Algebraic Geometry
Problem Sheet 2

Assessed coursework – Deadline 12 February 2018

Throughout, we let k be an algebraically closed field whose characteristic is not 2.

(1) Let $\varphi\colon V \to W$ be a regular map between affine algebraic sets. Prove that $\varphi^*\colon k[W] \to k[V]$ is injective if and only if the image of φ is dense in W.

(2) Prove that a regular function $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism if and only if it is given by a polynomial of degree 1. (Do not use differentiation because it does not behave nicely when the characteristic of the base field is not zero.)

(3) Let C denote the circle $\mathbb{V}(X^2 + Y^2 - 1) \subseteq \mathbb{A}^2$. Let φ be the rational function $(1 - Y)/X$ on C.

Find a point $\underline{a} \in C$ where φ is *not* regular (you need to prove that there is no fraction f/g which is equal to φ in $k(C)$ and satisfies $g(\underline{a}) \neq 0$).

Prove that φ is regular at all points of C other than $\underline{a}$.

(4) Determine (with proof) the irreducible components of the affine algebraic set

$$\mathbb{V}(Y^2 - XZ,\, X^2Y - Z) \subseteq \mathbb{A}^3.$$

(5) Let $V = \mathbb{V}(XY)$ and $W = \mathbb{V}(Y(Y - X^2))$.

(a) Describe the irreducible components of W. (A proof is not required.)

(b) Which points lie in more than one irreducible component of W?

(c) Consider the function $f\colon W \to k$ defined by

$$f(x, 0) = x^2, \quad f(x, y) = 0 \text{ if } y \neq 0.$$

Prove that f is a regular function on W by finding a polynomial $F \in k[X, Y]$ such that $F_{|W} = f$.

(d) Prove that the function $g\colon W \to k$ defined by

$$g(x, 0) = x, \quad g(x, y) = 0 \text{ if } y \neq 0$$

is not regular.

(e) Write down polynomials in $k[X, Y]$ which define a surjective regular map $V \to W$.

Note: you only need to write down one such map, and you do not need to prove that its image is equal to W.

(f) Is W isomorphic to V? Give a proof.

You may use any facts or examples from the lecture notes without proof.

(6) Let $C = \mathbb{V}(Y^2 - X^3) \subseteq \mathbb{A}^2$. (You may use without proof the fact that C is irreducible.) Let $\varphi \in k(C)$ denote the rational function Y/X.

Prove that φ is not regular at $(0, 0)$.

Suggested method (you are free to use a different method if you wish):

Suppose that we can write $\varphi = f/g$ for some regular functions $f, g \in k[C]$. Observe that for every $t \in k$, we have $(t^2, t^3) \in C$ and use this to prove that $f(T^2, T^3) = T\, g(T^2, T^3)$ in the ring of polynomials $k[T]$. Deduce that T^2 divides $f(T^2, T^3)$ and then that $g(0, 0) = 0$.