]$ one of the irreducible polynomials f_i does not divide g. It follows that f_i does not divide any power of g and hence f does not divide any power of g. Thus $g\notin\operatorname{rad}I$.

We conclude that $\operatorname{rad}I$ is the ideal generated by $f_1f_2\cdots f_m$.

We know that V is irreducible if and only if $\mathbb{I}(V)$ is a prime ideal.

If $m>1$, then f_1 and $f_2\cdots f_m$ are polynomials which are not in $\mathbb{I}(V)$ but whose product is in $\mathbb{I}(V)$ so $\mathbb{I}(V)$ is not prime.

If $m=1$, then if we have $g_1g_2\in\mathbb{I}(V)=(f_1)$, unique factorisation tells us that either f_1 divides g_1 or f_1 divides g_2, so either $g_1\in\mathbb{I}(V)$ or $g_2\in\mathbb{I}(V)$. Thus $\mathbb{I}(V)$ is prime.

We have shown that V is irreducible if and only if $m=1$, that is, if and only if f is a power of an irreducible polynomial.

Question 5. *Use problem (4) to prove that the following sets are irreducible:*
(a) the parabola $\mathbb{V}(Y - X^2) \subseteq \mathbb{A}^2$*;*
(b) the circle $\mathbb{V}(X^2 + Y^2 - 1) \subseteq \mathbb{A}^2$*.*

Note: for (b), we need to assume that the characteristic of k *is not* 2 *(I omitted this condition from the original problem sheet).*

(a) We prove that the polynomial $f(X,Y) = Y - X^2$ is irreducible. By problem (4), it follows that $\mathbb{V}(f)$ is irreducible.

Consider f as a polynomial in Y with coefficients in the ring $k[X]$. Viewed as a polynomial in Y, f has degree 1. So if we can factorise f, one factor must have degree 1 in Y and the other factor must have degree 0 in Y. Thus we can write:

$$Y - X^2 = \big(a(X)\,Y + b(X)\big) \cdot c(X).$$

Comparing the Y terms of either side, we see that $1 = a(X) \cdot c(X)$. Therefore $a(X)$ and $c(X)$ are both constant polynomials.

Thus the factor $c(X)$ in our factorisation of $Y - X^2$ is a constant. Since this applies to any possible factorisation of $Y - X^2$, we conclude that $Y - X^2$ is irreducible.

(b) We prove that the polynomial $g(X,Y) = X^2 + Y^2 - 1$ is irreducible.

Consider g as a polynomial in Y with coefficients in the ring $k[X]$: it has degree 2. Thus g could factorise in two different ways:

(i) $Y^2 + X^2 - 1 = \big(a(X)\,Y^2 + b(X)\,Y + c(X)\big) \cdot d(X)$.

(ii) $Y^2 + X^2 - 1 = \big(a(X)\,Y + b(X)\big) \cdot \big(c(X)\,Y + d(X)\big)$.

In case (i), the same argument as in (a) gives that $a(X) \cdot d(X) = 1$ and so $d(X)$ is constant.

In case (ii), comparing Y^2 terms gives $a(X) \cdot c(X) = 1$. Thus $a(X)$ and $c(X)$ are constants. We can multiply one factor by a scalar and divide the other by the same scalar, so we may assume (without loss of generality) that $a(X) = c(X) = 1$. Thus we get

$$Y^2 + (X^2 - 1) = \big(Y + b(X)\big) \cdot \big(Y + d(X)\big).$$

Comparing Y terms gives $0 = b(X) + d(X)$ and so $d(X) = -b(X)$, giving

$$Y^2 + (X^2 - 1) = \big(Y + b(X)\big) \cdot \big(Y - b(X)\big) = Y^2 - b(X)^2.$$

Thus $b(X)^2 = 1 - X^2$. But the irreducible factorisation of $1 - X^2$ is $(1 - X)(1 + X)$. Since $k[X]$ is a UFD, we conclude that $1 - X^2$ is not a square, giving a contradiction.

Thus no factorisation of the form (ii) exists, while in form (i) one factor is a constant. Therefore $Y^2 + X^2 - 1$ is irreducible.

