Algebraic Geometry

Problem Sheet 5

(1) Let $V \subseteq \mathbb{P}^n$ be a projective algebraic set in which all components have dimension $n-1$. Prove that V is a hypersurface.

(2) Let $H \subseteq \mathbb{P}^n$ be a hyperplane. Let $V \subseteq H$ be an irreducible projective algebraic set (note: $V \subseteq H$, rather than $V \subseteq \mathbb{P}^n$).

Let x be a point of $\mathbb{P}^n \setminus H$. Let C be the union of all lines joining V to x (this is called the **cone** over V with vertex x).

Prove that C is a projective algebraic set, that it is irreducible and that $\dim C = \dim V + 1$.

(3) Let $V_{n,d}$ denote the vector space of homogeneous polynomials of degree d in $k[X_0, \dots, X_n]$ and let $P_{n,d}$ denote the projective space attached to $V_{n,d}$.

(a) Show that, for each e such that $0 \leq e \leq d$, multiplication of polynomials $V_{n,e} \times V_{n,d-e} \to V_{n,d}$ induces a regular map

$$\mu_{n,d,e} \colon P_{n,e} \times P_{n,d-e} \to P_{n,d}.$$

(b) By applying completeness to $\mu_{n,d,e}$, show that the set

$$P_{n,d,\mathrm{irr}} = \{[f] \in P_{n,d} : f \text{ is irreducible}\}$$

is an open subset of $P_{n,d}$.

(c) Let $n \geq 2$. Show that $P_{n,d,\mathrm{irr}}$ is non-empty. (Easy approach: write down an irreducible homogeneous polynomial of degree d. Harder approach, making use of dimension: Calculate the dimension of $P_{n,e} \times P_{n,d-e}$ for each e and compare with $\dim P_{n,d}$. What goes wrong when $n = 1$ and $d \geq 2$?)

(d) By considering maps of the form

$$P_{n,a} \times P_{n,b} \to P_{n,d} \;:\; ([f],[g]) \mapsto [f^r g]$$

for all triples a, b, r such that $ar + b = d$ and $r \geq 2$, show that the set

$$P_{n,d,\mathrm{rad}} = \{[f] \in P_{n,d} : f \text{ generates a radical ideal}\}$$

is an open subset of $P_{n,d}$.

(e) Give an example (for some n, d) of two polynomials f, g such that $[f], [g] \in P_{n,d} \setminus P_{n,d,\mathrm{rad}}$ and f and g define the same hypersurface in $\mathbb{P}^n$, but f is not a scalar multiple of g.

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(4) Let $V \subseteq \mathbb{P}^n$ be an irreducible projective algebraic set of dimension d.

(a) Let

$$\Sigma = \{(p, q, r) \in V \times V \times \mathbb{P}^n : p \neq q \text{ and } r \in L_{pq}\}$$

and let $\overline{\Sigma}$ denote the Zariski closure of Σ. You may assume that $\overline{\Sigma}$ is irreducible, and that

$$\overline{\Sigma} \cap \{(p, q, r) \in V \times V \times \mathbb{P}^n : p \neq q\} = \Sigma.$$

Prove that the projection $\pi_{1,2} \colon \overline{\Sigma} \to V \times V$ (onto the first two factors) is surjective.

(b) By applying the fibre dimension theorem to the projection $\pi_{1,2} \colon \overline{\Sigma} \to V \times V$, show that $\dim \overline{\Sigma} \leq 2d + 1$.

(c) Let

$$S = \bigcup_{\substack{(p,q) \in V \times V \\ p \neq q}} L_{pq}$$

and let π_3 denote the projection onto the last factor $\overline{\Sigma} \to \mathbb{P}^n$. Prove that $\pi_3(\Sigma) = S$ and deduce that S is contained in a closed subset of $\mathbb{P}^n$ of dimension at most $2d + 1$.

(d) Deduce that if $2d + 1 < n$, then there exists a point $r \in \mathbb{P}^n$ such that every line through r intersects V in at most one point.