1. Introduction

Practical information about the course.

Participation is encouraged. Please talk to each other, please interrupt me and ask questions.

Problem classes – 29 Jan, 19 Feb, 27 Feb, 19 Mar, 20 Mar
Coursework – two pieces, each worth 5% (problem sheets 2 and 4)
  Deadlines: 12 February, 12 March
Problem sheets and coursework will be available on my web page:
  `http://wwwf.imperial.ac.uk/~morr/2017-8/alg-geom`
My email address: `m.orr@imperial.ac.uk`
Office hours: Tue 11-12, Thu 2-3 in Huxley 681

Course outline.

(1) Affine varieties – definition, examples, maps between varieties, translating between geometry and commutative algebra (the Nullstellensatz)
(2) Projective varieties – definition, examples, maps between varieties, rigidity and images of maps
(3) Dimension – several different definitions (all equivalent, but useful for different purposes), calculating dimensions of examples
(4) Smoothness and singularities – definition, examples, key theorems
(5) Examples of varieties (depending on how much time is left)

What is not in the course?

(1) Schemes
(2) Sheaves and cohomology
(3) Curves, divisors and the Riemann–Roch theorem

The base field. Let k be an algebraically closed field.

We are going to be thinking about solutions to polynomials, so everything is much simpler over algebraically closed fields. Number theorists might be interested in other fields, but you generally have to start by understanding the algebraically closed case first. In this course we will stop with the algebraically closed case too.

Apart from being algebraically closed, it usually does not matter much which field we use to do algebraic geometry – except sometimes it matters whether the characteristic is zero or positive. In this course I will take care to mention results which depend on the characteristic, and sometimes we might consider only the characteristic zero case. You will not lose much if you just assume that $k = \mathbb{C}$ throughout the course (except when it will be explicitly something else).

Indeed it is often useful to think about $k = \mathbb{C}$ because then you can use your usual geometric intuition. When I draw pictures on the whiteboard, I am usually only drawing the real solutions because it is hard to draw shapes in $\mathbb{C}^2$. This is cheating but it is often very useful – the real solutions are not the full picture but in many cases we can still see the important features there.

Affine space.

Definition. Algebraic geometers write $\mathbb{A}^n$ to mean k^n, and call it **affine** n**-space**.

You may think of this as just a funny choice of notation, but there are at least two reasons for it:

(i) When we write k^n, it makes us think of a vector space, equipped with operations of addition and scalar multiplication. But $\mathbb{A}^n$ means just a set of points, described by coordinates $(x_1, \ldots, x_n)$ with $x_i \in k$, without the vector space structure.

(ii) Because it usually doesn't matter much what our base field k is (as long as it is algebraically closed), it is convenient to have notation which does not prominently mention k.

On occasions when it *is* important to specify which field k we are using, we write $\mathbb{A}^n_k$ for affine n-space.

2. Affine algebraic sets

Definition. An **affine algebraic set** is a subset $V \subseteq \mathbb{A}^n$ which consists of the common zeros of some finite set of polynomials $f_1, \ldots, f_m$ with coefficients in k.

More formally, an **affine algebraic set** is a set of the form

$$V = \{(x_1, \ldots, x_n) \in \mathbb{A}^n : f_1(x_1, \ldots, x_n) = \cdots = f_m(x_1, \ldots, x_n) = 0\}$$

for some polynomials $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$.

Examples of affine algebraic sets.

Exercise 2.1. Think of some examples and non-examples of affine algebraic sets.

These are the examples and non-examples you came up with in lectures.

Examples.

(1) The whole space $\mathbb{A}^n$, defined by the polynomial $f_1 = 0$ (or by the empty set of polynomials).
(2) The set $\{2\}$, defined by the polynomial $X - 2$. More generally, any point in $\mathbb{A}^1$.
(3) Any union of finitely many affine algebraic sets (see proof below). Combining with (2), we deduce that any finite subset of $\mathbb{A}^1$ is an affine algebraic set.
(4) An algebraic curve in $\mathbb{A}^2$, that is, a set of the form

$$\{(x_1, x_2) \in \mathbb{A}^2 : f(x_1, x_2) = 0\}$$

for some polynomial $f \in k[X_1, X_2]$.

Non-examples.

(1) Any infinite subset of $\mathbb{A}^1$ (other than $\mathbb{A}^1$ itself). This is because a one-variable polynomial with infinitely many roots must be the zero polynomial.

Here are some additional examples of affine algebraic sets.

Further examples.

(5) Any point in $\mathbb{A}^n$. The single-point set $\{(a_1, \ldots, a_n)\}$ is defined by the equations

$$X_1 - a_1 = 0, \ldots, X_n - a_n = 0.$$

Using (3), we see that any finite subset of $\mathbb{A}^n$ is an affine algebraic set.
(6) Embeddings of $\mathbb{A}^m$ in $\mathbb{A}^n$ where $m < n$:

$$\{(x_1, \ldots, x_m, 0, \ldots, 0) \in \mathbb{A}^n\} = \{(x_1, \ldots, x_n) \in \mathbb{A}^n : x_{m+1} = \cdots = x_n = 0\}.$$

More generally, the image of a linear map $\mathbb{A}^m \to \mathbb{A}^n$:

$$\{(x_1, \ldots, x_n) \in \mathbb{A}^n : \text{some linear conditions}\}.$$

Further non-example.

Example (6) does not generalise to images of maps where each coordinate is given by a polynomial. For example, consider the map

$$\phi\colon \mathbb{A}^2 \to \mathbb{A}^2 \text{ where } f(x,y) = (x, xy).$$

The image of ϕ is

$$S = \mathbb{A}^2 \setminus \{(0,y)\} \cup \{(0,0)\}.$$

To prove that S is not an affine algebraic set, consider a polynomial $g(X,Y) \in k[X,Y]$ which vanishes on S. For each fixed $y \in k$, the one-variable polynomial $g(X,y)$ vanishes at all $x \neq 0$. This implies that $g(X,y)$ is the zero polynomial. Thus $g(x,y) = 0$ for all $x, y \in k^2$, that is, g is the zero polynomial.

Philosophical remark. This remark might seem obscure for now; we will come back to it later.

The words "affine variety" mean more or less the same thing as "affine algebraic set" but there is an ontological difference. "Affine algebraic set" means a subset which lives inside $\mathbb{A}^n$ and knows how it lives inside $\mathbb{A}^n$, while "affine variety" means an object in its own right which is considered outside of $\mathbb{A}^n$. I will try to use these words consistently, but the difference is quite subtle and books may not always use it consistently. For the first few weeks, we will talk about "affine algebraic sets" only.

Note that some books (e.g. Reid, Hartshorne) have another difference between affine varieties and affine algebraic sets – they require varieties to be irreducible (which we will define next time). Other books (e.g. Shafarevich) do not require varieties to be irreducible. In this course we will *not* require varieties to be irreducible.

Unions and intersections of affine algebraic sets. One of the examples was a union of finitely many affine algebraic sets. Now we prove that the union of two affine algebraic sets is an affine algebraic set.

Lemma 2.1. If $V, W \subseteq \mathbb{A}^n$ are affine algebraic sets, then their union $V \cup W \subseteq \mathbb{A}^n$ is also an affine algebraic set.

Proof. We have to take the product for each possible pair of defining polynomials: if

$$V = \{(x_1, \dots, x_n) \in \mathbb{A}^n : f_1(\underline{x}) = \dots = f_r(\underline{x}) = 0\},$$
$$W = \{(x_1, \dots, x_n) \in \mathbb{A}^n : g_1(\underline{x}) = \dots = g_s(\underline{x}) = 0\},$$

then

$$V \cup W = \{\underline{x} \in \mathbb{A}^n : f_i(\underline{x})g_j(\underline{x}) = 0 \text{ for all } i, j \text{ where } 1 \leq i \leq r, 1 \leq j \leq s\}.$$

Note that we really need all the pairs $f_i g_j$, not just for example $f_1 g_1$, $f_2 g_2$, etc. It is obvious that if $\underline{x} \in V \cup W$, then all the products $f_i g_j$ vanish at $\underline{x}$.

The reverse is a little trickier. Suppose that we have $\underline{x} \in \mathbb{A}^n$ satisfying $f_i(\underline{x})g_j(\underline{x}) = 0$ for all i and j. Looking just at f_1, we get:

$$\begin{aligned} f_1g_1(\underline{x}) = 0, &\text{ so } f_1(\underline{x}) = 0 \text{ or } g_1(\underline{x}) = 0. \\ f_1g_2(\underline{x}) = 0, &\text{ so } f_1(\underline{x}) = 0 \text{ or } g_2(\underline{x}) = 0. \\ &\vdots \\ f_1g_s(\underline{x}) = 0, &\text{ so } f_1(\underline{x}) = 0 \text{ or } g_s(\underline{x}) = 0. \end{aligned}$$

Putting these all together, we get

$$f_1(\underline{x}) = 0 \text{ or } g_j(\underline{x}) = 0 \text{ for every } j.$$

We can do the same thing for f_2 to get

$$f_2(\underline{x}) = 0 \text{ or } g_j(\underline{x}) = 0 \text{ for every } j$$

and so on for each f_i. Putting all these together, we get

$$f_i(\underline{x}) = 0 \text{ for every } i \text{ or } g_j(\underline{x}) = 0 \text{ for every } j.$$

This says precisely that $\underline{x} \in V \cup W$. □

It is even easier to check that the intersection of finitely many affine algebraic sets is an affine algebraic sets: if V is defined by polynomials $f_1, \dots, f_r$ and W is defined by polynomials $g_1, \dots, g_s$, then $V \cap W$ is simply the set where all the polynomials in both lists vanish i.e.

$$V \cap W = \{\underline{x} \in \mathbb{A}^n : f_1(\underline{x}) = \cdots = f_r(\underline{x}) = 0 \text{ and } g_1(\underline{x}) = \cdots = g_s(\underline{x}) = 0\}.$$

Questions.

(1) Is the union of infinitely many affine algebraic sets always an affine algebraic set?
(2) Is the intersection of infinitely many affine algebraic sets always an affine algebraic set?

Revise from Commutative Algebra: ideals, noetherian rings, Hilbert Basis Theorem.

3. Intersections and ideals

Answers to questions from previous lecture.

(1) No! The union of infinitely many algebraic sets is *not always* an affine algebraic set. (I don't mean that it is never an affine algebraic set, just that there exist counter-examples.) Indeed, any subset of $\mathbb{A}^n$ can be written as a union of single-point sets.
(2) Yes! The intersection of infinitely many algebraic sets *is* always an affine algebraic set.

If we try to prove (2) by combining the lists of defining equations, we run into a problem: in our definition of affine algebraic set we only allowed a *finite* list of polynomial equations.

To get round this, we use ideals.

Ideals. Let's introduce some notation.

Definition. For *any* set $S \subseteq k[X_1, \dots, X_n]$, let

$$\mathbb{V}(S) = \{\underline{x} \in \mathbb{A}^n : f(\underline{x}) = 0 \text{ for all } f \in S\}.$$

Lemma 3.1. If $S \subseteq k[X_1, \dots, X_n]$ generates the ideal I, then $\mathbb{V}(S) = \mathbb{V}(I)$.

Proof. We have $S \subseteq I$ and so it is easy to see that $\mathbb{V}(I) \subseteq \mathbb{V}(S)$.

Suppose that $\underline{x} \in \mathbb{V}(S)$, and $f \in \mathbb{V}(I)$. Then there are $f_1, \dots, f_m \in S$ and $q_1, \dots, q_m \in k[X_1, \dots, X_n]$ such that

$$f = q_1 f_1 + \dots + q_m f_m.$$

Since $f_1(\underline{x}) = \dots = f_m(\underline{x}) = 0$, it follows that $f(\underline{x}) = 0$.

Since this holds for every $f \in I$, $\underline{x} \in \mathbb{V}(I)$. □

Using the Hilbert Basis Theorem, we can deduce that the restriction to "finite" lists of polynomials in the definition of affine algebraic set is unnecessary:

Corollary 3.2. $\mathbb{V}(S)$ is an affine algebraic set for *any* set of polynomials $S \subseteq k[X_1, \dots, X_n]$.

Proof. Let I be the ideal in $k[X_1, \dots, X_n]$ generated by S. By the Hilbert Basis Theorem, $k[X_1, \dots, X_n]$ is noetherian and so we can choose a finite set $\{f_1, \dots, f_m\}$ which generates I. Then Lemma 3.1 tells us that

$$\mathbb{V}(S) = \mathbb{V}(I) = \mathbb{V}(f_1, \dots, f_m).$$ □

Corollary 3.3. The intersection of infinitely many affine algebraic sets is an affine algebraic set.

Proof. Combine the lists of defining polynomials for all the algebraic sets, and apply Corollary 3.2. □

Ideals and algebraic sets: back and forth. We can also go in the other direction: from affine algebraic sets to ideals.

Definition. If A is any subset of $\mathbb{A}^n$ (usually A will be an affine algebraic set), we define

$$\mathbb{I}(A) = \{f \in k[X_1, \dots, X_n] : f(\underline{x}) = 0 \text{ for all } \underline{x} \in A\}.$$

Note that $\mathbb{I}(A)$ is an ideal in $k[X_1, \dots, X_n]$.

We have now defined two functions

$$\mathbb{V} : \{\text{ideals in } k[X_1, \dots, X_n]\} \to \{\text{affine algebraic sets in } \mathbb{A}^n\},$$
$$\mathbb{I} : \{\text{affine algebraic sets in } \mathbb{A}^n\} \to \{\text{ideals in } k[X_1, \dots, X_n]\}.$$

These functions are not inverses of each other. For example, for the ideal $(X^2) \subseteq k[X]$:

$$\mathbb{I}\Big(\mathbb{V}\Big((X^2) \Big)\Big) = (X) \neq (X^2).$$

But composing $\mathbb{V}$ and $\mathbb{I}$ in the other order gives the identity.

Lemma 3.4. If V is an affine algebraic set, then $\mathbb{V}(\mathbb{I}(V)) = V$.

Proof. It is clear that $V \subseteq \mathbb{V}(\mathbb{I}(V))$ (and this works when V is any subset of $\mathbb{A}^n$, not necessarily algebraic).

For the reverse inclusion, we have to use the hypothesis that V is an affine algebraic set. By the definition of affine algebraic set, $V = \mathbb{V}(J)$ for some ideal $J \subseteq k[X_1, \dots, X_n]$.

Suppose that $\underline{y} \notin V$. We shall show that $\underline{y} \notin \mathbb{V}(\mathbb{I}(V))$.

Because $\underline{y} \notin V = \mathbb{V}(J)$, there exists $f \in J$ such that $f(\underline{y}) \neq 0$. By definition, $J \subseteq \mathbb{I}(V)$ and so $f \in \mathbb{I}(V)$. Hence $f(\underline{y}) \neq 0$ tells us that $\underline{y} \notin \mathbb{V}(\mathbb{I}(V))$. □

Chain condition for affine algebraic sets. What is the geometric interpretation of the Hilbert Basis Theorem?

It is clear that $\mathbb{V}$ and $\mathbb{I}$ reverse the direction of inclusions. Hence the ascending chain condition for ideals translates into the descending chain condition for affine algebraic sets.

Lemma 3.5. Let $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$ be a descending chain of affine algebraic sets in $\mathbb{A}^n$.

Then there exists N such that $V_n = V_N$ for all $n > N$.

Proof. The fact that

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$$

implies that

$$\mathbb{I}(V_1) \subseteq \mathbb{I}(V_2) \subseteq \mathbb{I}(V_3) \subseteq \cdots.$$

Because $k[X_1, \dots, X_n]$ is noetherian, there exists N such that $\mathbb{I}(V_n) = \mathbb{I}(V_N)$ for all $n > N$. By Lemma 3.4, $V_n = \mathbb{V}(\mathbb{I}(V_n))$ for every n and so this proves the proposition. □

Statement of the Nullstellensatz. When does $\mathbb{I}(\mathbb{V}(I)) = I$? It turns out that the only reason that this can fail is where elements of the ideal I have n-th roots which are not in I, just as with the example of $I = (X^2)$ where $X^2 \in I$ has a square root X which is not in I.

Recall the definition of the radical of an ideal from Commutative Algebra:

Definition. Let I be an ideal in a ring R. The **radical** of I is

$$\operatorname{rad} I = \sqrt{I} = \{f \in R : \exists n > 0 \text{ s.t. } f^n \in I\}.$$

We say that I is a **radical ideal** if $\operatorname{rad} I = I$.

Theorem 3.6 (Hilbert's Nullstellensatz)**.** Let I be any ideal in the polynomial ring $k[X_1, \dots, X_n]$ over an algebraically closed field k. Then

$$\mathbb{I}(\mathbb{V}(I)) = \operatorname{rad} I.$$

This is a substantial theorem, fundamental to algebraic geometry. We will prove it in a few lectures time, after developing some more tools.