Question 6. *Describe the irreducible components of the affine algebraic sets*

$$V = \mathbb{V}(XY, Z) \subseteq \mathbb{A}^3, \quad W = \mathbb{V}(XY, XZ) \subseteq \mathbb{A}^3.$$

Give generators for $\mathbb{I}(V)$ *and* $\mathbb{I}(W)$*.*

V has two irreducible components: the line $X = Z = 0$ and the line $Y = Z = 0$ (the two axes in the plane $Z = 0$).

We prove that $\mathbb{I}(V) = (XY, Z)$. Suppose that $f(X,Y,Z) \in \mathbb{I}(V)$. Then we can write

$$f(X,Y,Z) = f_0(X,Y) + Z.f_1(X,Y,Z)$$

where f_0 is a polynomial which only depends on X and Y. Since $Z \in \mathbb{I}(V)$, f_0 is also in $\mathbb{I}(V)$.

Consider V just as an affine algebraic set in $\mathbb{A}^2$ (forgetting the Z-coordinate). The polynomial XY generates a radical ideal (because it has no repeated irreducible factors) and so, by the Nullstellensatz,

$$\mathbb{I}(\mathbb{V}(XY)) = (XY) \subseteq k[X,Y].$$

Thus $f_0 \in (XY)$. Hence $f = f_0 + Z.f_1 \in (XY, Z)$.

W has two irreducible components: the plane $X = 0$ and the line $Y = Z = 0$.

We prove that $\mathbb{I}(W) = (XY, XZ)$. Let $g \in \mathbb{I}(W)$.

First we show that $g \in (X)$. Write

$$g(X,Y,Z) = g_0(Y,Z) + X.g_1(X,Y,Z)$$

Since g and X vanish everywhere on the plane $X = 0$, so does $g_0(Y,Z)$. Because g_0 depends only on Y and Z, this forces $g_0 = 0$ as a polynomial. We conclude that $g(X,Y,Z) = X.g_1(X,Y,Z)$.

We want to show that every term of g_1 is divisible by either Y or Z. Write

$$g_1(X,Y,Z) = a(X) + b(X,Y,Z)$$

where every term in $b(X,Y,Z)$ is divisible by either Y or Z. Then $b(x,0,0) = 0$ for all $x \in k$.

(*) But $x.g_1(x,0,0) = g(x,0,0) = 0$ for all $x \in k$ (because $(x,0,0) \in W$), and so $g_1(x,0,0) = 0$ for all $x \neq 0$. Hence $a(x) = 0$ for all $x \neq 0$. Because the field k is infinite, we conclude that $a(X)$ is the zero polynomial.

Therefore $g(X,Y,Z) = X.b(X,Y,Z)$ and so every term in $g(X,Y,Z)$ is divisible by eithet XY or XZ. Thus $g \in (XY, XZ)$.

Note. In the problem class, I got paragraph (*) wrong as I omitted the argument that $g_1(x,0,0)$ is 0 for $x \neq 0$.

Question 7. *Assume that the characteristic of the base field k is not 2. Find the irreducible components of the subset of $\mathbb{A}^3$ defined by the equations*

$$X^2 + Y^2 + Z^2 = 0, \quad X^2 - Y^2 - Z^2 + 2 = 0.$$

Let the set in question be called V.

Adding the two equations gives

$$2X^2 + 2 = 0$$

as another equation satisfied on V. Hence $x = \pm i$ on V.

Substituting back in gives

$$-1 + Y^2 + Z^2 = 0.$$

Thus V is the union of two circles

$$\{(+i, y, z) : y^2 + z^2 = 1\} \cup \{(-i, y, z) : y^2 + z^2 = 1\}.$$

Using question 5(c), we know that each of these circles is irreducible and hence they are the irreducible components of V.

Question 8. *Decompose into irreducible components the subset of* $\mathbb{A}^3$ *defined by the polynomials*

$$f = Y^2 - XZ, \quad g = Z^2 - Y^3.$$

(Start by factorising $Yf + g = h_1 h_2$*. Then describe* $V \cap \{h_1 = 0\}$ *and* $V \cap \{h_2 = 0\}$*.)*

We have

$$Yf + g = Y^3 - XYZ + Z^2 - Y^3 = Z(XY - Z).$$

Hence at every point of V, either $z = 0$ or $xy - z = 0$.

