1. Introduction

Practical information about the course.

Participation is encouraged. Please talk to each other, please interrupt me and ask questions.

Problem classes – 29 Jan, 19 Feb, 26 Feb, 19 Mar, 20 Mar
Coursework – two pieces, each worth 5% (problem sheets 2 and 4)
  Deadlines: 12 February, 12 March
Problem sheets and coursework will be available on my web page:
  `http://wwwf.imperial.ac.uk/~morr/2017-8/alg-geom`
My email address: `m.orr@imperial.ac.uk`
Office hours: Tue 11-12, Thu 2-3 in Huxley 681

Course outline.

(1) Affine varieties – definition, examples, maps between varieties, translating between geometry and commutative algebra (the Nullstellensatz)
(2) Projective varieties – definition, examples, maps between varieties, rigidity and images of maps
(3) Dimension – several different definitions (all equivalent, but useful for different purposes), calculating dimensions of examples
(4) Smoothness and singularities – definition, examples, key theorems
(5) Examples of varieties (depending on how much time is left)

What is not in the course?

(1) Schemes
(2) Sheaves and cohomology
(3) Curves, divisors and the Riemann–Roch theorem

The base field. Let k be an algebraically closed field.

We are going to be thinking about solutions to polynomials, so everything is much simpler over algebraically closed fields. Number theorists might be interested in other fields, but you generally have to start by understanding the algebraically closed case first. In this course we will stop with the algebraically closed case too.

Apart from being algebraically closed, it usually does not matter much which field we use to do algebraic geometry – except sometimes it matters whether the characteristic is zero or positive. In this course I will take care to mention results which depend on the characteristic, and sometimes we might consider only the characteristic zero case. You will not lose much if you just assume that $k = \mathbb{C}$ throughout the course (except when it will be explicitly something else).

Indeed it is often useful to think about $k = \mathbb{C}$ because then you can use your usual geometric intuition. When I draw pictures on the whiteboard, I am usually only drawing the real solutions because it is hard to draw shapes in $\mathbb{C}^2$. This is cheating but it is often very useful – the real solutions are not the full picture but in many cases we can still see the important features there.

Affine space.

Definition. Algebraic geometers write $\mathbb{A}^n$ to mean k^n, and call it **affine** n**-space**.

You may think of this as just a funny choice of notation, but there are at least two reasons for it:

(i) When we write k^n, it makes us think of a vector space, equipped with operations of addition and scalar multiplication. But $\mathbb{A}^n$ means just a set of points, described by coordinates $(x_1, \dots, x_n)$ with $x_i \in k$, without the vector space structure.

(ii) Because it usually doesn't matter much what our base field k is (as long as it is algebraically closed), it is convenient to have notation which does not prominently mention k.

On occasions when it *is* important to specify which field k we are using, we write $\mathbb{A}^n_k$ for affine n-space.

2. Affine algebraic sets

Definition. An **affine algebraic set** is a subset $V \subseteq \mathbb{A}^n$ which consists of the common zeros of some finite set of polynomials $f_1, \dots, f_m$ with coefficients in k.

More formally, an **affine algebraic set** is a set of the form

$$V = \{(x_1, \dots, x_n) \in \mathbb{A}^n : f_1(x_1, \dots, x_n) = \dots = f_m(x_1, \dots, x_n) = 0\}$$

for some polynomials $f_1, \dots, f_m \in k[X_1, \dots, X_n]$.

Examples of affine algebraic sets.

Exercise 2.1. Think of some examples and non-examples of affine algebraic sets.

These are the examples and non-examples you came up with in lectures.

Examples.

(1) The whole space $\mathbb{A}^n$, defined by the polynomial $f_1 = 0$ (or by the empty set of polynomials).
(2) The set $\{2\}$, defined by the polynomial $X - 2$. More generally, any point in $\mathbb{A}^1$.
(3) Any union of finitely many affine algebraic sets (see proof below). Combining with (2), we deduce that any finite subset of $\mathbb{A}^1$ is an affine algebraic set.
(4) An algebraic curve in $\mathbb{A}^2$, that is, a set of the form

$$\{(x_1, x_2) \in \mathbb{A}^2 : f(x_1, x_2) = 0\}$$

for some polynomial $f \in k[X_1, X_2]$.

Non-examples.

(1) Any infinite subset of $\mathbb{A}^1$ (other than $\mathbb{A}^1$ itself). This is because a one-variable polynomial with infinitely many roots must be the zero polynomial.

Here are some additional examples of affine algebraic sets.