Note that, to calculate $\operatorname{rad} I$, we need to add in n-th roots of all elements of I, not just the generators. For example, if $I = (X, Y^2 - X) \subseteq k[X, Y]$, then we can rewrite this as $I = (X, Y^2)$ and so $\operatorname{rad} I = (X, Y) \neq I$, even though neither of the original generators of I had any non-trivial n-th roots.

Products. Just a remark on one other way of constructing new affine algebraic sets from existing ones:

If $V \subseteq \mathbb{A}^m$ and $W \subseteq \mathbb{A}^n$ are affine algebraic sets, then their Cartesian product $V \times W \subseteq \mathbb{A}^{m+n}$ is an affine algebraic set. Write

$$V = \{(x_1, \dots, x_m) \in \mathbb{A}^m : f_1(\underline{x}) = \dots = f_r(\underline{x}) = 0\},$$
$$W = \{(y_1, \dots, y_n) \in \mathbb{A}^n : g_1(\underline{y}) = \dots = g_s(\underline{y}) = 0\}.$$

Then

$$V \times W = \{(x_1, \dots, x_m, y_1, \dots, y_n) \in \mathbb{A}^{m+n} : f_1(\underline{x}) = \dots = f_r(\underline{x}) = g_1(\underline{y}) = \dots = g_s(\underline{y}) = 0\}.$$

This looks a bit like the equations defining $V \cap W$, but here the f_i involve different variables from the g_j, while for $V \cap W$ both used the same variables.

Zariski topology. We have seen that affine algebraic sets in $\mathbb{A}^n$ satisfy the following conditions:

(i) $\mathbb{A}^n$ and $\emptyset$ are affine algebraic sets. (The empty set is the vanishing set of a non-zero constant polynomial.)
(ii) A finite union of affine algebraic sets is an affine algebraic set.
(iii) An arbitrary intersection of affine algebraic sets is an affine algebraic set.

These are precisely the conditions satisfied by the *closed* sets in a topological space. Therefore, we can define a topological space in which the underlying set is $\mathbb{A}^n$ and the closed sets are the affine algebraic sets. This is called the **Zariski topology**.

For any affine algebraic set $V \subseteq \mathbb{A}^n$, we define the **Zariski topology** on V to be the subspace topology on V induced by the Zariski topology on $\mathbb{A}^n$.

4. Zariski topology and irreducible sets

Basic facts about the Zariski topology. We defined the Zariski topology on an affine algebraic set $V \subseteq \mathbb{A}^n$ to be the subset topology induced by the Zariski topology on $\mathbb{A}^n$. Thus: a subset of V is Zariski closed in V if and only if it is Zariski closed in $\mathbb{A}^n$, but a Zariski open subset of V need not be Zariski open in $\mathbb{A}^n$. (For example: let V be the x-axis in $\mathbb{A}^2$. Then $V \setminus \{0\}$ is open in V, but not open in $\mathbb{A}^2$.)

Example. The Zariski topology on $\mathbb{A}^1$ is the same as the cofinite topology.

Thus we see that that Zariski topology has much fewer closed sets (or much fewer open sets) than for example the Euclidean topology.

Lemma 4.1. Suppose that $k = \mathbb{C}$ (so there is a Euclidean topology on $\mathbb{A}^n_{\mathbb{C}}$). If V is a Zariski closed subset of $\mathbb{A}^n_{\mathbb{C}}$, then V is closed in the Euclidean topology. ("The Euclidean topology is finer than the Zariski topology.")

Proof. Let $f \in \mathbb{C}[X_1, \dots, X_n]$ be a polynomial. It is a continuous function $\mathbb{A}^n_{\mathbb{C}} \to \mathbb{C}$ for the Euclidean topology. Since $\{0\}$ is a closed subset of $\mathbb{C}$, $\mathbb{V}(f) = f^{-1}(0)$ is a closed subset of $\mathbb{A}^n_{\mathbb{C}}$ in the Euclidean topology. We conclude by noting that intersections of closed sets are closed. □

The open subsets of the Zariski topology are all "very big." This is made precise (for $\mathbb{A}^1$) by the following lemma.

Lemma 4.2. Prove that every pair U_1, U_2 of non-empty open sets in $\mathbb{A}^1$ has a non-empty intersection $U_1 \cap U_2$.

Hence the Zariski topology on $\mathbb{A}^1$ is not Hausdorff.

A subset of $\mathbb{A}^1$ is dense in the Zariski topology if and only if it is infinite.

At the moment, the Zariski topology is likely to seem very strange. It might also seem like: what is the point of such a strange topology? We will not use it in a very deep way, it is just a convenient language to be able to talk about open and closed sets. (It does get used more seriously in the theory of schemes.)

Connected and irreducible sets.

Question. Consider the following affine algebraic sets in $\mathbb{A}^2$. Do they have 1 or 2 pieces? (I have deliberately not specified what I mean by "pieces." There are multiple sensible interpretations, so there is not always a unique "correct" answer.)

(1) The union of two disjoint lines $\mathbb{V}(X(X-1))$.
(2) The union of two intersecting lines $\mathbb{V}(XY)$.
(3) The hyperbola $\mathbb{V}(XY-1)$.

Answer.

(1) $\mathbb{V}(X(X-1))$ unambiguously has 2 pieces: the two lines $X = 0$ and $X = 1$.

Recall that a topological space is **connected** if it is not possible to write it as the union of two disjoint non-empty closed sets. This notion makes sense for the Zariski topology.

$\mathbb{V}(X(X-1))$ is *not connected* because it is $\mathbb{V}(X) \cup \mathbb{V}(X-1)$.

(2) This has more than one answer. The two axes form 2 pieces. However they intersect at the origin, joining them into 1 piece. The set $\mathbb{V}(XY)$ is *connected* but *reducible*.

Definition. A topological space S is **reducible** if it is empty, or there exist closed sets $S_1, S_2 \subseteq S$ such that $S = S_1 \cup S_2$, and neither S_1 nor S_2 is equal to S.

The opposite: A topological space S is **irreducible** if it is non-empty and it is not possible to write it as the union $S_1 \cup S_2$ of two closed sets, unless at least one of S_1 and S_2 is equal to S itself. (Change from the definition of *connected*: S_1 and S_2 are not required to be disjoint.)

Irreducibility is not a very useful notion for the topological spaces we consider in analysis. For example, considering the real line with the Euclidean topology, we can write it as a union of proper closed subsets:

$$\mathbb{R} = \{x \in \mathbb{R} : x \leq 0\} \cup \{x \in \mathbb{R} : x \geq 0\}$$

These subsets are not disjoint because they intersect at 0.

(3) A drawing of $\mathbb{V}(XY - 1)$ in $\mathbb{R}^2$ looks like it has two pieces. But (as mentioned before) we are missing a lot by only looking at real solutions. Over $\mathbb{C}$ it unambiguously has one piece.

One way to visualise this is to note that $\mathbb{V}(XY - 1)$ "looks like" the set $\mathbb{A}^1 \setminus \{0\}$ (projecting onto the x coordinate is a bijection between these sets). This is not a formal statement – we have not yet defined a notion of isomorphism of affine algebraic sets, and even if we had, $\mathbb{A}^1 \setminus \{0\}$ is not an affine algebraic set. In a few weeks we will develop technology to make this into a rigorous statement.

But for now we use it as a heuristic. $\mathbb{R} \setminus \{0\}$ unambiguously has 2 pieces, but $\mathbb{C} \setminus \{0\}$ unambiguously has 1 piece. So the hyperbola (over an algebraically closed field) should have only one piece. We prove below the lecture that $\mathbb{V}(XY - 1)$ is *irreducible* (and also *connected*).

Lemma 4.3. The hyperbola $H = \mathbb{V}(XY - 1)$ is irreducible.

Proof. We need to describe the Zariski closed subsets of H. So let $V \subseteq H$ be a proper Zariski closed subset. There must be some polynomial $f \in k[X, Y]$ which vanishes on V but does not vanish on all of H.

Because $V \subseteq H$ and $y = 1/x$ on H, we have

$$f(x, y) = f(x, 1/x) \text{ when } (x, y) \in V.$$

Now $f(X, 1/X)$ is almost a polynomial in the single variable X, except that it may contain negative powers of X:

$$f(X, 1/X) = \sum_{n \in \mathbb{Z}} a_n X^n.$$

We can multiply up by X^m where $-m$ is the lowest exponent of X which appears in this expression. Then $X^m f(X, 1/X)$ is a polynomial in X, which vanishes on V.

Furthermore $f(X, 1/X)$ is not identically zero because f does not vanish identically on H. Hence $X^m f(X, 1/X)$ is a non-zero single-variable polynomial, therefore it has only finitely many roots.

The roots of $X^m f(X, 1/X) = 0$ are the possible x-coordinates for points in V. For each value of x, there is at most one possible y such that $(x, y) \in V$ because $y = 1/x$ on V. Therefore V is finite.

Thus the Zariski topology on H is the cofinite topology, and we know that this is irreducible. □

Here's a bonus fact about connected sets in the Zariski topology which I didn't mention in the lecture. The proof is surprisingly hard.

Theorem. (Not part of the course.) Over $\mathbb{C}$, an affine algebraic set in is connected in the Zariski topology if and only if it is connected in the Euclidean topology.

Question. If V is an affine algebraic set, what condition on the ideal $\mathbb{I}(V)$ is equivalent to V being irreducible?

5. Irreducible components

Correction. In the last lecture, we defined reducible and irreducible topological spaces. These definitions were wrong about the empty set. According to a correct definition, the empty set is reducible: it should say "a set is reducible if it is empty, or ...". (I have corrected the definition in these notes.)

Note on topology. I have written some short notes on the topological definitions we need, available on Blackboard and on my web page. This lecture contains the most topology of any lecture in the course.

Prime ideals.

Definition. (from Commutative Algebra) An ideal I in a ring R is a **prime ideal** if $I \neq R$ and for every $f, g \in R$, if $fg \in I$, then $f \in I$ or $g \in I$ (or both).

Lemma 5.1. An affine algebraic set $V \subseteq \mathbb{A}^n$ is irreducible if and only if $\mathbb{I}(V)$ is a prime ideal in $k[X_1, \dots, X_n]$.

Proof. First suppose that V is irreducible. Suppose we have $f, g \in k[X_1, \dots, X_n]$ such that $fg \in \mathbb{I}(V)$. Let

$$V_1 = \{\underline{x} \in V : f(\underline{x}) = 0\}, \ V_2 = \{\underline{x} \in V : g(\underline{x}) = 0\}.$$

For every $\underline{x} \in V$, $f(\underline{x})g(\underline{x}) = 0$ and hence either $f(\underline{x}) = 0$ or $g(\underline{x}) = 0$. Thus for every $\underline{x} \in V$, either $\underline{x} \in V_1$ or $\underline{x} \in V_2$. In other words, $V = V_1 \cup V_2$. Furthermore V_1 and V_2 are closed subsets of V. Hence as V is irreducible, either $V_1 = V$ or $V_2 = V$. If $V_1 = V$ then $f \in \mathbb{I}(V)$ and if $V_2 = V$ then $g \in \mathbb{I}(V)$.

Now suppose that V is reducible. Then we can write it as a union $V_1 \cup V_2$ of proper closed subsets. Since V_1 is a proper closed subset of V, there exists some $f \in k[X_1, \dots, X_n]$ vanishing on V_1 but not on all of V. Similarly there exists g vanishing on V_2 but not on all of V. Thus neither f nor g is in $\mathbb{I}(V)$, but the product fg vanishes on $V_1 \cup V_2$ and hence we have $fg \in \mathbb{I}(V)$. Thus $\mathbb{I}(V)$ is not prime.

V is empty if and only if $\mathbb{I}(V) = k[X_1, \dots, X_n]$, which is explicitly defined to not be a prime ideal. So it was OK to ignore this case above. □

Definition. A **hypersurface** is an affine algebraic set in $\mathbb{A}^n$ defined by *one* polynomial equation, that is,

$$\{\underline{x} \in \mathbb{A}^n : f(\underline{x}) = 0\}$$

for some $f \in k[X_1, \dots, X_n]$.

It follows from Lemma 5.1 together with Hilbert's Nullstellensatz that a hypersurface defined by a polynomial f is irreducible if and only if f is a power of an irreducible polynomial (see problem sheet 1).

Irreducible spaces and open sets.

It can often be convenient to rewrite the definition of irreducible spaces in terms of open sets instead of closed sets:

Lemma 5.2. The following conditions on a topological space S are equivalent to irreducibility:

(i) S is non-empty, and every pair of non-empty open subsets $U_1, U_2 \subseteq S$ have non-empty intersection $U_1 \cap U_2$.
(ii) S is non-empty, and every non-empty open subset of S is dense in S.

Corollary 5.3. Let S be a irreducible topological space and $U \subseteq S$ a non-empty open subset. Then U is irreducible (in the subspace topology).

(Proofs of the lemma and corollary are just manipulations of the topological definitions.)

Corollary 5.3(1) says that "irreducible" is a very long way from "Hausdorff": the Hausdorff condition says that a space has lots of pairs of disjoint non-empty open subsets, while an irreducible space has none. For example, we saw that $\mathbb{R}$ (with the Euclidean topology) is reducible in many ways.

Corollary 5.3 implies that $\mathbb{A}^1 \setminus \{0\}$ is irreducible (in the subspace topology induced by the Zariski topology on $\mathbb{A}^1$), because it is open in $\mathbb{A}^1$. This lends support to the heuristic argument that the hyperbola $\mathbb{V}(XY - 1)$ is irreducible, but it is not a proof – checking that the subspace topology on $\mathbb{A}^1 \setminus \{0\}$ is the same as the Zariski topology on $\mathbb{V}(XY - 1)$ would require as much work as the proof that $\mathbb{V}(XY - 1)$ is irreducible.

Irreducible components.

Just like the definition of connected components, we can define:

Definition. Let S be a topological space. An **irreducible component** of S is a maximal irreducible subset of S.

Unlike connected components, irreducible components need not be disjoint. For example, the irreducible components of $\{(x, y) : xy = 0\}$ are the lines $x = 0$ and $y = 0$, which intersect in $\{(0, 0)\}$.

More generally, the irreducible components of a hypersurface $\mathbb{V}(f)$ correspond to the irreducible factors of f: if $f = f_1^{a_1} \cdots f_m^{a_m}$ (where the f_i are distinct irreducible polynomials), then the irreducible components of $\mathbb{V}(f)$ are $\mathbb{V}(f_1), \dots, \mathbb{V}(f_m)$.

Irreducible components have the following key properties:

Proposition 5.4. Let V be an affine algebraic set. Then:

(1) The union of the irreducible components of V is all of V.
(2) V has only finitely many irreducible components.

(1) matches a property of connected components. (2) does not apply to the connected components of an arbitrary topological space: for example, $\mathbb{Z}$ or $\mathbb{Q}$ with the subspace topology from $\mathbb{R}$. Note that (2) *does* imply that an affine algebraic set has only finitely many connected components for the Zariski topology, because each connected component must be a union of irreducible components.

Proposition 5.4(2) is a "finiteness" statement, so it is not surprising that it follows from the noetherian property (the descending chain condition on closed subsets). The key idea in the proof is as follows: If an affine algebraic set is reducible, then we can write it as a union of proper closed subsets. If these subsets are reducible, then we can write them in turn as unions of proper closed subsets. The following lemma says that this process eventually stops: after finitely many steps, we reach irreducible sets.

Lemma 5.5. Every affine algebraic set can be written as a union of finitely many irreducible closed subsets.

Proof. Suppose that V is an affine algebraic set which cannot be written as a union of finitely many irreducible closed subsets.

V must be reducible (otherwise we could write it as a union of one irreducible closed subset!) So $V = V_1 \cup W_1$, with V_1 and W_1 proper closed subsets of V.

V_1 and W_1 cannot both be unions of finitely many irreducible closed subsets, because taking the union of those decompositions would give us V as a union of finitely many irreducible closed subsets.

Thus at least one of V_1 and W_1 does not satisfy the lemma. Without loss of generality, we may suppose that V_1 does not satisfy the lemma.

Then V_1 must be reducible, so we can write $V_1 = V_2 \cup W_2$. We can repeat the argument: at least one of V_2 and W_2 does not satisfy the lemma, without loss of generality V_2, etc.

Thus we build up a chain of closed subsets $V \supset V_1 \supset V_2 \supset V_3 \supset \cdots$ where all these sets do not satisfy the lemma, and all the inclusions are strict. This contradicts Lemma 3.5 (the descending chain condition for affine algebraic sets). □

In order to prove Proposition 5.4, we want to show that the finitely many irreducible closed subsets in Lemma 5.5 *are* the irreducible components. There is just one wrinkle: Consider $V = \mathbb{V}(XY)$. The irreducible components are $\mathbb{V}(X)$ and $\mathbb{V}(Y)$. But we could write V as a union of finitely many irreducible closed subsets by saying:

$$V = \mathbb{V}(X) \cup \mathbb{V}(Y) \cup \{(0, 2)\}.$$

Thus we can always add in extra sets to a decomposition as in Lemma 5.5, where the extra sets are contained in one of the other sets in the decomposition.