The intersection of V with the plane $Z = 0$ is described by the polynomials

$$f(X, Y, 0) = Y^2, \quad g(X, Y, 0) = Y^3$$

(as well as $Z = 0$). In other words, this part of V is defined by the equations

$$Y = 0, \quad Z = 0.$$

The intersection of V with the algebraic set defined by the equation $XY - Z = 0$ is described by the polynomials

$$f(X, Y, XY) = Y^2 - X^2Y = Y(Y - X^2),$$
$$g(X, Y, XY) = X^2Y^2 - Y^3 = Y^2(X^2 - Y)$$

(as well as $XY - Z = 0$). We conclude that at every point of V where $XY - Z = 0$, either $Y = 0$ (in which case $XY - Z = 0$ forces Z to be zero also, so we are back in the previous case) or else $Y - X^2 = 0$. In the latter case $y = x^2$ and $z = xy = x^3$.

Conversely, if $y = x^2$ and $z = x^3$, then (x, y, z) is indeed in V.

Thus we guess that V has two irreducible components: the line $Y = Z = 0$ and the twisted cubic curve $S = \{(x, y, z) : y = x^2, z = x^3\}$.

The line $Y = Z = 0$ is certainly irreducible because it is a line. It remains to check that S is irreducible. To see this, consider a Zariski closed subset $T \subseteq S$ and a polynomial $h \in k[X, Y, Z]$ which vanishes on T. Now

$$h(x, x^2, x^3) = 0$$

is a single-variable polynomial, so it has only finitely many solutions (unless it vanishes everywhere on the twisted cubic). Thus any Zariski closed subset of S is either finite or S itself. Hence S is irreducible (same argument as for $\mathbb{A}^1$).

We conclude that the irreducible components of V are the line $Y = Z = 0$ and $S = \{(x, x^2, x^3) : x \in k\}$.

Note. The formal logical structure of the above argument is as follows. Let L be the line $Y = Z = 0$. Most of the argument above was showing that, if $(x, y, z) \in V$, then either $(x, y, z) \in L$ or $(x, y, z) \in S$. It is clear that $L \subseteq V$ and $S \subseteq V$. We conclude that $V = L \cup S$.

We checked at the end of the argument that L and S are irreducible. Furthermore, each of L and S is Zariski closed (because we constructed them by intersecting zero-sets of polynomials). Finally, it is obvious that neither $L \subseteq S$ nor $S \subseteq L$.

Hence Proposition 5.6 from the lectures tells us that L and S are the irreducible components of V. (And this proposition guarantees that, if we started by finding a different combination of f and g to factorise, we would eventually reach the same list of irreducible comopnents, although perhaps by a different route.)

Question 9. *Show that the algebraic set in $\mathbb{A}^3$ defined by the equations*

$$Y^2 - XZ = 0, \quad X^3 - YZ = 0$$

has two irreducible components, one of which is the set

$$C = \{(t^3, t^4, t^5) : t \in k.$$

Determine the other irreducible component.

(There was a mistake in this question as printed in the problem sheet: it said connected components where it should say irreducible components.)

Let V be the algebraic set defined by the polynomials

$$f = Y^2 - XZ, \quad g = X^3 - YZ.$$

Observe first that C is indeed contained in V.

To determine the irreducible components, we try to form a combination of f and g which factorises. We have

$$X^2 f + Zg = X^2Y^2 - X^3Z + X^3Z - YZ^2 = Y(X^2Y - Z^2).$$

Thus on V, either $Y = 0$ or $X^2Y - Z^2 = 0$.

Consider first the case $Y = 0$. Then V is defined by

$$f(X, 0, Z) = -XZ, \quad g(X, 0, Z) = X^3.$$

Thus we must have $X = 0$ (and no further conditions). Thus the z-axis ($X = Y = 0$) is contained in V.

Now we show that the rest of V is contained in C. In other words, we have to show that if we have $(x, y, z) \in k$ such that

$$y^2 - xz = x^3 - yz = x^2y - z^2 = 0, \qquad (*)$$

then $(x, y, z) \in C$. Observe that we can recover t as y/x unless $x = 0$.

