Algebraic Geometry
Problem Sheet 3

(1) (a) Let $\varphi\colon \mathbb{P}^n \to \mathbb{P}^m$ be a regular map. Prove that there exist homogeneous polynomials $f_0, \dots, f_m \in k[X_0, \dots, X_n]$ such that the expression

$$[f_0(x_0, \dots, x_n) : \cdots : f_m(x_0, \dots, x_n)]$$

defines φ at every point of $\mathbb{P}^n$. (Use the fact that $k[X_0, \dots, X_n]$ is a UFD.)

(b) Prove that, if $\varphi\colon \mathbb{P}^n \to \mathbb{P}^1$ is a non-constant regular map, then the image of φ is not contained in $\mathbb{A}^1$.

(c) Prove that every rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is regular.

(2) (a) Prove that, if $\varphi\colon \mathbb{A}^2 \dashrightarrow k$ is a rational function, then there exist polynomials $f, g \in k[X, Y]$ such that $\varphi = f/g$ and such that $g(x) \neq 0$ on all of $\operatorname{dom} \varphi$.

(b) Deduce that, if $\varphi\colon \mathbb{A}^2 \dashrightarrow k$ is regular on $\mathbb{A}^2 \setminus \{(0,0)\}$, then it is regular at $(0,0)$ too.

(c) Consider the quasi-projective algebraic set $V = \mathbb{A}^2 \setminus \{(0,0)\}$. Prove that the ring of regular functions on V is the polynomial ring $k[X, Y]$.

(d) Prove that V is not isomorphic to any affine algebraic set. (Use the fact that, in an affine algebraic set W, every proper ideal in $k[W]$ defines a non-empty subset.)

(3) Consider the example from lecture 15:

$$\overline{C} = \{[w : x : y] \in \mathbb{P}^2 : w^2 y = x^3\}$$

and $\varphi\colon \overline{C} \dashrightarrow \mathbb{P}^1$ is the rational map represented by $[w : x : y] \mapsto [w : x]$. Prove that φ is not regular at $[0 : 0 : 1]$.

This is similar to sheet 2, question 6. There is a regular map $\mathbb{A}^1 \to C$ given by $t \mapsto (t, t^3)$. Homogenise to get a regular map $\overline{\psi}\colon \mathbb{P}^1 \to \overline{C}$.

If φ is represented by some polynomials $f, g \in k[W, X, Y]$, then the equivalence relation between different representations of a rational map says

$$Xf = Wg.$$

Composing with $\overline{\psi}$, we get

$$s^2 t f(s^3, s^2 t, t^3) = s^3 g(s^3, s^2 t, t^3).$$

This holds for all $(s, t) \neq (0, 0)$, so it is an identity in the polynomial ring $k[S, T]$. Using the same trick as sheet 2, question 6, show that S^2 divides $f \circ \overline{\psi}$ and then that S divides $g \circ \overline{\psi}$. Conclude that $g(0, 0, 1) = f(0, 0, 1) = 0$.

Alternatively: one could deduce this directly from sheet 2, question 6 by considering the intersection of $\overline{C}$ with the affine space $y \neq 0$ (but the above method will build familiarity with polynomials in homogeneous coordinates).

(4) Let

$$f(X_0, \dots, X_n) = \sum_{i,j=0}^{n} a_{ij} X_i X_j \in k[X_0, \dots, X_n]$$

be a homogeneous polynomial of degree 2 (where a_{ij} are not all zero).

The projective algebraic set $V \subseteq \mathbb{P}^n$ defined by the equation $f = 0$ is called a **quadric**. We aim to show that, if V is irreducible, then it is birational to $\mathbb{P}^{n-1}$.

(a) Prove that V is irreducible if and only if f is irreducible.

(b) Let $p = [0 : \cdots : 0 : 1] \in \mathbb{P}^n$. What condition on the coefficients a_{ij} is equivalent to: $p \in V$?

From now on, we assume that V is irreducible and that $p \in V$.

(c) Let H be the hyperplane defined by the equation $\mathbb{P}^n$. For each point $x \in H$, let L_x denote the line through x and p. Prove that either $L_x \subseteq V$ or $L_x \cap V$ contains 1 or 2 points.

(d) Let $U = \{x \in H : \#(L_x \cap V) = 2\}$. By finding equations for $H \setminus U$, prove that U is a Zariski open subset of H.

(e) What condition on the coefficients a_{ij} is equivalent to: $U \neq \emptyset$? (You may find it helpful to use the fact that H is irreducible.)

From now on, we assume that $U \neq \emptyset$.

(f) For $x \in U$, write $L_x \cap V = \{p, \psi(x)\}$. Find the coordinates of the point $\psi(x)$. Conclude that ψ is a rational map $H \dashrightarrow V$.

(g) By considering the projection from p to H, deduce that V is birational to H.

(h) Determine the domain of definition of ψ. What is $\psi(x)$ for $x \in \operatorname{dom} \psi \setminus U$?

(i) Apply this to the quadric defined by the equation $XY = WZ$ in $\mathbb{P}^3$: write down the rational map $\psi \colon H \dashrightarrow V$ and its domain of definition for this case.

(j) Let $\pi \colon V \dashrightarrow H$ denote the projection from p. What is $\operatorname{dom} \pi$?
The answer is different for $n = 2$ and $n \geq 3$. For $n = 2$, give an algebraic proof and an informal geometric explanation of what is going on. For $n \geq 3$, give either a geometric or an algebraic proof.

(5) A quadric in $\mathbb{P}^2$ is called a **conic**. Let

$$f(X, Y, Z) = aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ \in k[X, Y, Z]$$

(where a, b, c, d, e, f are not all zero). Let $V \subseteq \mathbb{P}^2$ be the conic defined by f.

(a) Prove that two quadratics f and g define the same conic if and only if $f = \lambda g$ for some $\lambda \in k \setminus \{0\}$.

Hence we can identify the set of conics with the projective space $\mathbb{P}^5$.

(b) For any point $p \in \mathbb{P}^2$, prove that the set of conics which contain p corresponds to a hyperplane in $\mathbb{P}^5$.

(c) Prove that, given any 5 points in $\mathbb{P}^2$, there exists a conic passing through all 5 points.

(d) Find conditions on 5 points in $\mathbb{P}^2$ which are equivalent to:

   (i) There exists an irreducible conic passing through the 5 points.

   (ii) There exists a unique conic passing through the 5 points.