Of course we can always throw away these empty sets: let $V = V_1 \cup \cdots \cup V_r$, as in Lemma 5.5. By throwing away any V_i which is contained in another V_j, we can assume that $V_i \not\subseteq V_j$ whenever $i \neq j$, and still the union of the V_js will be V. After doing this we can prove:

Proposition 5.6. Let V be an affine algebraic set. Write $V = V_1 \cup \cdots \cup V_r$, where the V_i are irreducible closed subsets and $V_i \not\subseteq V_j$ for $i \neq j$.

Then $V_1, \ldots, V_r$ are precisely the irreducible components of V.

Proof. First we show that each V_i is an irreducible component. By hypothesis, V_i is irreducible. So if V_i is not an irreducible component, it is not a *maximal* irreducible set and must be contained in a larger irreducible set $W \subseteq V$.

But then $W = (V_1 \cap W) \cup \cdots \cup (V_r \cap W)$, where $V_1 \cap W, \dots, V_r \cap W$ are closed subsets of W. Because W is irreducible, we must have $W = V_j \cap W$ for some j. Thus $V_i \subseteq W \subseteq V_j$. By the condition $V_i \not\subseteq V_j$ for any $j \neq i$, we must have $i = j$ and $W = V_i$. Thus V_i is an irreducible component of V.

Conversely, let C be an irreducible component of V. Then $C = (V_1 \cap C) \cup \cdots \cup (V_r \cap C)$. By the same argument as before, the irreducibility of C implies that $C \subseteq V_i$ for some i. Then the maximality of C implies that $C = V_i$. □

The combination of Lemma 5.5 and Proposition 5.6 proves Proposition 5.4 (both (1) and (2)).

Primary decomposition of ideals.

(Not part of the course)

The irreducible component decomposition of an affine algebraic set can give a geometric understanding of the primary decomposition of ideals in the noetherian ring $k[X_1, \dots, X_n]$. However, the irreducible decomposition gives only partial information about the primary decomposition of an ideal, because ideals contain more information than affine algebraic sets (recall that the algebraic set depends only on the radical of the ideal).

For example: $I = (X^2, XY) \subseteq k[X, Y]$. Then $\mathbb{V}(I)$ is simply the line $X = 0$, which of course is irreducible. However a primary decomposition of I is

$$I = (X) \cap (X^2, XY, Y^2).$$

Here (X) is the ideal of the line $X = 0$, the unique irreducible component of $V = \mathbb{V}(I)$. The ideal (X^2, XY, Y^2) defines the point $\{(0, 0)\}$, which is contained in V so is not an irreducible component.

Thus the *minimal* associated primes of the primary decomposition of I corespond to the irreducible components of $\mathbb{V}(I)$, while non-minimal associated primes correspond to additional smaller sets strictly contained in the irreducible components (called "embedded components"). In scheme theory, we can think of $\mathbb{V}(I)$ as containing "multiple copies" of these embedded components. For example, the ideal $I = (X^2, XY)$ corresponds (in the world of schemes) to the line $X = 0$ with "two copies of the origin."

6. Regular functions and regular maps

Regular functions.

So far we have only considered algebraic sets as sets, sitting individually. Now we look at functions between them. Just as one uses continuous functions for topological spaces, holomorphic functions for complex manifolds, homomorphisms for groups, etc., so algebraic geometry has its own type of functions – regular functions. Of course, these are given by polynomials.

Definition. Let $V \subseteq \mathbb{A}^n$ be an affine algebraic set. A **regular function** on V is a function $f\colon V \to k$ such that there exists a polynomial $F \in k[X_1, \dots, X_n]$ with $f(\underline{x}) = F(\underline{x})$ for all $x \in V$.

Note that the polynomial F is not uniquely determined by the function f: $F, G \in k[X_1, \dots, X_n]$ determine the same regular function on V if and only if $F - G$ vanishes on V, that is iff $F - G \in \mathbb{I}(V)$.

Definition. The regular functions on V form a k-algebra: they can be added and multiplied by each other and multiplied by scalars in k. This is called the **coordinate ring** of V and denoted $k[V]$.

There is a ring homomorphism $k[X_1, \dots, X_n] \to k[V]$ which sends a polynomial F to the function $F_{|V}$ which it defines on V. This homomorphism is surjective and its kernel is $\mathbb{I}(V)$, so

$$k[V] \cong k[X_1, \dots, X_n]/\mathbb{I}(V).$$

Example. What are the coordinate rings of the following affine algebraic sets?

(i) $\mathbb{A}^n$.
(ii) A point.
(iii) $\{x \in \mathbb{A}^1 : x(x-1) = 0\}$ (two points).
(iv) $\{(x, y) \in \mathbb{A}^2 : xy = 0\}$ (two intersecting lines).
(v) $\{(x, y) \in \mathbb{A}^2 : xy - 1 = 0\}$ (hyperbola).

Answers.

(i) $k[X_1, \dots, X_n]$.
(ii) k. A regular function on a point is just a single value.
(iii) $k \times k$. A regular function on two points is determined by two scalars, namely its value on each of the two points. For any pair of values $(a, b) \in k \times k$, one can easily write down a polynomial $f \in k[X]$ such that $f(1) = a$ and $f(0) = b$. Alternatively, one can check algebraically that the map

$$(a, b) \mapsto (a-1)X + b \bmod (X(X-1))$$

is a k-algebra isomorphism $k \times k \to k[X_1, \dots, X_n]/(X(X-1))$.
(iv) $\{(f, g) \in k[X] \times k[Y] : f(0) = g(0)\}$.
To prove this, note that

$$k[X, Y]/(XY) \cong \Big\{a_0 + \sum_{r=1}^{m} b_r X^r + \sum_{s=1}^{n} c_s Y^s : a_0, b_1, \dots, b_m, c_1, \dots, c_n \in k, m, n \in \mathbb{N}\Big\}.$$

We can compare these two descriptions by observing that

$$k[X] = \Big\{a_0 + \sum_{r=1}^{m} b_r X^r\Big\}, \quad k[Y] = \Big\{a_0 + \sum_{s=1}^{n} c_s Y^s\Big\},$$

and the condition that $f(0) = g(0)$ is equivalent to insisting that these two polynomials have the same constant coefficient a_0.

(v) The quotient ring $k[X, Y]/(XY - 1)$.

To describe this more explicitly, note that any term $a_{i,j} X^r Y^s$ of a two-variable polynomial is congruent (mod $XY - 1$) to either $a_{r,s} X^{r-s}$ (if $r \geq s$) or $a_{r,s} Y^{s-r}$ (if $s > r$). Thus every coset in $k[X, Y]/(XY - 1)$ has a representative of the form

$$\sum_{i=0}^{m} a_i X^i + \sum_{j=1}^{n} a_j Y^j.$$

The polynomials of this form determine different functions on V, so we have written down exactly one representative of each coset.

Since $XY = 1$ in $k[V]$, we may relabel Y as X^{-1}; then the multiplication rule will be what the notation leads us to expect. So we can write

$$k[V] = k[X, X^{-1}] = \Big\{ \sum_{j=-n}^{m} a_j X^m : a_{-n}, \dots, a_m \in k, m, n \in \mathbb{N} \Big\}.$$

Example (iii) generalises: if V is a disconnected affine algebraic set, we can write V as a union $V_1 \cup V_2$ of disjoint Zariski closed subsets, and then

$$k[V] = k[V_1] \times k[V_2].$$

On the other hand, if V is reducible but connected, so that the sets V_1 and V_2 are not disjoint, then $k[V]$ is a proper subset of $k[V_1] \times k[V_2]$ (see example (iv)).

Example (iv) does not generalise to arbitrary reducible algebraic sets: we may have $V = V_1 \cup V_2$ where V_1 and V_2 are closed subsets, but

$$k[V] \neq \{(f, g) \in k[V_1] \times k[V_2] : f_{|V_1 \cap V_2} = g_{|V_1 \cap V_2}\}.$$

There will be an example of this on problem sheet 2.

Lemma 6.1. An affine algebraic set V is irreducible if and only if $k[V]$ is an integral domain.

Proof. V is irreducible if and only if $\mathbb{I}(V)$ is a prime ideal in $k[X_1, \dots, X_n]$. □

Regular maps. A regular function goes from an algebraic set V to the field k. We can also define regular maps, which go from one algebraic set V to another algebraic set W.

Definition. Let $V \subseteq \mathbb{A}^m$ and $W \subseteq \mathbb{A}^n$ be affine algebraic sets. A **regular map** $\varphi \colon V \to W$ is a function $V \to W$ such that there exist polynomials $F_1, \dots, F_n \in k[X_1, \dots, X_m]$ such that

$$\varphi(\underline{x}) = (F_1(\underline{x}), \dots, F_n(\underline{x}))$$

for all $\underline{x} \in V$.

Regular maps are often called **morphisms**.

In order to check that a given list of polynomials $F_1, \dots, F_n$ defines a regular map $V \to W$, it is necessary to check that $(F_1(\underline{x}), \dots, F_n(\underline{x})) \in W$ for every $\underline{x} \in V$. Equivalently, we need to check that the regular functions $F_{1|V}, \dots, F_{n|V} \in k[V]$ satisfy the equations

$$g(F_{1|V}, \dots, F_{n|V}) = 0$$

in the coordinate ring $k[V]$, for each polynomial $g \in \mathbb{I}(W)$.

Examples.

(1) Let $V \subseteq \mathbb{A}^m$ be an affine algebraic set. For any $n < m$, the projection $\pi \colon V \to \mathbb{A}^n$ defined by

$$\pi(x_1, \dots, x_m) = (x_1, \dots, x_n)$$

is a regular map.

(2) A regular function on V is the same thing as a regular map $V \to \mathbb{A}^1$.

(3) Consider SL_n, the set of $n \times n$ matrices with determinant 1. This is an affine algebraic set in $\mathbb{A}^{n^2}$ because the determinant is a polynomial in the entries of a matrix. The map $a \mapsto a^{-1}$ is a regular map $\mathrm{SL}_n \to \mathrm{SL}_n$: Cramer's rule tells us how to write each entry of a^{-1} as a polynomial in the entries of a divided by $\det a$, and because we are only considering $a \in \mathrm{SL}_n$ we can drop the division by $\det a$.

Regular maps and Zariski topology. A regular map $\varphi \colon V \to W$ is a continuous function with respect to the Zariski topology. This is because, if $A \subseteq W$ is a Zariski closed subset defined by polynomials $f_1, \dots, f_r$, then $\varphi^{-1}(A)$ is the zero set of the polynomials $f_1 \circ \varphi, \dots, f_r \circ \varphi$ and therefore $\varphi^{-1}(A)$ is Zariski closed. In complex analysis, "holomorphic" is a much stricter condition than "continuous in the Euclidean topology," and similarly "regular" is much stricter than "continuous in the Zariski topology."

The following fact is very useful:

Lemma 6.2. Let $\varphi, \psi \colon V \to W$ be regular maps. If there exists a Zariski dense subset $A \subseteq V$ such that $\varphi_{|A} = \psi_{|A}$, then $\varphi = \psi$ on all of A.

Note that, if X and Y are Hausdorff topological spaces, then any continuous maps $X \to Y$ which agree on a dense set must agree everywhere. However the lemma does not follow immediately from the fact that regular maps are continuous, because the Zariski topology is not Hausdorff! (And the lemma is definitely false if we try to generalise it to all continuous maps with respect to the Zariski topology.) Thus in order to prove the lemma, we have to use something special about regular maps as opposed to general continuous maps.

Proof. Write $\varphi = (F_1, \dots, F_m), \psi = (G_1, \dots, G_m)$, where $F_1, \dots, F_m, G_1, \dots, G_m$ are polynomials. Then $F_i - G_i$ is also a polynomial for each i, and so

$$V_{\mathrm{eq}} = \{\underline{x} \in V : \varphi(\underline{x}) = \psi(\underline{x})\} = \{\underline{x} \in V : (F_i - G_i)(\underline{x}) = 0 \text{ for all } i\}$$

is a Zariski closed subset of V. But we know that V_{eq} contains A, which is Zariski dense in V. Hence $V_{\mathrm{eq}} = V$. □

7. Regular maps and algebra; rational functions

Isomorphisms.

Definition. A regular map $\varphi\colon V \to W$ is an **isomorphism** if there exists a regular map $\psi\colon W \to V$ such that $\psi \circ \varphi = \mathrm{id}_V$ and $\varphi \circ \psi = \mathrm{id}_W$.

Example. If V is the parabola $\{(x, y) : y - x^2 = 0\}$, then the regular map $\varphi\colon V \to \mathbb{A}^1$ given by

$$\varphi(x, y) = x$$

is an isomorphism because it has an inverse $\psi\colon \mathbb{A}^1 \to V$ given by

$$\psi(x) = (x, x^2).$$

Example. On the other hand, if H is the hyperbola $\{(x, y) : xy = 1\}$, then the projection $(x, y) \mapsto x$ is not an isomorphism $H \to \mathbb{A}^1$ because it is not surjective so it cannot possibly have an inverse. This is not enough to prove that H is not isomorphic to $\mathbb{A}^1$, because maybe there is some other regular map $H \to \mathbb{A}^1$ which is an isomorphism. (We will soon prove that H is not isomorphic to $\mathbb{A}^1$.)

Example. Consider the affine algebraic set $W = \{(x, y) : y^2 - x^3 = 0\}$. The regular map $\varphi\colon \mathbb{A}^1 \to W$ given by

$$\varphi(t) = (t^2, t^3)$$

is a bijection but it is not an isomorphism. Note that we should expect W not to be isomorphic to $\mathbb{A}^1$ because it has a singularity at the origin.

To prove that $\varphi\colon \mathbb{A}^1 \to W$ is not an isomorphism: Consider a regular map $\psi\colon W \to \mathbb{A}^1$. It must be given by a polynomial $g(X, Y) \in k[X, Y]$ and so

$$\psi \circ \varphi(t) = \psi(t^2, t^3)$$

is a polynomial in t which can have a constant term and terms of degree 2 or greater, but no term of degree 1. Hence we cannot find ψ such that $\psi \circ \varphi(t) = t$.

Regular maps and the coordinate ring.

Suppose we have a regular map $\varphi\colon V \to W$ between affine algebraic sets. For each regular function g on W, we get a regular function $\varphi^* g$ on V defined by

$$(\varphi^* g)(x) = g(\varphi(x)).$$

We call $\varphi^* g \in k[V]$ the **pull-back** of $g \in k[W]$.

Thus φ induces a k-algebra homomorphism

$$\varphi^*\colon k[W] \to k[V].$$

Note that φ^* goes in the opposite direction to φ.

If we have two regular maps $\varphi\colon V \to W$ and $\psi\colon W \to Z$, then we can compose them to get $\psi \circ \varphi\colon V \to Z$. One can easily check that the associated pullback maps on coordinate rings satisfy

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*\colon k[Z] \to k[V]. \qquad (*)$$

For those who know category theory, we say that $V \mapsto k[V]$ is a contravariant functor

$$\{\text{affine algebraic sets}\} \to \{k\text{-algebras}\}.$$

In particular, (*) tells us that if $\varphi\colon V \to W$ is an isomorphism with inverse $\psi\colon W \to V$, then $\psi^* \circ \varphi^* = \mathrm{id}$ and $\varphi^* \circ \psi^* = \mathrm{id}$. Thus if V and W are isomorphic affine algebraic sets, then their coordinate rings $k[V]$ and $k[W]$ are isomorphic as k-algebras.

Example. Now we can prove that the hyperbola H is not isomorphic to $\mathbb{A}^1$: we know that $k[H] = k[X, X^{-1}]$, and this is not isomorphic to $k[\mathbb{A}^1] = k[X]$ because in $k[X]$ the only invertible elements are the scalars, while $k[X, X^{-1}]$ contains non-scalar invertible elements, such as X.

Rational functions.

Informally, rational functions are "functions" on varieties defined by polynomial fractions, for example the "function" $x \mapsto 1/x$ on $\mathbb{A}^1$. Observe that this is not really a function $\mathbb{A}^1 \to \mathbb{A}^1$ because it is not defined at $x = 0$, but it is a genuine function on the Zariski open subset $\mathbb{A}^1 \setminus \{0\}$. (These are analogues of meromorphic functions in complex analysis.)

Let V be an irreducible affine algebraic set.

Definition. The **function field** of V is the field of fractions of the coordinate ring $k[V]$. We denote this by $k(V)$.

(We make this definition only for irreducible affine algebraic sets because they have integral domains as their coordinate rings, and it is only integral domains which have field of fractions.)

For example, the function field of $\mathbb{A}^1$ is $k(X)$, the fraction field of the polynomial ring $k[X]$.