Further examples.

(5) Any point in $\mathbb{A}^n$. The single-point set $\{(a_1, \dots, a_n)\}$ is defined by the equations

$$X_1 - a_1 = 0, \dots, X_n - a_n = 0.$$

Using (3), we see that any finite subset of $\mathbb{A}^n$ is an affine algebraic set.
(6) Embeddings of $\mathbb{A}^m$ in $\mathbb{A}^n$ where $m < n$:

$$\{(x_1, \dots, x_m, 0, \dots, 0) \in \mathbb{A}^n\} = \{(x_1, \dots, x_n) \in \mathbb{A}^n : x_{m+1} = \dots = x_n = 0\}.$$

More generally, the image of a linear map $\mathbb{A}^m \to \mathbb{A}^n$:

$$\{(x_1, \dots, x_n) \in \mathbb{A}^n : \text{some linear conditions}\}.$$

Further non-example.

Example (6) does not generalise to images of maps where each coordinate is given by a polynomial. For example, consider the map

$$\phi\colon \mathbb{A}^2 \to \mathbb{A}^2 \text{ where } f(x, y) = (x, xy).$$

The image of ϕ is

$$S = \mathbb{A}^2 \setminus \{(0, y)\} \cup \{(0, 0)\}.$$

To prove that S is not an affine algebraic set, consider a polynomial $g(X, Y) \in k[X, Y]$ which vanishes on S. For each fixed $y \in k$, the one-variable polynomial $g(X, y)$ vanishes at all $x \neq 0$. This implies that $g(X, y)$ is the zero polynomial. Thus $g(x, y) = 0$ for all $x, y \in k^2$, that is, g is the zero polynomial.

Philosophical remark. This remark might seem obscure for now; we will come back to it later.

The words "affine variety" mean more or less the same thing as "affine algebraic set" but there is an ontological difference. "Affine algebraic set" means a subset which lives inside $\mathbb{A}^n$ and knows how it lives inside $\mathbb{A}^n$, while "affine variety" means an object in its own right which is considered outside of $\mathbb{A}^n$. I will try to use these words consistently, but the difference is quite subtle and books may not always use it consistently. For the first few weeks, we will talk about "affine algebraic sets" only.

Note that some books (e.g. Reid, Hartshorne) have another difference between affine varieties and affine algebraic sets – they require varieties to be irreducible (which we will define next time). Other books (e.g. Shafarevich) do not require varieties to be irreducible. In this course we will *not* require varieties to be irreducible.

Unions and intersections of affine algebraic sets. One of the examples was a union of finitely many affine algebraic sets. Now we prove that the union of two affine algebraic sets is an affine algebraic set.

Lemma 2.1. If $V, W \subseteq \mathbb{A}^n$ are affine algebraic sets, then their union $V \cup W \subseteq \mathbb{A}^n$ is also an affine algebraic set.

Proof. We have to take the product for each possible pair of defining polynomials: if

$$V = \{(x_1, \dots, x_n) \in \mathbb{A}^n : f_1(\underline{x}) = \dots = f_r(\underline{x}) = 0\},$$
$$W = \{(x_1, \dots, x_n) \in \mathbb{A}^n : g_1(\underline{x}) = \dots = g_s(\underline{x}) = 0\},$$

then

$$V \cup W = \{\underline{x} \in \mathbb{A}^n : f_i(\underline{x}) g_j(\underline{x}) = 0 \text{ for all } i, j \text{ where } 1 \leq i \leq r, 1 \leq j \leq s\}.$$

Note that we really need all the pairs $f_i g_j$, not just for example $f_1 g_1$, $f_2 g_2$, etc. It is obvious that if $\underline{x} \in V \cup W$, then all the products $f_i g_j$ vanish at $\underline{x}$.

The reverse is a little trickier. Suppose that we have $\underline{x} \in \mathbb{A}^n$ satisfying $f_i(\underline{x})g_j(\underline{x}) = 0$ for all i and j. Looking just at f_1, we get:

$$\begin{aligned} f_1 g_1(\underline{x}) = 0, &\text{ so } f_1(\underline{x}) = 0 \text{ or } g_1(\underline{x}) = 0. \\ f_1 g_2(\underline{x}) = 0, &\text{ so } f_1(\underline{x}) = 0 \text{ or } g_2(\underline{x}) = 0. \\ &\vdots \\ f_1 g_s(\underline{x}) = 0, &\text{ so } f_1(\underline{x}) = 0 \text{ or } g_s(\underline{x}) = 0. \end{aligned}$$

