Algebraic Geometry
Problem Sheet 4

Assessed coursework 2 – Deadline 12 March 2018

(1) (a) Prove that the rational map $\varphi\colon [x : y : z] \mapsto [xy : yz : zx]$ is dominant and gives a birational equivalence $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Write down a formula for a rational inverse ψ of φ.

(b) What are the domains of definition of φ and ψ?

(c) Write down open subsets $A, B \subseteq \mathbb{P}^2$ such that φ induces an isomorphism $A \to B$.

(2) (a) Prove that $\mathbb{P}^1 \times \mathbb{A}^1$ is not isomorphic to either an affine or a projective algebraic set.

(b) Write down homogeneous polynomials defining closed subsets $V, Z \subseteq \mathbb{P}^3$ such that $\mathbb{P}^1 \times \mathbb{A}^1$ is isomorphic to $V \cap (\mathbb{P}^3 \setminus Z)$. (Use the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$.)

(3) Let $H \subseteq \mathbb{P}^n$ be a hypersurface, defined by a homogeneous polynomial f of degree d. Let $V \subseteq \mathbb{P}^n$ be an irreducible projective algebraic set such that $V \cap H$ is empty.

(a) Prove that the functions $X_i X_j^{d-1}/f$ are constant on V for every i, j. (Make clear how you are using the condition that $V \cap H$ is empty.)

(b) Deduce that V is a point.

(4) By considering the rank of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ y_0 & y_1 & \cdots & y_n \\ z_0 & z_1 & \cdots & z_n \end{pmatrix}$$

or otherwise, show that the set

$$\{(x, y, z) \in \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n : \text{there exists a line in } \mathbb{P}^n \text{ containing all of } x, y, z\}$$

is a closed subset of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$. (You may assume without proof that closed subsets of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ are described by "trihomogeneous" polynomials, i.e. polynomials in $k[X_0, \dots, X_n, Y_0, \dots, Y_n, Z_0, \dots, Z_n]$ which are separately homogeneous in the X-variables, in the Y-variables and in the Z-variables.)

(5) Let $\Sigma_{1,2}$ denote the image of the Segre embedding $\sigma_{1,2}\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$.
Consider two distinct points in $\mathbb{P}^1 \times \mathbb{P}^2$:

$$(a, x) = ([a_0 : a_1], [x_0 : x_1 : x_2]) \text{ and } (b, y) = ([b_0 : b_1], [y_0 : y_1 : y_2]).$$

Let $p = \sigma_{1,2}(a, x)$ and $q = \sigma_{1,2}(b, y)$.

(a) Write down three homogeneous polynomials of degree 2 which define $\Sigma_{1,2}$ as a subset of $\mathbb{P}^5$.

(b) Write down a regular map $\mathbb{P}^1 \to \mathbb{P}^5$ whose image is the line $L_{pq} \subseteq \mathbb{P}^5$ which passes through p and q (no proofs are required).

(c) Find (with proof) three polynomial equations, homogeneous of degree 1 with respect to each of a, x, b, y, which must be satisfied by (a, x) and (b, y) if $L_{pq} \subseteq \Sigma_{1,2}$.

(d) By factorising the equations from (c), conclude that if $L_{pq} \subseteq \Sigma_{1,2}$, then either $a = b$ or $x = y$.