Definition. A **rational function** on V is an element of the function field $k(V)$. Thus a rational function can be written in the form f/g, where f and g are regular functions. There may be many different choices for f and g which define the same rational function f/g.

Definition. We say that a rational function $\varphi \in k(V)$ is **regular** at a point $x \in V$ if there exist regular functions $f, g \in k[V]$ such that $\varphi = f/g$ and $g(x) \neq 0$.

Thus regular points are precisely the points at which we can assign a value to $\varphi(x)$: if $g(x) \neq 0$, then we can define $\varphi(x) = f(x)/g(x)$.

We are allowed to choose different fractions f/g representing φ at different points $x \in V$, in order to show that those points are regular. For example, consider the algebraic set defined by the equation $XY = ZT$ in $\mathbb{A}^4$. Let

$$\varphi = X/Z \in k(V).$$

The defining equation implies that we also have

$$\varphi = T/Y.$$

Looking at the fraction X/Z shows us that φ is regular wherever $Z \neq 0$, and looking at the fraction T/Y shows us that φ is regular wherever $Y \neq 0$. On the other hand, φ is not regular on the closed subset $Y = Z = 0$. (One can verify that there is no other fraction representing φ which is non-zero on this closed subset.)

8. Domain of definition and rational maps

Domain of definition of a rational function.

Let V be an irreducible affine algebraic set. Let $\varphi \in k(V)$ be a rational function.

Definition. The set of points where φ is regular is called the **domain of definition** of φ, and denoted $\operatorname{dom} \varphi$.

This is the set of points where it makes sense to assign a value to $\varphi(x)$. For $x \in \operatorname{dom} \varphi$, the value $\varphi(x)$ is independent of which fraction f/g we choose to represent φ (as long as $g(x) \neq 0$).

Lemma 8.1. The domain of definition of a rational function $\varphi \in k(V)$ is a non-empty Zariski open subset of V.

Proof. Consider the set of all possible fractions f/g with $f, g \in k[V]$ representing $\varphi \in k(V)$. The set of points at which φ is *not* regular is the intersection of the Zariski closed sets $\{x \in V : g(x) = 0\}$ across all these fractions. Hence the set of points at which φ is not regular is a Zariski closed subset of V. The domain of definition is the complement of this set, and therefore is Zariski open.

To show that the domain of definition is non-empty, pick a single fraction f/g representing $\varphi \in k(V)$. The regular function g is not equal to zero as an element of $k[V]$ (by the definition of the field of fractions), so $\{x \in V : g(x) = 0\}$ is a proper closed subset of V. The domain of definition contains the complement of this set, namely $\{x \in V : g(x) \neq 0\}$, and hence is non-empty. □

Note that every regular function $f \in k[V]$ is also a rational function $f/1 \in k(V)$, and its domain of definition is all of V. The converse also holds:

Lemma 8.2. Let $\varphi \in k(V)$ be a rational function whose domain of definition is equal to V. Then φ is a regular function on V.

Proof. Since $\operatorname{dom} \varphi = V$, for each point $x \in V$, we can choose regular functions $f_x, g_x \in k[V]$ such that $\varphi = f_x/g_x$ and $g_x(x) \neq 0$. Let $I \subseteq k[V]$ denote the ideal generated by the functions g_x. Because $k[V]$ is noetherian, we can pick finitely many of these functions $g_{x_1}, \dots, g_{x_m}$ which still generate I.

For each $x \in V$, there is some $g_x \in I$ which is non-zero at x. Hence the Zariski closed set

$$\{x \in V : h(x) = 0 \text{ for all } h \in I\}$$

is empty. Then the Nullstellensatz implies that I is all of $k[V]$ (there are a lot more steps involved in applying the Nullstellensatz here than I realised – we will see how to do this in a couple of lectures time).

In particular, $1 \in I$. Since $I = (g_{x_1}, \dots, g_{x_m})$, there exist $u_1, \dots, u_m \in k[V]$ such that

$$1 = u_1 g_{x_1} + \dots + u_m g_{x_m} \text{ in } k[V].$$

We can now calculate

$$\begin{aligned}\varphi = 1.\varphi &= (u_1 g_{x_1} + \cdots + u_m g_{x_m})\varphi \\ &= u_1 g_{x_1}\frac{f_{x_1}}{g_{x_1}} + \cdots u_m g_{x_m}\frac{f_{x_m}}{g_{x_m}} \\ &= u_1 f_{x_1} + \cdots + u_m f_{x_m}.\end{aligned}$$

Since $u_i, f_{x_i} \in k[V]$, so is φ. □

Rational maps.

Let $V \subseteq \mathbb{A}^m$, $W \subseteq \mathbb{A}^n$ be irreducible affine algebraic sets.

Definition. A **rational map** $\varphi\colon V \dashrightarrow W$ is an n-tuple of rational functions $\varphi_1, \dots, \varphi_n \in k(V)$ such that, for every point $x \in V$ where $\varphi_1, \dots, \varphi_n$ are all regular, the point $(\varphi_1(x), \dots, \varphi_n(x))$ is in W.

We use the broken arrow symbol $\dashrightarrow$ instead of the usual arrow because a rational map is not a function on V in the usual set-theoretic sense. It only defines a genuine function $U \to W$, where U is the domain of definition of φ. This is defined as follows.

Definition. The **domain of definition** of a rational map $\varphi\colon V \dashrightarrow W$ is the intersection of the domains of definition of the component rational functions $(\varphi_1, \dots, \varphi_n)$.

The two lemmas we proved for rational functions also hold for rational maps: the domain of definition of a rational map $\varphi\colon V \dashrightarrow W$ is a non-empty Zariski open subset of V, and if a rational map is regular everywhere then it is a regular map. In order to prove that the domain of definition of a rational map is non-empty, we have to use the fact that V is irreducible (and therefore every open subset of V is dense).

Example. An important example of a rational map is projection from a point onto a hyperplane.

Let H be a hyperplane in $\mathbb{A}^n$ (that is, a set defined by a single *linear* equation). Let p be a point in $\mathbb{A}^n \setminus H$. For simplicity, we shall assume that p is the origin and that

$$H = \{(x_1, \dots, x_n) \in \mathbb{A}^n : x_n = 1\}.$$

(We could always reduce to this case by a suitable change of coordinates.)

Let us write H_0 for the hyperplane through p parallel to H, that is,

$$H_p = \{(x_1, \dots, x_n) \in \mathbb{A}^n : x_n = 0\}.$$

For each point $x \in \mathbb{A}^n \setminus H_p$, let L_x denote the line which passes through p and x. Since $x \notin H_p$, L_x intersects H in exactly one point. Call this point $\varphi(x)$.

We can write this algebraically as

$$\varphi(x_1, \dots, x_n) = (x_1/x_n, \dots, x_{n-1}/x_n, 1)$$

and so φ is a rational map $\mathbb{A}^n \dashrightarrow H$. This map is called **projection from** p **onto** H.

We have $\operatorname{dom}\varphi = \mathbb{A}^n \setminus H_p$. (Note that we have not proved this, because we have not proved that there is no other list of fractions which define the same rational map but have non-zero denominators at points in H_p. One can prove this.)

For any affine algebraic set $V \subseteq \mathbb{A}^n$ such that $V \not\subseteq H_0$, we can restrict π to get a rational map $V \dashrightarrow H$. (Note that p might be in V, or it might not.)

Example. Let V be the circle $\{(x, y) : x^2 + y^2 = 1\}$. Consider the projection from the point $p = (1, 0)$ on to the line $x = 0$. This is a rational map $\pi\colon V \dashrightarrow \mathbb{A}^1$ with the formula

$$\pi(x, y) = y/(1 - x).$$

We can see geometrically that this projection induces a bijection between the circle (excluding p) and the line (at least for real points). If we compute the formula for the inverse map, we get

$$\psi(t) = \Big(\frac{t^2 - 1}{t^2 + 1}, \frac{2t}{t^2 + 1}\Big),$$

a well-known parameterisation of the circle. Thus we see that the inverse is a rational map $\psi\colon \mathbb{A}^1 \dashrightarrow V$. Note that ψ is not regular at $t = \pm i$ – we don't see this on the picture, which only shows the real points.

Next time we will define formally what it means to say that the rational maps π and ψ are inverse to each other, taking into account that they are not true functions between the sets V and $\mathbb{A}^1$ because they are not regular everywhere.

9. Composing rational maps; from algebra to geometry

Composing rational maps.

Last time we defined rational maps $\pi\colon V \dashrightarrow \mathbb{A}^1$ and $\psi\colon \mathbb{A}^1 \dashrightarrow V$ where V is the circle. These maps are inverses in that composing them (either way round) gives the identity, if we ignore the points where the maps are not regular.

In order to rigorously define composition of rational maps, we need to notice that sometimes the set of points where a composite map is undefined is "everywhere" and exclude that situation. For example, consider the rational map $\mathbb{A}^2 \dashrightarrow \mathbb{A}^1$ defined by

$$\xi(x,y) = \frac{1}{1-x^2-y^2}.$$

This map is not regular anywhere on the circle V, and hence it does not make sense to try to define the composite map $\xi \circ \psi\colon \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$ (it is not defined anywhere!).

This problem can occur because the image of ψ is not dense in $\mathbb{A}^2$. So to rule it out, we make the following definition.

Definition. The **image** of a rational map $\varphi\colon V \dashrightarrow W$ is the set of points

$$\{\varphi(x) \in W : x \in \operatorname{dom}\varphi\}.$$

A rational map is **dominant** if its image is Zariski dense in W.

For example, ψ from the end of the previous lecture is dominant if we consider it as a rational map $\mathbb{A}^1 \dashrightarrow V$ but it is not dominant if we consider it as a rational map $\mathbb{A}^1 \dashrightarrow \mathbb{A}^2$. (This is like surjectivity: whether a function is surjective or not depends on what codomain you declare it to have.)

Let V, W, T be irreducible affine algebraic sets. If $\varphi\colon V \dashrightarrow W$ is a *dominant* rational map and $\psi\colon W \dashrightarrow T$ is a rational map (ψ is not required to be dominant), then it makes sense to compose them because we know that $\operatorname{dom}\psi$ is a Zariski open subset of W, while $\operatorname{im}\varphi$ is a Zariski dense subset of W and so

$$\operatorname{dom}\psi \cap \operatorname{im}\varphi \neq \emptyset.$$

Thus there are at least some points where $\psi \circ \varphi$ is defined. One can check (by writing out ψ in terms of fractions of polynomials, then substituting in fractions of polynomials representing φ) that $\psi \circ \varphi$ is a rational map $V \dashrightarrow T$.

Definition. Rational maps $\varphi\colon V \dashrightarrow W$ and $\psi\colon W \dashrightarrow V$ are **rational inverses** if both are dominant and $\varphi\circ\psi = \mathrm{id}_W$ and $\psi\circ\varphi = \mathrm{id}_V$ (everywhere these composite rational maps are well-defined).

A rational map $\varphi\colon V \dashrightarrow W$ is a **birational equivalence** if it is dominant and has a rational inverse.

We say that irreducible algebraic sets V and W are **birational** (or **birationally equivalent**) if there exists a birational equivalence $V \dashrightarrow W$.

Our example from the previous lecture showed that the circle is birational to $\mathbb{A}^1$.

Another example is the cuspidal cubic

$$W = \{(x,y) : y^2 = x^3\}.$$

This is also birational to $\mathbb{A}^1$, as shown by the rational maps

$$W \dashrightarrow \mathbb{A}^1 : (x, y) \mapsto y/x,$$
$$\mathbb{A}^1 \dashrightarrow W : t \mapsto (t^2, t^3).$$

Birationally equivalent affine algebraic sets look the same "almost everywhere." For example, the cuspidal cubic is the same as the affine line everywhere *except* at the origin.

On the other hand, $\mathbb{A}^1$ is not birationally equivalent to $\mathbb{A}^2$ or to an elliptic curve

$$\{(x, y) : y^2 = f(x)\} \text{ where } f \text{ is a cubic polynomial with no repeated roots.}$$

We will prove this later in the course once we have more tools.

If $\varphi\colon V \dashrightarrow W$ is a dominant rational map, then we can use it to pull back rational functions from W to V (just like we earlier used regular maps to pull back regular functions). We get a k-homomorphism of fields

$$\varphi^*\colon k(W) \to k(V)$$

defined by $\varphi^*(g) = g \circ \varphi$. (A k-homomorphism means that φ^* restricts to the identity on the copies of k which are contained in $k(W)$ and $k(V)$, namely the constant functions.)

If φ is a birational equivalence, then φ^* is a k-isomorphism of fields.

From algebra homomorphisms to regular maps.

We have seen that each regular map $f\colon V \to W$ induces a k-algebra homomorphism $f^*\colon k[W] \to k[V]$, and that each dominant rational map $\varphi\colon V \dashrightarrow W$ induces a k-field homomorphism $\varphi^*\colon k(W) \to k(V)$. We can also carry out these constructions in the reverse direction: starting with a k-algebra homomorphism and getting a regular map, or similarly for rational maps.

Observe that if $f\colon V \to W$ is a regular map and $W \subseteq \mathbb{A}^n$, we can recover f from $f^*\colon k[W] \to k[V]$ by taking the coordinate functions $X_1, \dots, X_n \in k[W]$ on W and pulling them back to get

$$f_1 = f^*(X_1),\ \dots,\ f_n = f^*(X_n) \in k[V].$$

These are precisely the regular functions on V such that $f = (f_1, \dots, f_n)$.

This procedure works for any k-algebra homomorphism $\alpha\colon k[W] \to k[V]$: define a regular map $s\colon V \to W$ by

$$s = (\alpha(X_1), \dots, \alpha(X_n)).$$

(Here $\alpha(X_1), \dots, \alpha(X_n) \in k[V]$.) Then $\alpha = s^*$.

Thus every k-algebra homomorphism $k[W] \to k[V]$ is the pull back by some regular map $V \to W$. We conclude:

Proposition 9.1. $\varphi \mapsto \varphi^*$ is a bijection

$$\{\text{regular maps } V \to W\} \longrightarrow \{k\text{-algebra homomorphisms } k[W] \to k[V]\}.$$

Corollary 9.2. Affine algebraic sets V and W are isomorphic if and only if their coordinate rings $k[V]$ and $k[W]$ are isomorphic as k-algebras.

The moral is: if we only care about affine algebraic sets up to isomorphism, then coordinate rings contain exactly the same information as algebraic sets themselves (in the language of category theory, the functor $V \mapsto k[V]$ is fully faithful).

One can do the same thing for rational maps:

Proposition 9.3. $\varphi \mapsto \varphi^*$ is a bijection

$$\{\text{dominant rational maps } V \dashrightarrow W\} \longrightarrow \{k\text{-field homomorphisms } k(W) \to k(V)\}.$$

Corollary 9.4. Irreducible affine algebraic sets V and W are birationally equivalent if and only if their function fields $k(V)$ and $k(W)$ are k-isomorphic.

Dictionary between algebraic subsets and ideals.

Can we do something similar with Zariski closed subsets of V, and work them out from the algebra of $k[V]$?

Suppose that $V \subseteq \mathbb{A}^n$.

In $\mathbb{A}^n$: the Nullstellensatz tells us that the functions $\mathbb{I}$ and $\mathbb{V}$ are bijections

$$\{\text{Zariski closed subsets of } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } k[X_1, \dots, X_n]\}.$$

Since $\mathbb{I}$ and $\mathbb{V}$ reverse the direction of inclusions, we deduce that they restrict to bijections

$$\{\text{Zariski closed subsets of } V\} \longleftrightarrow \{\text{radical ideals in } k[X_1, \dots, X_n] \text{ containing } \mathbb{I}(V)\}.$$

We know that

$$k[V] \cong k[X_1, \dots, X_n]/\mathbb{I}(V).$$

It is a basic algebraic fact that

$$\{\text{ideals in } k[X_1, \dots, X_n] \text{ containing } \mathbb{I}(V)\} \longleftrightarrow \{\text{ideals in } k[X_1, \dots, X_n]/\mathbb{I}(V)\}.$$

Under this correspondence, radical ideals on one side correspond to radical ideals on the other side and similarly for prime ideals.

We conclude that the natural maps are bijections

$$\{\text{Zariski closed subsets of } V\} \longleftrightarrow \{\text{radical ideals in } k[V]\}$$

and

$$\{\text{irreducible Zariski closed subsets of } V\} \longleftrightarrow \{\text{prime ideals in } k[V]\}.$$

Questions. Can we see the points of V through the algebra of $k[V]$?

Which k-algebras can occur as $k[V]$ where V is an affine algebraic set?

10. Equivalence of algebra and geometry; Nullstellensatz

Points and maximal ideals.

Can we describe the points of an affine algebraic set V in terms of the algebra of $k[V]$? The points of V are the smallest non-empty Zariski closed subsets. Since the bijection between Zariski closed subsets and ideals reverses direction of inclusion, they correspond to maximal ideals:

$$\{\text{points of } V\} \longleftrightarrow \{\text{maximal ideals in } k[V]\}.$$

Reduced finitely generated k-algebras.