Putting these all together, we get

$$f_1(\underline{x}) = 0 \text{ or } g_j(\underline{x}) = 0 \text{ for every } j.$$

We can do the same thing for f_2 to get

$$f_2(\underline{x}) = 0 \text{ or } g_j(\underline{x}) = 0 \text{ for every } j$$

and so on for each f_i. Putting all these together, we get

$$f_i(\underline{x}) = 0 \text{ for every } i \text{ or } g_j(\underline{x}) = 0 \text{ for every } j.$$

This says precisely that $\underline{x} \in V \cup W$. □

It is even easier to check that the intersection of finitely many affine algebraic sets is an affine algebraic sets: if V is defined by polynomials $f_1, \ldots, f_r$ and W is defined by polynomials $g_1, \ldots, g_s$, then $V \cap W$ is simply the set where all the polynomials in both lists vanish i.e.

$$V \cap W = \{\underline{x} \in \mathbb{A}^n : f_1(\underline{x}) = \cdots = f_r(\underline{x}) = 0 \text{ and } g_1(\underline{x}) = \cdots = g_s(\underline{x}) = 0\}.$$

Questions.

(1) Is the union of infinitely many affine algebraic sets always an affine algebraic set?
(2) Is the intersection of infinitely many affine algebraic sets always an affine algebraic set?

Revise from Commutative Algebra: ideals, noetherian rings, Hilbert Basis Theorem.

3. Intersections and ideals

Answers to questions from previous lecture.

(1) No! The union of infinitely many algebraic sets is *not always* an affine algebraic set. (I don't mean that it is never an affine algebraic set, just that there exist counter-examples.) Indeed, any subset of $\mathbb{A}^n$ can be written as a union of single-point sets.
(2) Yes! The intersection of infinitely many algebraic sets *is* always an affine algebraic set.

If we try to prove (2) by combining the lists of defining equations, we run into a problem: in our definition of affine algebraic set we only allowed a *finite* list of polynomial equations.

To get round this, we use ideals.

Ideals. Let's introduce some notation.

Definition. For *any* set $S \subseteq k[X_1, \dots, X_n]$, let

$$\mathbb{V}(S) = \{\underline{x} \in \mathbb{A}^n : f(\underline{x}) = 0 \text{ for all } f \in S\}.$$

Lemma 3.1. If $S \subseteq k[X_1, \dots, X_n]$ generates the ideal I, then $\mathbb{V}(S) = \mathbb{V}(I)$.

Proof. We have $S \subseteq I$ and so it is easy to see that $\mathbb{V}(I) \subseteq \mathbb{V}(S)$.

Suppose that $\underline{x} \in \mathbb{V}(S)$, and $f \in \mathbb{V}(I)$. Then there are $f_1, \dots, f_m \in S$ and $q_1, \dots, q_m \in k[X_1, \dots, X_n]$ such that

$$f = q_1 f_1 + \cdots + q_m f_m.$$

Since $f_1(\underline{x}) = \cdots = f_m(\underline{x}) = 0$, it follows that $f(\underline{x}) = 0$.

Since this holds for every $f \in I$, $\underline{x} \in \mathbb{V}(I)$. $\square$

Using the Hilbert Basis Theorem, we can deduce that the restriction to "finite" lists of polynomials in the definition of affine algebraic set is unnecessary:

Corollary 3.2. $\mathbb{V}(S)$ is an affine algebraic set for *any* set of polynomials $S \subseteq k[X_1, \dots, X_n]$.

Proof. Let I be the ideal in $k[X_1, \dots, X_n]$ generated by S. By the Hilbert Basis Theorem, $k[X_1, \dots, X_n]$ is noetherian and so we can choose a finite set $\{f_1, \dots, f_m\}$ which generates I. Then Lemma 3.1 tells us that

$$\mathbb{V}(S) = \mathbb{V}(I) = \mathbb{V}(f_1, \dots, f_m).$$ $\square$

Corollary 3.3. The intersection of infinitely many affine algebraic sets is an affine algebraic set.

Proof. Combine the lists of defining polynomials for all the algebraic sets, and apply Corollary 3.2. $\square$

Ideals and algebraic sets: back and forth. We can also go in the other direction: from affine algebraic sets to ideals.

Definition. If A is any subset of $\mathbb{A}^n$ (usually A will be an affine algebraic set), we define

$$\mathbb{I}(A) = \{f \in k[X_1, \ldots, X_n] : f(\underline{x}) = 0 \text{ for all } \underline{x} \in A\}.$$

Note that $\mathbb{I}(A)$ is an ideal in $k[X_1, \ldots, X_n]$.

