Algebraic Geometry

Problem Sheet 1

(1) Let A be any subset of $\mathbb{A}^n$. Prove that $\mathbb{V}(\mathbb{I}(A))$ is the Zariski closure of A.

(2) Determine all Zariski closed subsets of the union of two lines $\mathbb{V}(XY) \subseteq \mathbb{A}^2$. Deduce that $\mathbb{V}(XY)$ is connected in the Zariski topology.

(3) Let V be the affine algebraic set in $\mathbb{A}^2$ defined by the polynomials

$$f = X^2 + Y^2 - 1, \quad g = X - 1.$$

Describe the set V and find $\mathbb{I}(V)$. Is $\mathbb{I}(V) = (f, g)$?

(4) Prove that a hypersurface $\{\underline{x} \in \mathbb{A}^n : f(\underline{x}) = 0\}$ is irreducible if and only if f is a power of an irreducible polynomial. (You will need to use the Nullstellensatz and the fact that $k[X_1, \dots, X_n]$ is a unique factorisation domain.)

(5) Use problem (4) to prove that the following sets are irreducible:
(a) the parabola $\mathbb{V}(Y - X^2) \subseteq \mathbb{A}^2$;
(b) the circle $\mathbb{V}(X^2 + Y^2 - 1) \subseteq \mathbb{A}^2$.
Note: for (b), we need to assume that the characteristic of k is not 2.

(6) Describe the irreducible components of the affine algebraic sets

$$V = \mathbb{V}(XY, Z) \subseteq \mathbb{A}^3, \quad W = \mathbb{V}(XY, XZ) \subseteq \mathbb{A}^3.$$

Give generators for $\mathbb{I}(V)$ and $\mathbb{I}(W)$.

(7) Assume that the characteristic of the base field k is not 2. Find the irreducible components of the subset of $\mathbb{A}^3$ defined by the equations

$$X^2 + Y^2 + Z^2 = 0, \quad X^2 - Y^2 - Z^2 + 2 = 0.$$

(8) Decompose into irreducible components the algebraic set $V \subseteq \mathbb{A}^3$ defined by the polynomials

$$f = Y^2 - XZ, \quad g = Z^2 - Y^3.$$

(Start by factorising $Yf + g = h_1 h_2$. Then describe $V \cap \{h_1 = 0\}$ and $V \cap \{h_2 = 0\}$.)

(9) Show that the algebraic set in $\mathbb{A}^3$ defined by the equations

$$Y^2 - XZ = 0, \quad X^3 - YZ = 0$$

has two irreducible components, one of which is the set

$$C = \{(t^3, t^4, t^5) : t \in k\}.$$

Determine the other ~~connected~~ irreducible component.

(10) Let $V \subseteq \mathbb{A}^m$ and $W \subseteq \mathbb{A}^n$ be irreducible affine algebraic sets. Prove that the product $V \times W$ is irreducible.