To fully understand the relationship between affine algebraic sets and k-algebras, there is one more question to answer: Which k-algebras can occur as $k[V]$ where V is an affine algebraic set?

We write down some algebraic properties which obviously hold for $A = k[V]$:

(1) A is finitely generated, because if $V \subseteq \mathbb{A}^n$ then A is generated by the coordinate functions $X_1, \dots, X_n$.

(2) A is reduced (meaning that if $f \in A$ and $f^k = 0$ for some $k > 0$, then $f = 0$). This is because A is a ring of functions in the usual set-theoretic sense: if $f^k = 0$ then $f(x)^k = 0$ for all $x \in V$, so $f(x) = 0$ for all $x \in V$.

Using the Nullstellensatz, we can prove that these properties are enough to characterise the k-algebras which are coordinate rings of affine algebraic sets.

Proposition 10.1. Let A be a finitely generated reduced k-algebra. Then there exists an affine algebraic set V such that $k[V] \cong A$.

Proof. Pick a finite set $f_1, \dots, f_n \in A$ which generates A as a k-algebra. We can define a homomorphism $\alpha\colon k[X_1, \dots, X_n] \to A$ by $X_1 \mapsto f_1, \dots, X_n \mapsto f_n$.

Let $I = \ker \varphi$ and let $V = \mathbb{V}(I) \subseteq \mathbb{A}^n$.

The homomorphism α is surjective because $f_1, \dots, f_n$ generate A, and so

$$A \cong k[X_1, \dots, X_n]/I.$$

Thus $k[X_1, \dots, X_n]/I$ is a reduced k-algebra. It follows that I is a radical ideal.

Hence the Nullstellensatz tells us that $I = \mathbb{I}(V)$. Thus

$$k[V] \cong k[X_1, \dots, X_n]/\mathbb{I}(V) \cong k[X_1, \dots, X_n]/I \cong A. \qquad \square$$

The notion of affine variety.

Often in mathematics, it is convenient to consider objects only "up to isomorphism." For example, one might talk about "the group with 7 elements," ignoring the fact that there are many different groups with 7 elements because they are all isomorphic to each other (and therefore they all behave in the same ways).

Similarly, in algebraic geometry we often want to consider affine algebraic sets up to isomorphism. But affine algebraic sets are always defined in a concrete way: they are a subset of some specific affine space $\mathbb{A}^n$. (It is as if we had defined all finite groups to be subgroups of a symmetric group S_n.) And we have seen that affine algebraic sets can be isomorphic even when they appear to be quite different as subsets of affine space, for example the line $\mathbb{A}^1$ is isomorphic to the parabola $\mathbb{V}(Y - X^2) \subseteq \mathbb{A}^2$. Thus it is useful to use different terminology: we talk about "affine algebraic sets" when we mean subsets of $\mathbb{A}^n$, and we talk about "affine varieties" when we mean an affine algebraic set up to isomorphism, forgetting its embedding into $\mathbb{A}^n$.

Proposition 10.1 is more naturally stated in terms of affine varieties rather than affine algebraic sets: in the proof we had to choose a generating set for A, for which there is no distinguished choice. Different choices of generating set would lead to isomorphic affine algebraic sets, but embedded differently into affine space. So it is better to say that each finitely generated reduced k-algebra A is the coordinate ring of some affine variety V, with no distinguished choice of embedding into $\mathbb{A}^n$. (I mentioned this philosophy about affine varieties before in lecture 2, and I will mention it again after we have defined quasi-projective varieties.)

For those who know some fancy categorical language, we can sum up all the results on the equivalence between affine geometric objects and their coordinate rings by saying that $V \mapsto k[V]$ is an *equivalence of categories*

$$\{\text{affine varieties over } k\} \longrightarrow \{\text{reduced finitely generated } k\text{-algebras}\}^{\text{op}}$$

where the superscript "op" indicates that the directions of morphisms are reversed.

The Weak and Strong Nullstellensatz.

Now we aim to prove Hilbert's Nullstellensatz. There are many different proofs, all of which require some difficult algebra. We will roughly follow the method in Shafarevich (Appendix A), which incorporates the hard algebra into one statement which we can quote, and then do the rest as geometrically as possible.

Recall the statement of Hilbert's Nullstellensatz, also called the Strong Nullstellensatz.

Theorem 10.2 (Strong Nullstellensatz)**.** Let I be any ideal in the polynomial ring $k[X_1, \dots, X_n]$ over an algebraically closed field k. We have

$$\mathbb{I}(\mathbb{V}(I)) = \operatorname{rad} I.$$

In order to prove this, we will first prove a weaker version, which is called the Weak Nullstellensatz, then use that to deduce the Strong Nullstellensatz.

Theorem 10.3 (Weak Nullstellensatz)**.** Let I be an ideal in the polynomial ring $k[X_1, \dots, X_n]$ over an algebraically closed field k. If $\mathbb{V}(I) = \emptyset$, then $I = k[X_1, \dots, X_n]$.

This is a statement about the existence of solutions to polynomial equations, so it is necessary to require k to be algebraically closed. As an example to show that it fails when k is not algebraically closed, consider the ideal $(X^2 + Y^2 + 1)$ in $\mathbb{R}[X, Y]$. This ideal is not the full polynomial ring, but there are no real solutions to the equation $x^2 + y^2 + 1 = 0$.

Note that the Strong Nullstellensatz easily implies the Weak Nullstellensatz: if $\operatorname{rad} = k[X_1, \dots, X_n]$ then $1 \in \operatorname{rad} I$ so $1 \in I$ so $I = k[X_1, \dots, X_n]$.

Proof that Weak Nullstellensatz implies Strong Nullstellensatz. We use a method called the Rabinowitsch trick, introducing an extra variable.

Let I be an ideal in $k[X_1, \dots, X_n]$ and let $V = \mathbb{V}(I) \subseteq \mathbb{A}^n$.

It is easy to see that $\operatorname{rad} I \subseteq \mathbb{I}(V)$. Thus we have to prove that $\mathbb{I}(V) \subseteq \operatorname{rad} I$.

Let $f \in \mathbb{I}(V)$. Define a new polynomial g with an extra variable Y by:

$$g(X_1, \dots, X_n, Y) = f(X_1, \dots, X_n) \cdot Y - 1.$$

Let J be the ideal in $k[X_1, \dots, X_n, Y]$ generated by I and g, and consider the affine algebraic set $W = \mathbb{V}(J) \subseteq \mathbb{A}^{n+1}$.

Every point of $(x_1, \dots, x_n, y) \in W$ satisfies $f(x_1, \dots, x_n) \neq 0$ (necessary so that $f(x_1, \dots, x_n)y$ can equal 1). Since $I \subseteq J$, points of W also satisfy $(x_1, \dots, x_n) \in V$. Therefore, if $\pi \colon \mathbb{A}^{n+1} \to \mathbb{A}^n$ is the projection map (forgetting the Y coordinate), then

$$\pi(W) \subseteq \{(x_1, \dots, x_n) \in V : f(x_1, \dots, x_n) \neq 0\}.$$

Since $f \in \mathbb{I}(V)$, this set is empty.

This implies that W itself is empty. Therefore, by the Weak Nullstellensatz,

$$J = k[X_1, \dots, X_n, Y].$$

In particular, $1 \in J$ and thus

$$1 = a + bg \quad \text{for some } a \in I.k[X_1, \dots, X_n, Y], \; b \in k[X_1, \dots, X_n, Y].$$

Expand out a and b as sums over powers of Y:

$$a = \sum_{j \geq 0} a_j Y^j \text{ where } a_j \in I,$$
$$b = \sum_{j \geq 0} b_j Y^j \text{ where } b_j \in k[X_1, \dots, X_n].$$

Expanding the equation $1 = a + bg$ and picking out the terms of degree j in Y, we get:

$$\begin{aligned} 1 &= a_0 - b_0 && \text{for } j = 0, \\ 0 &= a_j + b_{j-1} f - b_j && \text{for } j \geq 1. \end{aligned}$$

By induction on j, these imply that

$$b_j \in I - f^j \text{ for all } j \geq 0$$

(where $I - f^j$ means the coset $\{t - f^j : t \in I\}$.

But b is a polynomial, so $b_j = 0$ once j gets large enough. Thus for some j, we get $0 \in I - f^j$, that is, $f^j \in I$. This proves that $f \in \operatorname{rad} I$. □

11. Proof of the Weak Nullstellensatz

We can restate the Weak Nullstellensatz in elementary terms as: if $f_1, \dots, f_m \in k[X_1, \dots, X_n]$ are a finite set of polynomials, and the ideal I which they generate is not the whole polynomial ring, then there exists a common solution $(x_1, \dots, x_n) \in k^n$ to the equations

$$f_1(x_1, \dots, x_n) = 0, \ \dots, \ f_m(x_1, \dots, x_n) = 0.$$

We prove this in two steps:

(1) there exists some larger field K containing k such that these equations have a common solution in K^n.
(2) if the equations have a common solution in K^n, then they also have a common solution in k^n.

Finding a solution in a bigger field.

The proof of step (1) is fairly short, and relies on Zorn's lemma.

Lemma 11.1. Let $f_1, \dots, f_m$ be polynomials in $k[X_1, \dots, X_n]$, such that the ideal $I = (f_1, \dots, f_m)$ is not equal to $k[X_1, \dots, X_n]$.

Then there exists a field K which is a finitely generated extension of k such that the equations

$$f_1(x_1, \dots, x_n) = 0, \ \dots, \ f_m(x_1, \dots, x_n) = 0$$

have a common solution $(x_1, \dots, x_n) \in K^n$.

Proof. Because $I \neq k[X_1, \dots, X_n]$, we can use Zorn's lemma to show that I is contained in some maximal ideal $M \subseteq k[X_1, \dots, X_n]$. (This is a natural way to start: we are trying to show that $\mathbb{V}(I)$ has a point, and last time we saw that points in $\mathbb{V}(I)$ correspond to maximal ideals in $k[X_1, \dots, X_n]$ containing I. We can't just quote the correspondence from the previous lecture because we used the Nullstellensatz in proving that correspondence, but this justifies why obtaining a maximal ideal is a good first step.)

Let $K = k[X_1, \dots, X_n]/M$. Let $x_1, \dots, x_n$ denote the images of $X_1, \dots, X_n$ in K. K is a field because M is a maximal ideal, and it is finitely generated as an extension of k because it is generated by $x_1, \dots, x_n$.

Since $f_j(X_1, \dots, X_n) \in I \subseteq M$, we get that $f_j(x_1, \dots, x_n) = 0$ in K for each j. Thus $(x_1, \dots, x_n)$ is the required common solution to $f_1, \dots, f_m$ in K^n. □

Shrinking the field required. Before proving step (2), we begin by quoting an algebraic result.

Lemma 11.2. Let k be an algebraically closed field and let K be a finitely generated extension field of k. Then there exist $t_1, \dots, t_d, u \in K$ such that

(i) $K = k(t_1, \dots, t_d, u)$;
(ii) $t_1, \dots, t_d$ are algebraically independent over k (that is, there is no non-zero polynomial in d variables with coefficients in k whose value at $(t_1, \dots, t_d)$ is zero); and

(iii) u is algebraic over $k(t_1, \dots, t_d)$ (that is, there exists a non-zero polynomial in one variable with coefficients in the field $k(t_1, \dots, t_d)$ which is zero at u).

Proof. This follows from the primitive element theorem in field theory. For a full proof, see Proposition A.7 in the Appendix of Shafarevich, *Basic Algebraic Geometry 1.* □

This proposition has a nice geometric interpretation: every finitely generated extension of k is isomorphic to the field of fractions of a hypersurface. We need to use the Nullstellensatz to prove this geometric interpretation, so that is postponed until after we have finished the proof of the Nullstellensatz.

Theorem 11.3. Let k be an algebraically closed field and let K be a finitely generated extension field of k. Let $f_1, \dots, f_m \in k[X_1, \dots, X_n]$.

Suppose there exists a common solution $(x_1, \dots, x_n) \in K^n$ to the equations

$$f_1(x_1, \dots, x_n) = \dots = f_m(x_1, \dots, x_n) = 0.$$

Then there exists a common solution $(y_1, \dots, y_n) \in k^n$ to the equations

$$f_1(y_1, \dots, y_n) = \dots = f_m(y_1, \dots, y_n) = 0.$$

Proof. Write $K = k(t_1, \dots, t_d, u)$ as in Lemma 11.2.

Let $K' = k(t_1, \dots, t_d)$. Because $t_1, \dots, t_d$ are algebraically independent, we can identify K' with $k(T_1, \dots, T_d)$, the field of fractions of the polynomial ring $k[T_1, \dots, T_d]$. This will allow us to substitute a vector $\underline{z} \in k^d$ into an element $\alpha \in K'$ and get out an element $\alpha(\underline{z}) \in k$, as long as the denominator of α does not vanish at $\underline{z}$.

We use two facts about the finite algebraic extension K/K':

(i) There exists a minimal polynomial $p(U) \in K'[U]$ for u; that is, $p(u) = 0$, p has leading coefficient 1, and p divides every other polynomial $q(U) \in K'[U]$ such that $q(u) = 0$.

(ii) Every element of K can be written in the form $a(u)$ for some polynomial $a(U) \in K'[U]$.

Informal outline of proof. The idea of the proof is to consider the "almost hypersurface" $H = \{(z_1, \dots, z_d, s) \in k^{d+1} : p(z_1, \dots, z_d, s) = 0\}$ (the "almost" is because p is not a polynomial in $k[T_1, \dots, T_d, U]$ but rather may have denominators, so we have to ignore the places where these denominators vanish). Then we construct a rational map $\varphi\colon H \dashrightarrow \mathbb{V}(f_1, \dots, f_m)$. The domain of definition of φ is an open subset of an "almost hypersurface", and we can easily check that this is non-empty. Then a point in the image of φ gives us a point in $\mathbb{V}(f_1, \dots, f_m)$, as desired.

Return to formal proof. We apply fact (ii) to $x_1, \dots, x_n \in K$ (our common solution to $f_1 = \dots = f_m = 0$): we can write $x_i = a_i(u)$ where $a_i(U) \in K'[U]$.

(In the informal outline, these $a_i \in k(T_1, \dots, T_d)[U]$ define a rational map $\varphi\colon H \dashrightarrow \mathbb{A}^n$). Next we check that the image of this rational map is contained in $\mathbb{V}(f_1, \dots, f_m)$.)

We know that $(x_1, \ldots, x_n)$ is a common solution to the polynomials $f_1, \ldots, f_m$. Hence

$$f_j(a_1(u), \ldots, a_n(u)) = 0 \text{ in } K \text{ for each } j.$$

In other words, the single-variable polynomial $f_j(a_1(U), \ldots, a_n(U)) \in K'[U]$ has u as a root. Therefore, fact (i) tells us that this polynomial is divisible by $p(U)$. Thus there exist polynomials $q_1, \ldots, q_m \in K'[U]$ such that

$$f_j(a_1(U), \ldots, a_n(U)) = q_j(U)\, p(U) \text{ in } K'[U]. \qquad (*)$$

Now, if $(z_1, \ldots, z_d, s) \in k^{d+1}$ satisfies $p(z_1, \ldots, z_d, s) = 0$, then (*) implies that

$$f_j(a_1(\underline{z}, s), \ldots, a_n(\underline{z}, s)) = 0 \text{ for } j = 1, \ldots, m$$

so long as all the denominators involved are non-zero. Thus we just have to find $(\underline{z}, s)$ where all these denominators will be non-zero.

So consider the polynomials $p(U), a_i(U), q_j(U)$: their coefficients are elements of the field K' which we are identifying with the field of fractions of $k[T_1, \ldots, T_d]$. Let $\sigma \in k[T_1, \ldots, T_d]$ denote the product of the denominators of all these fractions. Because the denominator of a fraction is never zero, σ is not the zero polynomial in. Therefore, there exists $(z_1, \ldots, z_d) \in k^d$ such that

$$\sigma(z_1, \ldots, z_d) \neq 0.$$

Then the denominators of the coefficients of p, a_i, q_j do not vanish at $s_1, \ldots, s_d$, so we can substitute $(s_1, \ldots, s_d)$ into each of these coefficients (as elements of $k(t_1, \ldots, t_d)$) and get out values in k. Thus we get new polynomials

$$\tilde{p}(U), \tilde{a}_i(U), \tilde{q}_j(U) \in k[U].$$

The leading coefficient of $p(U)$ is 1, which is unchanged by this process. So $\tilde{p}(U)$ has the same degree as $p(U)$. In particular $\tilde{p}(U)$ is not a constant polynomial. Hence as k is algebraically closed, there exists $s \in k$ such that $\tilde{p}(s) = 0$.