We have now defined two functions

$$\begin{aligned} \mathbb{V} &: \{\text{ideals in } k[X_1, \ldots, X_n]\} \to \{\text{affine algebraic sets in } \mathbb{A}^n\}, \\ \mathbb{I} &: \{\text{affine algebraic sets in } \mathbb{A}^n\} \to \{\text{ideals in } k[X_1, \ldots, X_n]\}. \end{aligned}$$

These functions are not inverses of each other. For example, for the ideal $(X^2) \subseteq k[X]$:

$$\mathbb{I}\Big(\mathbb{V}\Big(\,(X^2)\,\Big)\Big) = (X) \neq (X^2).$$

But composing $\mathbb{V}$ and $\mathbb{I}$ in the other order gives the identity.

Lemma 3.4. If V is an affine algebraic set, then $\mathbb{V}(\mathbb{I}(V)) = V$.

Proof. It is clear that $V \subseteq \mathbb{V}(\mathbb{I}(V))$ (and this works when V is any subset of $\mathbb{A}^n$, not necessarily algebraic).

For the reverse inclusion, we have to use the hypothesis that V is an affine algebraic set. By the definition of affine algebraic set, $V = \mathbb{V}(J)$ for some ideal $J \subseteq k[X_1, \ldots, X_n]$.

Suppose that $\underline{y} \notin V$. We shall show that $\underline{y} \notin \mathbb{V}(\mathbb{I}(V))$.

Because $\underline{y} \notin V = \mathbb{V}(J)$, there exists $f \in J$ such that $f(\underline{y}) \neq 0$. By definition, $J \subseteq \mathbb{I}(V)$ and so $f \in \mathbb{I}(V)$. Hence $f(\underline{y}) \neq 0$ tells us that $\underline{y} \notin \mathbb{V}(\mathbb{I}(V))$. □

Chain condition for affine algebraic sets. What is the geometric interpretation of the Hilbert Basis Theorem?

It is clear that $\mathbb{V}$ and $\mathbb{I}$ reverse the direction of inclusions. Hence the ascending chain condition for ideals translates into the descending chain condition for affine algebraic sets.

Lemma 3.5. Let $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$ be a descending chain of affine algebraic sets in $\mathbb{A}^n$.

Then there exists N such that $V_n = V_N$ for all $n > N$.

Proof. The fact that

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$$

implies that

$$\mathbb{I}(V_1) \subseteq \mathbb{I}(V_2) \subseteq \mathbb{I}(V_3) \subseteq \cdots.$$

Because $k[X_1, \ldots, X_n]$ is noetherian, there exists N such that $\mathbb{I}(V_n) = \mathbb{I}(V_N)$ for all $n > N$. By Lemma 3.4, $V_n = \mathbb{V}(\mathbb{I}(V_n))$ for every n and so this proves the proposition. □

Statement of the Nullstellensatz. When does $\mathbb{I}(\mathbb{V}(I)) = I$? It turns out that the only reason that this can fail is where elements of the ideal I have n-th roots which are not in I, just as with the example of $I = (X^2)$ where $X^2 \in I$ has a square root X which is not in I.

Recall the definition of the radical of an ideal from Commutative Algebra:

Definition. Let I be an ideal in a ring R. The **radical** of I is

$$\operatorname{rad} I = \sqrt{I} = \{f \in R : \exists n > 0 \text{ s.t. } f^n \in I\}.$$

We say that I is a **radical ideal** if $\operatorname{rad} I = I$.

Theorem 3.6 (Hilbert's Nullstellensatz)**.** Let I be any ideal in the polynomial ring $k[X_1, \dots, X_n]$ over an algebraically closed field k. Then

$$\mathbb{I}(\mathbb{V}(I)) = \operatorname{rad} I.$$

This is a substantial theorem, fundamental to algebraic geometry. We will prove it in a few lectures time, after developing some more tools.

Note that, to calculate $\operatorname{rad} I$, we need to add in n-th roots of all elements of I, not just the generators. For example, if $I = (X, Y^2 - X) \subseteq k[X, Y]$, then we can rewrite this as $I = (X, Y^2)$ and so $\operatorname{rad} I = (X, Y) \neq I$, even though neither of the original generators of I had any non-trivial n-th roots.

