Algebraic Geometry

Problem Sheet 6

(Mastery material)

(1) Suppose that k does not have characteristic 2 or 3.

For $a \in k$, let V_a denote the surface in $\mathbb{A}^3$ defined by the equation

$$x^3 + y^3 + z^3 - 3a(x^2 + y^2 + z^2) - a^2 = 0.$$

You may assume that this polynomial generates the ideal $\mathbb{I}(V_a)$.

For which values of a does V_a have singular points? For each a, find all the singular points of V_a.

(2) Let V, W be affine varieties. Let $v \in V$ and $w \in W$.

Prove that $V \times W$ is non-singular at (v, w) if and only if V is non-singular at v and W is non-singular at w.

(3) Let $V \subseteq \mathbb{A}^n$ be a reducible affine algebraic set, with irreducible components V_1 and V_2. Let $x \in V_1 \cap V_2$. Prove that

$$T_x V_1 + T_x V_2 \subseteq T_x V.$$

Is $T_x V_1 + T_x V_2$ always equal to $T_x V$?

(Here, $T_x V_1 + T_x V_2$ means the vector space spanned by $T_x V_1$ and $T_x V_2$.)

(4) Let f be a non-zero polynomial in $k[X_1, \dots, X_n]$ with no repeated factors. Let $V \subseteq \mathbb{A}^n$ be the hypersurface defined by f.

(a) Let x be a point in V and let u be a vector in k^n.
Prove that the polynomial

$$f(x + Tu) \in k[T]$$

has a repeated root at $T = 0$ if and only if $u \in T_x V$.

(b) Suppose that $\deg f = 2$. Prove that if x is a singular point of V and $L \subseteq \mathbb{A}^n$ is a line through x, then either $L \subseteq V$ or $L \cap V = \{x\}$.