Let

$$y_i = \tilde{a}_i(s) \in k.$$

Then (*) tells us that

$$f_j(y_1, \ldots, y_n) = \tilde{q}_j(s)\, \tilde{p}(s) \text{ for each } j.$$

But we chose s such that $\tilde{p}(s) = 0$, and so we conclude that $(y_1, \ldots, y_n) \in k^n$ is a common solution to

$$f_1(y_1, \ldots, y_n) = \cdots = f_m(y_1, \ldots, y_n) = 0. \qquad \square$$

Combining Lemma 11.2 and Theorem 11.3 proves the Weak Nullstellensatz.

Hypersurfaces and birational equivalence. Now we prove the geometrical interpretation of Lemma 11.2.

Proposition 11.4. Let K be a finitely generated extension of k. Then there exists an irreducible hypersurface $H \subseteq \mathbb{A}^{d+1}$ for some d such that K is isomorphic to the field of functions $k(H)$.

Corollary 11.5. Let $V \subseteq \mathbb{A}^n$ be an irreducible affine algebraic set. Then there exists an irreducible hypersurface $H \subseteq \mathbb{A}^{d+1}$ for some d such that V is birationally equivalent to H.

Corollary 11.5 tells us that, even if V is a complicated algebraic set defined by many equations, provided we only care about properties of V which are preserved by birational equivalence, we can replace V by a simpler set defined by just one equation, that is, a hypersurface. Note that it is *not* true that every irreducible affine algebraic set is *isomorphic* to a hypersurface.

Proof of Proposition 11.4. Write $K = k(t_1, \dots, t_d, u)$ as in Lemma 11.2, and let $K' = k(t_1, \dots, t_d)$. Let $p(U) \in K'[U]$ be the minimal polynomial of u over K'.

Each coefficient of $p(U)$ is a fraction whose numerator and denominator are polynomials in $t_1, \dots, t_d$. We can multiply up by a suitable element of $k[t_1, \dots, t_d]$ to clear the denominators, and also replace $t_1, \dots, t_d$ by indeterminates $T_1, \dots, T_d$ to get a polynomial $g \in k[T_1, \dots, T_d, U]$ such that

$$g(t_1, \dots, t_d, u) = 0 \text{ in the field } K.$$

Assuming we multiplied up by a lowest common denominator for the coefficients of p, g is irreducible.

Let H be the hypersurface $\mathbb{V}(g) \subseteq \mathbb{A}^{d+1}$. Because g is irreducible, it generates a radical ideal in $k[X_1, \dots, X_n]$ and so the (Strong) Nullstellensatz implies that

$$\mathbb{I}(H) = (g).$$

Thus the coordinate ring is given by

$$k[H] = k[T_1, \dots, T_d, U]/(g).$$

There is a k-algebra homomorphism $\alpha\colon k[T_1, \dots, T_d, U] \to K$ defined by

$$T_1 \mapsto t_1,\ \dots,\ T_d \mapsto t_d,\ U \mapsto u.$$

A little algebra (using Gauss's lemma) shows that the kernel of α is generated by g, so α induces an injection $k[H] \hookrightarrow K$. Furthermore, the image of α generates K as a field, so α induces an isomorphism from the fraction field of $k[H]$ to K.

The fraction field of $k[H]$ is the function field $k(H)$. Thus we have shown that $k(H) \cong k(V)$. By Corollary 9.4, this implies that V is birationally equivalent to H. □

Proof of Corollary 11.5. Apply Proposition 11.4 to $K = k(V)$. □

12. Projective algebraic sets

Projective space.

Projective space consists of affine space together with "points at infinity," one for each direction. The purpose for adding extra points is that it avoids special cases where a point "disappears to infinity." For example, a pair of parallel lines do not intersect in affine space but they do intersect at a point at infinity in projective space.

Definition. Projective n-space, $\mathbb{P}^n$, is the quotient of $k^{n+1} \setminus \{(0, \ldots, 0)\}$ by the equivalence relation

$$(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n) \text{ where } \lambda \in k \setminus \{0\}.$$

We call a representative for an equivalence class the **homogeneous coordinates** of that point in $\mathbb{P}^n$ (and there are many choices for each point, by scaling by λ). To avoid confusion between homogeneous coordinates for $\mathbb{P}^n$ and ordinary coordinates for $\mathbb{A}^n$, we usually write homogeneous coordinates as

$$[x_0 : x_1 : \cdots : x_n].$$

Observe that we can embed $\mathbb{A}^n$ into $\mathbb{P}^n$ by the map

$$(x_1, \ldots, x_n) \mapsto [1 : x_1 : \cdots : x_n].$$

Any other homogeneous coordinates where the first coordinate is non-zero can be re-scaled to have first coordinate 1. So we are left with the points with first coordinate equal to 0: these are the "points at infinity." A point $[0 : x_1 : \cdots : x_n]$ can be seen as a point in $\mathbb{P}^{n-1}$, by just dropping the initial zero. Thus

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}.$$

Similarly

$$\mathbb{P}^1 = \mathbb{A}^1 \cup \{\text{a point}\}.$$

Thinking about projective space as affine space plus points at infinity can be useful if we want to make use of our geometric intuition about affine space or the algebraic tools we have developed for working with affine algebraic sets. On the other hand, thinking about projective space in terms of homogeneous coordinates emphasises that all points of projective space look the same: we can only distinguish points at infinity from points in affine space after choosing a convention for how we embed $\mathbb{A}^n$ into $\mathbb{P}^n$ (for example, we could have used $[x_1 : \cdots : x_n : 1]$ instead); throughout this lecture we will use the convention above.

Projective algebraic sets.

A projective algebraic set is a subset of projective space defined by the vanishing of a finite list of polynomials. What does it mean for a polynomial to vanish at a point in projective space? Because a single point in $\mathbb{P}^n$ can be represented by many different homogeneous coordinates, it does not make sense to evaluate a polynomial in $k[X_0, \ldots, X_n]$ at a point of $\mathbb{P}^n$. We have to restrict attention to homogeneous polynomials.

Definition. A polynomial $f \in k[X_0, \dots, X_n]$ is **homogeneous** if every term of f has the same degree.

For example, $X_0^3 + X_0^2 X_1 + 3X_2^3 - X_0 X_1 X_2$ is homogeneous of degree 3 while $X_0 X_1 - X_2$ is not homogeneous because it has a term of degree 2 and a term of degree 1.

If $[x_0 : \dots : x_n]$ and $[y_0 : \dots : y_n]$ represent the same point $p \in \mathbb{P}^n$, then

$$(x_0, \dots, x_n) = \lambda(y_0, \dots, y_n) \text{ with } \lambda \in k \setminus \{0\}.$$

Hence if $f \in k[X_0, \dots, X_n]$ is a homogeneous polynomial of degree d, then

$$f(x_0, \dots, x_n) = \lambda^d f(y_0, \dots, y_n)$$

Thus the actual value of f at p is not well-defined, but it is well-defined whether or not f is zero at p.

Definition. A **projective algebraic set** is a set of the form

$$\{[x_0 : \dots : x_n] \in \mathbb{P}^n : f_1(x_0, \dots, x_n) = 0,\ \dots,\ f_m(x_0, \dots, x_n) = 0\}$$

for some finite list of homogeneous polynomials $f_1, \dots, f_m \in k[X_0, \dots, X_n]$.

By definition, a projective algebraic set is the vanishing of finitely many homogeneous polynomials. We can use the Hilbert Basis Theorem to show that the vanishing set of an infinite collection of homogeneous polynomials is a projective algebraic set. (This is similar to the analogous result for affine algebraic sets, but a little trickier due to the word "homogeneous.")

Example. An example of a projective algebraic set is

$$V = \{[w : x : y] \in \mathbb{P}^2 : wx - y^2 = 0\}.$$

What is $V \cap \mathbb{A}^2$? (Using the embedding $\mathbb{A}^2 \to \mathbb{P}^2$ which we considered before.) To find this, we just substitute $w = 1$ into the equation for V:

$$V \cap \mathbb{A}^2 = \{(x, y) \in \mathbb{A}^2 : x - y^2 = 0\},$$

that is, a parabola.

We can also work out the intersection of V with the "$\mathbb{P}^1$ at infinity:" it is points where $w = 0$. Substituting that into the equation for V, we get

$$\{[x : y] \in \mathbb{P}^1 : -y^2 = 0\} = \{[1 : 0]\}.$$

Thus V consists of the parabola together with a point at infinity "in the direction $(1, 0)$", i.e. along the x-axis (informally, the two arms of the parabola close up at infinity).

Homogenisation.

Example. We would like to reverse this process, and go from an affine algebraic set to a projective algebraic set. Consider the affine hyperbola $H = \{(x, y) \in \mathbb{A}^2 : xy - 1 = 0\}$.

We need to turn the polynomial $XY - 1$ into a homogeneous polynomial, using a new variable W. To do this, note that the highest degree term in $XY - 1$ has degree 2. We multiply each term by an appropriate power of W to get all terms of degree 2: thus we get $XY - W^2 = 0$. Let

$$H' = \{[w : x : y] \in \mathbb{A}^2 : xy - w^2 = 0\}.$$

When $w = 1$, we can substitute that in and see that we get back H'. When $w = 0$, the equation becomes $xy = 0$, so we now get two points at infinity: either $x = 0$, giving the point $[0 : 0 : 1] \in \mathbb{P}^2$, or $y = 0$, giving the point $[0 : 1 : 0] \in \mathbb{P}^2$. Thus

$$H' = H \cup \{[0 : 0 : 1], [0 : 1 : 0]\}.$$

Geometrically, H' consists of H together with points at infinity along the x- and y-axes. These axes are the asymptotes of H.

Example. Here's a more complex example (the twisted cubic curve). Let

$$C = \{(t, t^2, t^3) \in \mathbb{A}^3\} = \mathbb{V}(Y - X^2,\ Z - XY).$$

Homogenising the polynomials, we get

$$C' = \{[w : x : y : z] \in \mathbb{P}^3 : wy - x^2 = wz - xy = 0\}.$$

It is still true that we can reverse this by just setting $w = 1$, so $C' \cap \mathbb{A}^3 = C$.

But what happens at infinity? Subsituting in $w = 0$, we get

$$\{[0 : x : y : z] \in \mathbb{P}^3 : -x^2 = -xy = 0\} = \{[0 : 0 : y : z] \in \mathbb{P}^3\}.$$

Thus the intersection of C' with the plane at infinity is a copy of $\mathbb{P}^1$. This is not what we should expect, if C' were the smallest possible projective algebraic set containing C: the dimension of the intersection with the plane at infinity should be smaller than the dimension of the initial affine algebraic set (speaking informally).

In fact, the smallest possible projective algebraic set containing C' is

$$C'' = \{[w : x : y : z] \in \mathbb{P}^3 = wy - x^2 = wz - xy = zx - y^2 = 0\}.$$

The extra polynomial involves only x, y, z and is in the ideal generated by $Y - X^2$ and $Z - XY$. You can calculate:

$$C'' = C \cup \{[0 : 0 : 0 : 1]\}.$$

(I am not giving a procedure to find the smallest projective algebraic set containing a given affine algebraic set - I just assert that this happens to work in this case. There is an algorithm but you would not want to have to use it by hand.)

The process we went through above to obtain V' from V and H' from H can be generalised.

Definition. For any polynomial $f \in k[X_1, \ldots, X_n]$, we define the **homogenisation** of f to be the polynomial in $\bar{f} \in k[X_0, \ldots, X_n]$ obtained by the following procedure: let d be the maximum degree of terms of f. Then multiply each term of f by X_0^{d-e}, where e is the degree of this term in f.

For example: if $f(X_1, X_2, X_3) = X_1^3 + 4X_1X_2X_3 - X_1^2 - X_2^2 + 5X_3 + 8$, then the homogenisation is

$$\bar{f}(X_0, X_1, X_2, X_3) = X_1^3 + 4X_1X_2X_3 - X_1^2X_0 - X_2^2X_0 + 5X_3X_0^2 + 8X_0^3.$$

Let $V \subseteq \mathbb{A}^n$ be an affine algebraic set. Let $W \subseteq \mathbb{P}^n$ be the set defined by the homogenisations of all polynomials in $\mathbb{I}(V)$. Then W is the smallest projective algebraic set containing V. When we substitute $x_0 = 1$ into the polynomials defining W, we just get back $\mathbb{I}(V)$, so

$$W \cap \{[1 : x_1 : \cdots : x_n]\} = V.$$

The example above shows that if we just use homogenisations of a generating set, instead of all of $\mathbb{I}(V)$, we still get a projective algebraic set V' such that $V' \cap \mathbb{A}^n = V$, but it might not be the smallest such set.

Zariski topology on $\mathbb{P}^n$.

We can define the Zariski topology on $\mathbb{P}^n$ by saying that the closed subsets are the projective algebraic sets. We have just shown that the Zariski topology on $\mathbb{A}^n$ is the subspace topology coming from the Zariski topology on $\mathbb{P}^n$.

Note that the "smallest projective algebraic set containing V" which we just described is the same as the closure of V in the Zariski topology on $\mathbb{P}^n$.

Projective Nullstellensatz.

Which homogeneous ideals can occur as the ideal of functions vanishing on a projective algebraic set? Clearly they have to be radical ideals. Is there a projective version of the Nullstellensatz?

Yes, but it turns out that there is an exceptional case to deal with. Consider the homogeneous ideal $I_1 = (X_0, \ldots, X_n) \subseteq k[X_0, \ldots, X_n]$. The only solution in k^{n+1} to the equations $x_0 = 0, \ldots, x_n = 0$ is $(0, \ldots, 0)$. But this is not the homogeneous coordinates of any point in $\mathbb{P}^n$. So the projective algebraic set defined by I_1 is the empty set. Thus the ideals I_1 and $k[X_0, \ldots, X_n]$ both define the empty set in $\mathbb{P}^n$, even though they are both radical homogeneous ideals. So we have to modify the statement of the Nullstellensatz slightly from the affine case.

This turns out to be the only special case.

Proposition 12.1 (Projective Weak Nullstellensatz)**.** Let $I \subseteq k[X_0, \ldots, X_n]$ be a homogeneous ideal such that $\operatorname{rad} I$ is not equal to either $k[X_0, \ldots, X_n]$ or $(X_0, \ldots, X_n)$. Then the projective algebraic set defined by I is non-empty.

Proof. If the projective algebraic set defined by I in $\mathbb{P}^n$ is empty, then the affine algebraic set $\mathbb{V}(I) \subseteq \mathbb{A}^{n+1}$ must be either $\emptyset$ or $\{(0, \ldots, 0)\}$. We can conclude by applying the usual (strong) Nullstellensatz to the affine set $\mathbb{V}(I)$. ☐

13. Regular maps between projective algebraic sets

Bijection between homogeneous ideals and algebraic sets.

Last time we saw the Projective Weak Nullstellensatz, and we saw that the radical homogeneous ideal $(X_0, \ldots, X_n)$ defines the empty projective algebraic set, the same as (1). However, this turns out to be the only exception to the bijection between radical homogeneous ideals and projective algebraic sets:

Theorem 13.1. The map sending a homogeneous ideal to the corresponding projective algebraic set is a bijection between the following sets:

$$\left\{ \begin{matrix} \text{radical homogeneous ideals} \\ \text{in } k[X_0, \ldots, X_n] \\ \text{other than } (X_0, \ldots, X_n) \end{matrix} \right\} \longrightarrow \{\text{projective algebraic sets in } \mathbb{P}^n\}.$$

Proof. Apply the affine Nullstellensatz to the set in $\mathbb{A}^{n+1}$ defined by the same ideal. □

A remark on compactness.

Over the complex numbers, every projective algebraic set is compact in the analytic topology. This is because they are closed subsets of $\mathbb{P}^n_{\mathbb{C}}$, which is compact. (In the Zariski topology, the notion of compactness is not very interesting: every algebraic set is compact in the Zariski topology, even affine algebraic sets. Affine algebraic sets do not behave in ways matching our intuition about compactness: this intuition only works in Hausdorff spaces.)

There is a converse to this, which tells us that there is a very close relationship between analytic and algebraic geometry in $\mathbb{P}^n_{\mathbb{C}}$:

Theorem 13.2 (Chow's theorem)**.** Let V be an analytic subset of $\mathbb{P}^n_{\mathbb{C}}$ which is closed in the analytic topology. Then V is a projective algebraic set.

I won't define analytic subsets here, but roughly it means a set defined by zeroes of holomorphic functions. This theorem requires too much comlpex analytic geometry to prove here.

One can prove analytically that every holomorphic function on a connected compact complex manifold is constant (for example, this holds on the Riemann sphere, which is equal to $\mathbb{P}^1_{\mathbb{C}}$). Polynomials are holomorphic, so every regular function on a connected projective algebraic set over $\mathbb{C}$ is constant. Once we define regular functions on projective algebraic sets, it will turn out that the same is true over any field.

Regular maps between projective algebraic sets.