Products. Just a remark on one other way of constructing new affine algebraic sets from existing ones:

If $V \subseteq \mathbb{A}^m$ and $W \subseteq \mathbb{A}^n$ are affine algebraic sets, then their Cartesian product $V \times W \subseteq \mathbb{A}^{m+n}$ is an affine algebraic set. Write

$$V = \{(x_1, \dots, x_m) \in \mathbb{A}^m : f_1(\underline{x}) = \dots = f_r(\underline{x}) = 0\},$$
$$W = \{(y_1, \dots, y_n) \in \mathbb{A}^n : g_1(\underline{y}) = \dots = g_s(\underline{y}) = 0\}.$$

Then

$$V \times W = \{(x_1, \dots, x_m, y_1, \dots, y_n) \in \mathbb{A}^{m+n} : f_1(\underline{x}) = \dots = f_r(\underline{x}) = g_1(\underline{y}) = \dots = g_s(\underline{y}) = 0\}.$$

This looks a bit like the equations defining $V \cap W$, but here the f_i involve different variables from the g_j, while for $V \cap W$ both used the same variables.

Zariski topology. We have seen that affine algebraic sets in $\mathbb{A}^n$ satisfy the following conditions:

(i) $\mathbb{A}^n$ and $\emptyset$ are affine algebraic sets. (The empty set is the vanishing set of a non-zero constant polynomial.)
(ii) A finite union of affine algebraic sets is an affine algebraic set.
(iii) An arbitrary intersection of affine algebraic sets is an affine algebraic set.

These are precisely the conditions satisfied by the *closed* sets in a topological space. Therefore, we can define a topological space in which the underlying set is $\mathbb{A}^n$ and the closed sets are the affine algebraic sets. This is called the **Zariski topology**.

For any affine algebraic set $V \subseteq \mathbb{A}^n$, we define the **Zariski topology** on V to be the subspace topology on V induced by the Zariski topology on $\mathbb{A}^n$.

4. Zariski topology and irreducible sets

Basic facts about the Zariski topology. We defined the Zariski topology on an affine algebraic set $V \subseteq \mathbb{A}^n$ to be the subset topology induced by the Zariski topology on $\mathbb{A}^n$. Thus: a subset of V is Zariski closed in V if and only if it is Zariski closed in $\mathbb{A}^n$, but a Zariski open subset of V need not be Zariski open in $\mathbb{A}^n$. (For example: let V be the x-axis in $\mathbb{A}^2$. Then $V \setminus \{0\}$ is open in V, but not open in $\mathbb{A}^2$.)

Example. The Zariski topology on $\mathbb{A}^1$ is the same as the cofinite topology.

Thus we see that that Zariski topology has much fewer closed sets (or much fewer open sets) than for example the Euclidean topology.

Lemma 4.1. Suppose that $k = \mathbb{C}$ (so there is a Euclidean topology on $\mathbb{A}^n_{\mathbb{C}}$). If V is a Zariski closed subset of $\mathbb{A}^n_{\mathbb{C}}$, then V is closed in the Euclidean topology. ("The Euclidean topology is finer than the Zariski topology.")

Proof. Let $f \in \mathbb{C}[X_1, \dots, X_n]$ be a polynomial. It is a continuous function $\mathbb{A}^n_{\mathbb{C}} \to \mathbb{C}$ for the Euclidean topology. Since $\{0\}$ is a closed subset of $\mathbb{C}$, $\mathbb{V}(f) = f^{-1}(0)$ is a closed subset of $\mathbb{A}^n_{\mathbb{C}}$ in the Euclidean topology. We conclude by noting that intersections of closed sets are closed. $\square$

The open subsets of the Zariski topology are all "very big." This is made precise (for $\mathbb{A}^1$) by the following lemma.

Lemma 4.2. Prove that every pair U_1, U_2 of non-empty open sets in $\mathbb{A}^1$ has a non-empty intersection $U_1 \cap U_2$.

Hence the Zariski topology on $\mathbb{A}^1$ is not Hausdorff.

A subset of $\mathbb{A}^1$ is dense in the Zariski topology if and only if it is infinite.

At the moment, the Zariski topology is likely to seem very strange. It might also seem like: what is the point of such a strange topology? We will not use it in a very deep way, it is just a convenient language to be able to talk about open and closed sets. (It does get used more seriously in the theory of schemes.)

Connected and irreducible sets.

Question. Consider the following affine algebraic sets in $\mathbb{A}^2$. Do they have 1 or 2 pieces? (I have deliberately not specified what I mean by "pieces." There are multiple sensible interpretations, so there is not always a unique "correct" answer.)