If $x = 0$, then we can immediately read off from the equations (*) that $y = z = 0$. Otherwise, let $t = y/x$. Rewriting the the equations $y^2 - xz = 0$ and $x^3 - yz = 0$ in terms of x, z, t we get

$$t^2x^2 - xz = 0 \text{ and } x^3 - txz = 0.$$

Combining these two equations and dividing out powers of x (which we are assuming to be non-zero), we get

$$x = t^3.$$

Then $t = y/x$ and $t^2x^2 - xz$ imply that $y = t^4$ and $z = t^5$.

Thus V is the union of C with the z axis. One can verify that C is irreducible in a similar way to what we did in question 6, so in fact C and the z axis are irreducible components.

(Note: these two components intersect in $(0, 0, 0)$ so they are not connected components as claimed on the problem sheet.)

Question 10. *Let $V \subseteq \mathbb{A}^m$ and $W \subseteq \mathbb{A}^n$ be irreducible affine algebraic sets. Prove that the product $V \times W$ is irreducible.*

Show that the Zariski topology on $\mathbb{A}^2$ is not the same as the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$ coming from the Zariski topology on $\mathbb{A}^1$ (consider the diagonal $\{(x, y) : x = y\}$.)

First we show that if V and W are irreducible, then $V \times W$ is irreducible. Suppose that we can write $V \times W$ as a union $Z_1 \cup Z_2$ of two Zariski closed subsets.

For each point $v \in V$, the sets

$$(\{v\} \times W) \cap Z_1, \quad (\{v\} \times W) \cap Z_2$$

are Zariski closed subsets of $\{v\} \times W$ whose union is $\{v\} \times W$. But $\{v\} \times W$ is a copy of W and hence is irreducible. Therefore either

$$(\{v\} \times W) \cap Z_1 = \{v\} \times W \quad \text{or} \quad (\{v\} \times W) \cap Z_2 = \{v\} \times W.$$

Let

$$V_1 = \{v \in V : \{v\} \times W \subseteq Z_1\}$$

and

$$V_2 = \{v \in V : \{v\} \times W \subseteq Z_2\}$$

We have just shown that every $v \in V$ lies in either V_1 or V_2, that is,

$$V = V_1 \cup V_2.$$

If we write

$$Z_1 = \{(\underline{x}, \underline{y}) \in \mathbb{A}^{m+n} : f_1(\underline{x}, \underline{y}) = \cdots = f_r(\underline{x}, \underline{y}) = 0\},$$

then V_1 consists of those $\underline{x} \in V$ for which all the $f_i(\underline{x}, \underline{y})$ (substituting in chosen values for $x_1, \ldots, x_m$, leaving $y_1, \ldots, y_n$ as variables) are identically zero. This condition asserts that certain polynomials in $x_1, \ldots, x_m$ are zero; in other words, V_1 is a Zariski closed subset of $\mathbb{A}^m$.

Similarly, V_2 is a Zariski closed subset of $\mathbb{A}^m$.

But then the fact that V is irreducible tells us that $V = V_1$ or $V = V_2$. If $V = V_1$, then $Z_1 = V \times W$ while if $V = V_2$, then $Z_2 = V \times W$. This proves that $V \times W$ is irreducible.

For the second part of the question, observe that the diagonal

$$D = \{(x, y) \in \mathbb{A}^2 : x = y\}$$

is an affine algebraic set (it is defined by the polynomial $X - Y$). Hence it is Zariski closed in $\mathbb{A}^2$.

We will show that the diagonal is not closed in the product of the Zariski topologies on $\mathbb{A}^1$. Recall that the product topology is defined as follows: a subset of $\mathbb{A}^1 \times \mathbb{A}^1$ is *open* in the product topology if and only if it is a union of sets of the form where $U_1 \times U_2$, where U_1 and U_2 open in $\mathbb{A}^1$.

We want to show that $U = \mathbb{A}^2 \setminus D$ is not of this form. It suffices to show that U does not contain any non-empty sets of the form $U_1 \times U_2$, where U_1 and U_2 are open. But because $\mathbb{A}^1$ is irreducible in the Zariski topology, the intersection $U_1 \cap U_2$ must be open. Taking $x \in U_1 \cap U_2$, we get $(x, x) \in U_1 \times U_2$ which contradicts the assumption that $U_1 \times U_2 \subseteq \mathbb{A}^2 \setminus D$. Hence D is not closed in the product topology coming from the Zariski topology on $\mathbb{A}^1$.