We want to define regular maps between projective algebraic sets. Let $V \subseteq \mathbb{P}^m$ and $W \subseteq \mathbb{P}^n$ be projective algebraic sets. We expect a regular map $\varphi\colon V \to W$ to be a function which can be expressed as polynomials in the homogeneous coordinates:

$$\varphi([x_0 : \cdots : x_m]) = [f_0(x_0, \ldots, x_m) : \cdots : f_n(x_0, \ldots, x_m)].$$

In order for this to be a well-defined function, all the f_i must be homogeneous polynomials of the same degree, so that

$$\begin{aligned}&[f_0(\lambda x_0,\dots,\lambda x_m):\cdots:f_n(\lambda x_0,\dots,\lambda x_m)]\\&=[\lambda^d f_0(x_0,\dots,x_m):\cdots:\lambda^d f_n(x_0,\dots,x_m)].\end{aligned}$$

All the coordinates are multiplied by λ^d, so this is the same point in $\mathbb{P}^n$ as $[f_0(x_0,\dots,x_m):\cdots:f_n(x_0,\dots,x_m)]$.

There is another condition which must be imposed to get a well-defined function $V\to\mathbb{P}^m$: we must never have

$$f_0(x_0,\dots,x_n)=\cdots=f_m(x_0,\dots,x_n)=0$$

because $[0:\cdots:0]$ is not the homogeneous coordinates of a point in $\mathbb{P}^m$.

This is a very strong condition and there are too few lists of polynomials which satisfy it. However, we can get round it to some extent by imitating rational maps between affine algebraic sets, and allowing different lists of polynomials to define the map at different points (so that, at each point, there is some list of polynomials which is always non-zero). The homogeneous nature of the coordinates allows us to do this in such a way that the different lists of polynomials define the same map wherever they overlap.

To help explain this, we consider an example.

Example. Let V be the projective closure of the parabola, i.e.

$$V=\{[w:x:y]\in\mathbb{P}^2:wy=x^2\}.$$

Let

$$V'=V\cap\{[w:x:y]:w\neq 0\}=\{(x,y)\in\mathbb{A}^2:y=x^2\}.$$

There is a regular map $\varphi'\colon V'\to\mathbb{A}^1$ given by

$$\varphi'(x,y)=x.$$

Does this extend to a regular map $\varphi\colon V\to\mathbb{P}^1$? (We guess it should send the point at infinity $[0:0:1]\in V$ to the point at infinity $[0:1]\in\mathbb{P}^1$.)

To attempt to construct such a map, write φ' in homogeneous coordinates using the embedding $\mathbb{A}^2\hookrightarrow\mathbb{P}^2$:

$$[1:x:y]\mapsto[1:x].$$

Now we homogenise, i.e. multiply by powers of the "extra" coordinate w to make all the polynomials homogeneous of degree 1:

$$[w:x:y]\mapsto[w:x].$$

This is maps $[0:0:1]$ to $[0:0]$ which is not allowed! But we can fix this by expressing the same map differently. Using the homogeneous nature of the coordinates, and the equation $x^2=wy$ defining V, we have

$$[w:x]=[wx:x^2]=[wx:wy]=[x:y]$$

whenever the values we multiplied/divided by (w and x) are non-zero.

The expression $[x : y]$ is well-defined at $[0 : 0 : 1]$, with value $[0 : 1]$. On the other hand, $[x : y]$ gives $[0 : 0]$ at the point $[1 : 0 : 0] \in V$, so we cannot use $[x : y]$ alone to define a map $V \to \mathbb{P}^1$.

At least one of these two expressions is defined everywhere on V, and they agree where they overlap, so the two expressions together give a well-defined regular map $\varphi\colon V \to \mathbb{P}^1$:

$$\varphi([w : x : y]) = \begin{cases} [w : x] \text{ if } w \neq 0, \\ [x : y] \text{ if } y \neq 0. \end{cases} \tag{*}$$

Note that each expression is defined on a *Zariski open* subset of V. This is important because it is how we ensure that the value of φ at each point is *polynomially* related to its value at nearby points. (Open sets are the natural way to talk about "nearby points" in a topological space. This still applies in the Zariski topology, even though open sets are very big.)

Note that questions 5 and 6 on problem sheet 2 give examples of regular maps defined everywhere except at a single point of an affine algebraic set, where there is an obvious value the map "should" take at the missing point, but the map is not regular at that point because there is no way to extend it to that point using polynomials. This is why we are not allowed just to write down polynomials on arbitrary (non-open) subsets of V and claim they define a regular map.

The formal definition of a regular map between projective algebraic sets $V \subseteq \mathbb{P}^m$ and $W \subseteq \mathbb{P}^n$ is:

Definition. A **regular map** $\varphi\colon V \to W$ is a function $V \to W$ such that for every point $x \in V$, there exists a Zariski open set $U \subseteq V$ containing x and a sequence of polynomials $f_0, \dots, f_n \in k[X_0, \dots, X_m]$ such that:

(i) $f_0, \dots, f_n$ are homogeneous of the same degree;
(ii) for every $y \in U$, $f_0, \dots, f_n$ are not all zero at y;
(iii) for every $y = [y_0 : \cdots : y_m] \in U$, $\varphi(y) = [f_0(y_0, \dots, y_m) : \cdots : f_n(y_0, \dots, y_m)]$.

In practice, every regular map can be written down by specifying lists of polynomials on just finitely many open sets, like φ (this follows ultimately from the Hilbert Basis Theorem). To check that a purported definition like (*) really does define a regular map $V \to W$, you have to check:

(1) each set on which an expression is defined is Zariski open;
(2) an expression never gives $[0 : \cdots : 0]$ on its associated set;
(3) two expressions agree wherever they are both defined;
(4) the image of the map is contained in W.

Example. As another example, let's try to extend the inverse of φ from affine to projective algebraic sets. On affine algebraic sets, the inverse of φ' is $\psi'\colon \mathbb{A}^1 \to V'$ given by

$$\psi'(t) = (t, t^2).$$

In projective coordinates, this is

$$[1 : t] \mapsto [1 : t : t^2].$$

Homogenising (inserting powers of s to make all the polynomials on RHS degree 2), we get

$$[s : t] \mapsto [s^2 : st : t^2].$$

Now s^2, st, t^2 are never simultaneously zero for $[s : t] \in \mathbb{P}^1$, so in this case the single expression $[s^2 : st : t^2]$ is enough to define a regular map $\varphi\colon \mathbb{P}^1 \to V$ (note that the image of φ is indeed contained in V).

The two maps $\varphi\colon V \to \mathbb{P}^1$ and $\psi\colon \mathbb{P}^1 \to V$ are inverses, so we conclude that the projective parabola V is isomorphic to $\mathbb{P}^1$.

Note that this homogenisation procedure does not always work. There are regular maps between affine algebraic sets which it is impossible to extend to regular maps between their projective closures (there are points for which it is impossible to avoid sending them to $[0 : \cdots : 0]$).

14. Quasi-projective algebraic sets and rational maps

Regular maps equal on a dense subset.

We already proved the following lemma for regular maps between affine algebraic sets. It is even more useful for regular maps between projective algebraic sets (we will need it in the definition of rational maps).

Lemma 14.1. Let $\varphi, \psi \colon V \to W$ be regular maps. If there exists a Zariski dense subset $A \subseteq V$ such that $\varphi_{|A} = \psi_{|A}$, then $\varphi = \psi$.

The lemma is especially useful if V is *irreducible*, because then Zariski open subsets of V are dense. So the lemma tells us that, given a list of polynomials on a Zariski open subset of V, there is *at most* one regular map which is given by that list of polynomials on that set.

Proof. Let $Z = \{x \in V : \varphi(x) = \psi(x)\}$. By hypothesis, Z contains a dense subset of V. Hence in order to show that $Z = V$, it suffices to show that Z is closed in V.

We will use the following topological fact:

Fact. Let S be any topological space. Let $\{U_\alpha\}$ be a collection of open subsets of S whose union is all of S. Let Z be any subset of S such that $Z \cap U_\alpha$ is closed in the subspace topology on U_α for every α. Then Z is closed as a subset of S.

From the definition of regular maps, we know that we can cover V by Zariski open sets U_α such that on each U_α, both φ and ψ are defined by sequences of homogeneous polynomials:

$$\varphi_{|U_\alpha} = [f_{\alpha,0} : \dots : f_{\alpha,m}], \quad \psi_{|U_\alpha} = [g_{\alpha,0} : \dots : g_{\alpha,m}].$$

By the topological fact, it suffices to show that $Z \cap U_\alpha$ is closed in the subspace topology on U_α for every α.

Now

$$Z \cap U_\alpha = \{x \in U_\alpha : [f_{\alpha,0}(x) : \dots : f_{\alpha,m}(x)] = [g_{\alpha,0}(x) : \dots : g_{\alpha,m}(x)]\}.$$

This is the same as the set of $x \in U_\alpha$ where the vectors $(f_{\alpha,0}(x), \dots, f_{\alpha,m}(x))$ and $(g_{\alpha,0}(x), \dots, g_{\alpha,m}(x))$ are proportional (for any choice of homogeneous coordinates for x). A little algebra shows that this condition is equivalent to

$$f_{\alpha,i}(x) g_{\alpha,j}(x) - f_{\alpha,j}(x) g_{\alpha,i}(x) = 0 \text{ for all } i, j \in \{0, \dots, m\}.$$

This last condition is given by homogeneous polynomials, and therefore defines a closed subset in the subspace topology on U_α. □

Quasi-projective algebraic sets.

So far, we have defined affine algebraic sets and projective algebraic sets, as separate types of object. It is very convenient to have a single notion that unifies both affine and projective algebraic sets (for example to save us from having to prove a lemma for affine algebraic sets, then the same lemma for projective algebraic sets).

Definition. A **quasi-projective algebraic set** is the intersection between an open subset and a closed subset of $\mathbb{P}^n$ (in the Zariski topology).

A projective algebraic set is quasi-projective (just take the open subset to be $\mathbb{P}^n$ itself). An affine algebraic set is quasi-projective: it is the intersection between $\mathbb{A}^n$ (which is open in $\mathbb{P}^n$) and a projective algebraic set $\overline{V}$. There are other quasi-projective algebraic sets, for example $\mathbb{A}^1 \setminus \{0\}$ which is an open subset of $\mathbb{P}^1$.

We define a regular map between quasi-projective algebraic sets by the same definition as a regular map between projective algebraic sets.

If V and W are affine algebraic sets, we now have two ways to define regular maps $V \to W$:

(a) the original definition of regular maps between affine algebraic sets;
(b) view V and W as quasi-projective algebraic sets, and use the new definition of regular maps between quasi-projective algebraic sets.

Fortunately, these two definitions turn out to be equivalent. One has to do a bit of work to check this (the problem is that a regular map of affine algebraic sets must be defined by the same list of polynomials at every point, but a regular map of quasi-projective algebraic sets may be defined by the same polynomials at every point; proving that actually one list of polynomials is enough if the set happens to be affine is similar to the proof of Lemma 8.2).

This gives us for free a notion of regular maps from a projective algebraic set to an affine algebraic set or vice versa: just view them both as quasi-projective algebraic sets. For example, we can now define a **regular function** on a projective algebraic set V to be a regular map $V \to \mathbb{A}^1$ (thus it is a function from the algebraic set V taking values in the base field k). As remarked last lecture, we will later prove that the only regular functions on a projective algebraic set are the constants.

We can now make rigorous the claim that "$\mathbb{A}^1 \setminus \{0\}$ looks the same as the affine hyperbola $H = \{(x, y) \in \mathbb{A}^2 : xy = 1\}$." The set

$$\mathbb{A}^1 \setminus \{0\} = \mathbb{P}^1 \setminus \{[1:0], [0:1]\}$$

is a Zariski open subset of $\mathbb{P}^1$, because its complement is finite. Hence $\mathbb{A}^1 \setminus \{0\}$ is a quasi-projective algebraic set. The map $\varphi\colon \mathbb{A}^1 \setminus \{0\} \to H$ given by $\varphi(t) = (t, 1/t)$ can be written in homogeneous coordinates as

$$\varphi([1:t]) = [1:t:1/t] = [t:t^2:1]$$

so homogenising, we get

$$\varphi([s:t]) = [st:t^2:s^2].$$

So long as $[s:t] \in \mathbb{A}^1 \setminus \{0\}$, this does give a point in

$$H = \{[w:x:y] \in \mathbb{P}^2 : xy = w^2\} \cap \mathbb{A}^2$$

so φ is a regular map $\mathbb{A}^1 \setminus \{0\} \to H$. The projection $(x, y) \mapsto x$ is a regular inverse to φ. Hence $\mathbb{A}^1 \setminus \{0\}$ and H are isomorphic as quasi-projective algebraic sets.

Varieties.

As mentioned previously, we use the word "variety" to mean an algebraic set considered up to isomorphism, not caring about how it is embedded into affine or projective space. For example, $\mathbb{A}^1 \setminus \{0\}$ is isomorphic (as a quasi-projective algebraic set) to the affine algebraic set H, so we may say that $\mathbb{A}^1 \setminus \{0\}$ is an affine variety, even though $\mathbb{A}^1 \setminus \{0\}$ is definitely not an affine algebraic set.

There exist quasi-projective algebraic sets which are not isomorphic to anything either projective or affine, for example $\mathbb{A}^2 \setminus \{(0,0)\}$ (see problem sheet 3).

Rational maps between quasi-projective algebraic sets.

Let V and W be irreducible quasi-projective algebraic sets. The formal definition of a rational map $V \dashrightarrow W$ looks quite complicated, but the underlying idea is the same as for regular maps: a rational map is almost a regular map, except that it is allowed to have some points where it is not defined. Rational maps of affine algebraic sets are non-regular at points where the denominator is zero; for quasi-projective algebraic sets, they are non-regular at points where the coordinates of the image become $[0 : \cdots : 0]$. (Note that there is no need to use fractions in the definition of rational maps between quasi-projective algebraic sets: because our coordinates are homogeneous, we can always multiply up by a common denominator.)

Somehow we have to make a definition which takes account of the fact that rational maps are defined by different lists of polynomials at different points. But unlike with regular maps of projective algebraic sets, we can't tie the different expressions together into a single object by saying "a rational map is a function $V \to W$ such that ..." because a rational map is not a function $V \to W$.

Instead we define rational maps as equivalence classes for a certain equivalence relation. Note that this happens under the hood in defining rational functions on affine algebraic sets too: they are elements of a field of fractions, and the field of fractions of an integral domain R is defined as the set of equivalence classes in the set

$$\{(a, b) \in R^2 : b \neq 0\}$$

for the equivalence relation

$$(a, b) \sim (c, d) \text{ if } ad = bc.$$

In the ring $R = \mathbb{Z}$, we can choose a unique representative for each fraction by reducing to "lowest terms." But if R is not a UFD, there are no "lowest terms" representatives – this matches the fact that we might need different expressions to define a rational map at different points.

Definition. Let $V \subseteq \mathbb{P}^m$ and $W \subseteq \mathbb{P}^n$ be irreducible quasi-projective algebraic sets.

Let S denote the set of sequences $(f_0, \dots, f_n) \in k[X_0, \dots, X_m]^{n+1}$ such that:

(1) $f_0, \dots, f_n$ are homogeneous of the same degree;
(2) $f_0, \dots, f_n$ are not all identically zero on V (note that this looks a little like the $b \neq 0$ condition in defining the field of fractions);
(3) there exists a non-empty Zariski open set $A \subseteq V$ such that, for all $x \in A$, $[f_0(x) : \cdots : f_n(x)] \in W$.

Define an equivalence relation $\sim$ on S by: $(f_0, \dots, f_n) \sim (g_0, \dots, g_n)$ if

$$[f_0(x) : \cdots : f_n(x)] = [g_0(x) : \cdots : g_n(x)] \in \mathbb{P}^n$$

for all $x \in V$ where both expressions make sense. We could write this more algebraically as: $(f_0, \ldots, f_n) \sim (g_0, \ldots, g_n)$ if

$$f_i g_j = f_j g_i \text{ for all } i, j.$$

Observe that this resembles the equivalence relation used in defining the field of fractions.

You can check that $\sim$ is an equivalence relation using Lemma 14.1 and the fact that V is irreducible.

A **rational map** $\varphi \colon V \dashrightarrow W$ is an equivalence class in S for $\sim$.

We usually specify rational maps by just giving one representative $[f_0 : \cdots : f_n]$ in S.

15. More on rational maps

Domain of definition.

Definition. A rational map $\varphi\colon V \to W$ is **regular** at a point $x \in V$ if there exists at least one list of polynomials $(f_0, \dots, f_n) \in S$ representing φ such that

$$[f_0(x) : \cdots : f_n(x)] \neq [0 : \cdots : 0] \text{ and } [f_0(x) : \cdots : f_n(x)] \in W.$$

If φ is regular at x, then the equivalence relation $\sim$ ensures that the value $\varphi(x)$ is well-defined (independent of the choice of polynomials representing φ, as well as independent of the choice of homogeneous coordinates for x).