(1) The union of two disjoint lines $\mathbb{V}(X(X-1))$.
(2) The union of two intersecting lines $\mathbb{V}(XY)$.
(3) The hyperbola $\mathbb{V}(XY-1)$.

Answer.

(1) $\mathbb{V}(X(X-1))$ unambiguously has 2 pieces: the two lines $X=0$ and $X=1$.

Recall that a topological space is **connected** if it is not possible to write it as the union of two disjoint non-empty closed sets. This notion makes sense for the Zariski topology.

$\mathbb{V}(X(X-1))$ is *not connected* because it is $\mathbb{V}(X) \cup \mathbb{V}(X-1)$.

(2) This has more than one answer. The two axes form 2 pieces. However they intersect at the origin, joining them into 1 piece. The set $\mathbb{V}(XY)$ is *connected* but *reducible*.

Definition. A topological space S is **reducible** if there exist closed sets $S_1, S_2 \subseteq S$ such that $S = S_1 \cup S_2$, and neither S_1 nor S_2 is equal to S.

The opposite: A topological space S is **irreducible** if it is not possible to write it as the union $S_1 \cup S_2$ of two closed sets, unless at least one of S_1 and S_2 is equal to S itself. (Change from the definition of *connected*: S_1 and S_2 are not required to be disjoint.)

Irreducibility is not a very useful notion for the topological spaces we consider in analysis. For example, considering the real line with the Euclidean topology, we can write it as a union of proper closed subsets:

$$\mathbb{R} = \{x \in \mathbb{R} : x \leq 0\} \cup \{x \in \mathbb{R} : x \geq 0\}$$

These subsets are not disjoint because they intersect at 0.

(3) A drawing of $\mathbb{V}(XY - 1)$ in $\mathbb{R}^2$ looks like it has two pieces. But (as mentioned before) we are missing a lot by only looking at real solutions. Over $\mathbb{C}$ it unambiguously has one piece.

One way to visualise this is to note that $\mathbb{V}(XY - 1)$ "looks like" the set $\mathbb{A}^1 \setminus \{0\}$ (projecting onto the x coordinate is a bijection between these sets). This is not a formal statement – we have not yet defined a notion of isomorphism of affine algebraic sets, and even if we had, $\mathbb{A}^1 \setminus \{0\}$ is not an affine algebraic set. In a few weeks we will develop technology to make this into a rigorous statement.

But for now we use it as a heuristic. $\mathbb{R} \setminus \{0\}$ unambiguously has 2 pieces, but $\mathbb{C} \setminus \{0\}$ unambiguously has 1 piece. So the hyperbola (over an algebraically closed field) should have only one piece. We prove below the lecture that $\mathbb{V}(XY - 1)$ is *irreducible* (and also *connected*).

Lemma 4.3. The hyperbola $H = \mathbb{V}(XY - 1)$ is irreducible.

Proof. We need to describe the Zariski closed subsets of H. So let $V \subseteq H$ be a proper Zariski closed subset. There must be some polynomial $f \in k[X, Y]$ which vanishes on V but does not vanish on all of H.

Because $V \subseteq H$ and $y = 1/x$ on H, we have

$$f(x, y) = f(x, 1/x) \text{ when } (x, y) \in V.$$

Now $f(X, 1/X)$ is almost a polynomial in the single variable X, except that it may contain negative powers of X:

$$f(X, 1/X) = \sum_{n \in \mathbb{Z}} a_n X^n.$$

We can multiply up by X^m where $-m$ is the lowest exponent of X which appears in this expression. Then $X^m f(X, 1/X)$ is a polynomial in X, which vanishes on V.

Furthermore $f(X, 1/X)$ is not identically zero because f does not vanish identically on H. Hence $X^m f(X, 1/X)$ is a non-zero single-variable polynomial, therefore it has only finitely many roots.

The roots of $X^m f(X, 1/X) = 0$ are the possible x-coordinates for points in V. For each value of x, there is at most one possible y such that $(x, y) \in V$ because $y = 1/x$ on V. Therefore V is finite.

Thus the Zariski topology on H is the cofinite topology, and we know that this is irreducible. $\square$

Here's a bonus fact about connected sets in the Zariski topology which I didn't mention in the lecture. The proof is surprisingly hard.

Theorem. (Not part of the course.) Over $\mathbb{C}$, an affine algebraic set in is connected in the Zariski topology if and only if it is connected in the Euclidean topology.

Question. If V is an affine algebraic set, what condition on the ideal $\mathbb{I}(V)$ is equivalent to V being irreducible?