Just as for affine algebraic sets, we define the **domain of definition** of a rational map to be the set of points where it is regular.

Note that the domain of definition of a rational map can change if we change the target set W. For example, consider the map $\mathbb{P}^1 \to \mathbb{P}^2$ defined by

$$[s : t] \mapsto [s^2 : st : t^2].$$

This is regular at every point. We could interpret the same formula as defining a rational map $\mathbb{P}^1 \dashrightarrow W$ where $W \subseteq \mathbb{P}^2$ is the open set

$$W = \{[w : x : y] : w \neq 0\}.$$

As a rational map $\mathbb{P}^1 \dashrightarrow W$, this is not regular at the point $[0 : 1]$ because this point maps to $[0 : 0 : 1] \notin W$.

Lemma 15.1. Let $\varphi\colon V \dashrightarrow W$ be a rational map. The domain of definition of φ is a non-empty Zariski open subset of V.

Proof. Similar to the affine case (Lemma 8.1). □

It follows immediately from the definition of regular maps between quasi-projective algebraic sets that if a rational map is regular at every point, then it is a regular map. (In the affine case (Lemma 8.2), we had to work to prove that if a rational map is regular at every point, then there is a *single* polynomial expression which defines the map everywhere. In the quasi-projective case, we don't need to do this because our definition of regular map allows different expressions at different points.)

Example. Let C denote the affine algebraic set

$$C = \{(x, y) \in \mathbb{A}^2 : y = x^3\}.$$

This has projective closure

$$\overline{C} = \{[w : x : y] \in \mathbb{P}^2 : w^2 y = x^3\} = C \cup \{[0 : 0 : 1]\}.$$

Consider the regular map of affine algebraic sets $\varphi\colon C \to \mathbb{A}^1$ given by

$$\varphi(x, y) = x.$$

If we try to extend this to a map of projective algebraic sets $\overline{\varphi}\colon \overline{C} \to \mathbb{P}^1$, we would say that for points $[1 : x : y] \in C \subseteq \overline{C}$,

$$\varphi([1 : x : y]) = [1 : x]$$

and this homogenises to

$$\overline{\varphi}([w : x : y]) = [w : x].$$

Thus $\overline{\varphi}$ is a rational map $\overline{C} \dashrightarrow \mathbb{P}^1$.

The above expression for $\overline{\varphi}$ is not defined at the point $[0 : 0 : 1] \in \overline{C}$. We can prove that there is no other expression for $\overline{\varphi}$ which is defined at that point, and so $\overline{\varphi}$ is not regular at $[0 : 0 : 1]$ (see problem sheet 3, question 3).

Thus a regular map of affine algebraic sets extends to a rational map between their projective closures, but the extended map is not necessarily regular at the points at infinity.

Birational maps.

Just as in the affine case, if we have irreducible quasi-projective sets V, W, T and rational maps $\varphi\colon V \dashrightarrow W$ and $\psi\colon W \dashrightarrow T$, if the image of φ is dense in W, then the composite $\psi \circ \varphi$ is a rational map $V \dashrightarrow T$.

The following definitions are the same as the affine case:

Definition. A rational map $\varphi\colon V \dashrightarrow W$ is **dominant** if its image is dense in W.

A rational map $\varphi\colon V \dashrightarrow W$ is a **birational equivalence** if it is dominant and there exists a dominant rational map $\psi\colon W \dashrightarrow V$ such that $\psi \circ \varphi = \mathrm{id}_V$ and $\varphi \circ \psi = \mathrm{id}_W$ (where these composite rational maps are defined).

Irreducible algebraic sets V and W are **birational** if there exists a birational equivalence $V \dashrightarrow W$.

Note that $\mathbb{A}^n$ is birational to $\mathbb{P}^n$: consider the regular map

$$\varphi\colon \mathbb{A}^n \to \mathbb{P}^n : (x_1, \dots, x_n) \mapsto [1 : x_1 : \cdots : x_n]$$

and the rational map

$$\psi\colon \mathbb{P}^n \dashrightarrow \mathbb{A}^n : [x_0 : \cdots : x_n] \mapsto (x_1/x_0, \dots, x_n/x_0).$$

Each of these is dominant and composing them in either direction gives the identity, so these are birational equivalences.

Observe that φ is an isomorphism from $\mathbb{A}^n$ to an open subset of $\mathbb{P}^n$. We can generalise this to the following stronger result, which makes precise the intuition that varieties are birational if and only if they are the same "almost everywhere."

Lemma 15.2. Irreducible quasi-projective varieties V and W are birational if and only if there exist non-empty Zariski open subsets $A \subseteq V$ and $B \subseteq W$ such that A is isomorphic to B (as quasi-projective varieties).

Proof. Let $\varphi\colon V \dashrightarrow W$ and $\psi\colon W \dashrightarrow V$ be an inverse pair of rational maps. Let $A_1 = \operatorname{dom} \varphi$ and $B_1 = \operatorname{dom} \psi$. B_1 is a non-empty open subset of W.

Since φ induces a continuous map $A_1 \to W$, $A = \varphi_{|A_1}^{-1}(B_1)$ is an open subset of V. Furthermore, since φ is dominant, its image intersects the open set $B_1 \subseteq W$. Therefore A is non-empty.

Similarly $B = \psi_{|B_1}^{-1}(A_1)$ is a non-empty open subset of W.

One can now check that $\varphi_{|A}$ and $\psi_{|B}$ form an inverse pair of isomorphisms between A and B. □

If V is a quasi-projective algebraic set, we define a **rational function** on V to be a rational map $\varphi\colon V \dashrightarrow \mathbb{A}^1$. By definition, this is the same as a rational map $\varphi'\colon V \dashrightarrow \mathbb{P}^1$ except that we declare φ to be non-regular at points where $\varphi'(x) = \infty = [0:1] \in \mathbb{P}^1$. We can therefore say

$$\varphi(x) = [f(x) : g(x)] = [1 : g(x)/f(x)] = \frac{g(x)}{f(x)} \in \mathbb{A}^1$$

whenever $f(x) \neq 0$, for suitable polynomials f, g. Of course, as always with rational maps, we might need to use different polynomials to evaluate it at different points.

The rational functions on V form a field $k(V)$. Just as in the affine case, V is birational to W if and only if $k(V)$ is k-isomorphic to $k(W)$.

Linear spaces in $\mathbb{P}^n$.

We want to define a fundamental example of a rational map: projection from a point to a hyperplane. First, we need to make a few other definitions.

Definition. A **hyperplane** in $\mathbb{P}^n$ is the projective algebraic set defined by a single homogeneous linear equation:

$$H = \{[x_0 : \dots : x_n] \in \mathbb{P}^n : h_0x_0 + \dots + h_nx_n = 0\}$$

for some $h_0, \dots, h_n \in k$, not all zero.

More generally, a **linear subspace** of $\mathbb{P}^n$ is a subset defined by any set of homogeneous linear equations.

Examples of linear subspaces are $\mathbb{P}^n$ itself (empty set of equations), $\emptyset$ (too many equations), and singletons. We can't define the singleton $\{[p_0 : \dots : p_n]\}$ by the equations $x_0 = p_0, \dots, x_n = p_n$ because these are not homogeneous. Instead, we can write homogeneous equations asserting that the ratios between pairs of coordinates are correct:

$$\{[p_0 : \dots : p_n]\} = \{[x_0 : \dots : x_n] \in \mathbb{P}^n : p_ix_j = p_jx_i \text{ for all } i, j\}.$$

If Λ is a linear subspace of $\mathbb{P}^n$, then the affine cone $C(\Lambda)$ (the set of points in $\mathbb{A}^{n+1}$ satisfying the same equations as Λ) is a vector subspace of k^{n+1}.

As a vector space, we know what is meant by $\dim C(\Lambda)$. We define

$$\dim \Lambda = \dim C(\Lambda) - 1.$$

(We have not yet defined the dimension of an arbitrary algebraic set; this definition is only for linear subspaces of projective space. The -1 is because $C(\Lambda)$ contains a line for each point in Λ.) For example, $\mathbb{P}^n$ has dimension n, a hyperplane has dimension $n-1$ and a point has dimension 0.

If Λ is a linear subspace of $\mathbb{P}^n$ of dimension d, then $C(\Lambda) \cong k^{d+1}$ (as a vector space) and

$$\Lambda = \big(C(\Lambda) \setminus \{0\}\big)/(\text{multiplying by scalars})$$

so $\Lambda \cong \mathbb{P}^d$.

Definition. A **line** in $\mathbb{P}^n$ is a linear subspace of dimension 1.

16. Projections and products

Lines in $\mathbb{P}^n$.

Lemma 16.1. For any two distinct points $p, q \in \mathbb{P}^n$, there exists a unique line L_{pq} through p and q.

Proof. One could prove this by saying: $\mathbb{P}^n$ can be written as a union $\mathbb{A}^n \cup \mathbb{P}^{n-1}$, and going through the cases $p, q \in \mathbb{A}^n$; $p, q \in \mathbb{P}^{n-1}$; $p \in \mathbb{A}^n$ and $q \in \mathbb{P}^{n-1}$. In order to make this into a full proof, we would need to check that a line in $\mathbb{P}^n$, intersected with $\mathbb{A}^n$, is the same as the ordinary definition of a line in $\mathbb{A}^n$ (which is true!)

Instead we shall give a proof using linear algebra. A benefit of this proof is that it gives a description of the homogeneous coordinates of points in the line L_{pq}.

Let $p = [p_0 : \cdots : p_n]$ and $q = [q_0 : \cdots : q_n]$. The affine cones $C(p)$ and $C(q)$ are the one-dimensional vector spaces of generated by $(p_0, \ldots, p_n)$ and $(q_0, \ldots, q_n)$ respectively. Since $p \neq q$, these vector spaces are linearly independent so there is a unique 2-dimensional vector subspace $W \subseteq k^{n+1}$ which contains $C(p)$ and $C(q)$. The image of $W \setminus \{0\}$ in $\mathbb{P}^n$ is the unique line through p and q.

Explicitly: W consists of all linear combinations of the vectors $(p_0, \ldots, p_n)$ and $(q_0, \ldots, q_n)$. It follows that

$$L_{pq} = \{[p_0 s + q_0 t : \cdots : p_n s + q_n t] \in \mathbb{P}^n : [s : t] \in \mathbb{P}^1\}. \qquad \square$$

Projections.

A fundamental example of a rational map is projection from a point to a hyperplane. Let $p = [p_0 : \cdots : p_n] \in \mathbb{P}^n$ and let $H \subseteq \mathbb{P}^n$ be a hyperplane such that $p \notin H$. To simplify the calculations, we shall assume that

$$H = \{[x_0 : \ldots : x_n] \in \mathbb{P}^n : x_n = 0\}.$$

Any line in $\mathbb{P}^n$ which is not contained in H meets H in exactly one point. This is geometrically clear; one can prove it algebraically via linear algebra using the affine cones, or by the following calculation:

Let $x \in \mathbb{P}^n \setminus \{p\}$. Then

$$L_{px} = \{[p_0 s + x_0 t : \cdots : p_n s + x_n t] \in \mathbb{P}^n : [s : t] \in \mathbb{P}^1\}. \qquad (*)$$

Hence, to find $L_{px} \cap H$, we need to choose $[s : t]$ such that $p_n s + x_n t = 0$: we can choose $[s : t] = [x_n : -p_n]$ (note that $p_n \neq 0$ because $p \notin H$, so we do not get $[0 : 0]$). Substituting in to (*), the unique point of $L_{px} \cap H$ is

$$[p_0 x_n - x_0 p_n : \cdots : p_{n-1} x_n - x_{n-1} p_n : 0].$$

The final $0 = p_n x_n - x_n p_n$ is what we expect for a point in H. Note that, if $p \neq x$, then this is not $[0 : \cdots : 0]$ so it is well-defined.

Thus, for $x \in \mathbb{P}^n \setminus \{p\}$, it makes sense to define $\pi(x)$ to be the unique point of $L_{px} \cap H$. The above calculation shows that π is a rational map $\mathbb{P}^n \dashrightarrow H$, regular on $\mathbb{P}^n \setminus \{p\}$. We show below that π is not regular at p.

This rational map is called **projection from p to H**. One could replace this particular fixed H by any hyperplane not containing p, and carry out the same recipe.

Lemma 16.2. Let $n \geq 2$. The projection of $\mathbb{P}^n$ from p to H is not regular at p.

Proof. Intuitively: there are many lines passing through p and p (not a typo!), so the projection would have to map p to "everywhere at once."

We can make this rigorous: Pick a point $s \in H$ and consider the line L_{ps}. For any $x \in L_{ps} \setminus \{p\}$, the geometric description of π shows that $\pi(x) = s$.

If we assume that π is regular at p, then it restricts to a regular map $L_{ps} \to H$. We have just shown that this map is constant on $L_{ps} \setminus \{p\}$ and therefore it is constant on L_{ps}. Hence $\pi(p) = s$.

We could pick another point $t \in H$ and repeat exactly the same argument using L_{pt}, so tha $\pi(p) = t$. This is a contradiction.

(The condition $n \geq 2$ is needed to ensure that $H \cong \mathbb{P}^{n-1}$ has two distinct points s and t. If $n = 1$, then H is just a point and π is a constant map, so it is regular everywhere.) □

Products of projective algebraic sets.

Many sets that we want to work with (for example, the graph of a regular map $V \to W$) are naturally defined as subsets of products $V \times W$ of algebraic sets. Therefore we would like to be able to say that the product of algebraic sets are also algebraic sets.

We saw that this is easy for affine algebraic sets: $V \times W$ is an affine algebraic subset of $\mathbb{A}^{m+n}$. The key point here is the isomorphism $\mathbb{A}^m \times \mathbb{A}^n \cong \mathbb{A}^{m+n}$.

For projective algebraic sets, things are harder because $\mathbb{P}^m \times \mathbb{P}^n \not\cong \mathbb{P}^{m+n}$. To see informally why $\mathbb{P}^1 \times \mathbb{P}^1 \not\cong \mathbb{P}^2$, recall that $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\text{pt}\}$ so

$$\mathbb{P}^1 \times \mathbb{P}^1 = (\mathbb{A}^1 \times \mathbb{A}^1) \cup (\mathbb{A}^1 \times \{\text{pt}\}) \cup (\{\text{pt}\} \times \mathbb{A}^1) \cup (\{\text{pt}\} \times \{\text{pt}\}) = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^1 \cup \{\text{pt}\}.$$

Meanwhile

$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1 = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \{\text{pt}\}.$$

Thus $\mathbb{P}^1 \times \mathbb{P}^1$ contains an extra copy of $\mathbb{A}^1$ compared to $\mathbb{P}^2$. This is only an informal argument: a rigorous proof will be on problem sheet 4.

We could try giving an ad hoc definition for $\mathbb{P}^m \times \mathbb{P}^n$. It is fairly clear what algebraic subsets of $\mathbb{P}^m \times \mathbb{P}^n$ should mean (sets defined by polynomials in the two sets of homogeneous coordinates $[x_0 : \cdots : x_m], [y_0 : \cdots : y_n]$; in order for the zero set of such a polynomial to be well-defined, it must be **bihomogeneous**, that is, homogeneous in the x variables and homogeneous in the y variables, but potentially of different degrees in x and y degrees can be different). Similarly, we could give a definition of regular maps between subvarieties of $\mathbb{P}^m \times \mathbb{P}^n$ involving bihomogeneous polynomials. But it would be annoying to have just defined quasi-projective varieties, unifying affine and projective varieties, and then immediately have to introduce *ad hoc* definitions for another different kind of variety. So we aim to construct the product in a way which makes it a (quasi-)projective set, and then we can just reuse the definitions from before.

To construct the product $\mathbb{P}^m \times \mathbb{P}^n$ as a projective algebraic set, we embed it inside some larger $\mathbb{P}^N$. The homogeneous coordinates of a point in $\mathbb{P}^m \times \mathbb{P}^n$ will be given by an $(m+1) \times (n+1)$ matrix, so we need

$$N = (m+1)(n+1) - 1 = mn + m + n.$$

Label the homogeneous coordinates of a point in $\mathbb{P}^N$ as if they were entries of a matrix:

$$[(z_{ij} : 0 \leq i \leq m, 0 \leq j \leq n)],$$

rather than the usual $[z_0 : \cdots : z_N]$.

Define a map $\sigma_{m,n} \colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ by sending $([x_0 : \cdots : x_m], [y_0 : \cdots : y_n])$ to the point in $\mathbb{P}^N$ whose homogeneous coordinates $[(z_{ij})]$ are given by

$$z_{ij} = x_i y_j$$

for each pair of indices i, j. Another way to describe this is to say that the homogeneous coordinates of $\sigma_{m,n}([x_0 : \cdots : x_m], [y_0 : \cdots : y_n])$ are given by the product matrix

$$\begin{pmatrix} x_0 \\ \vdots \\ x_m \end{pmatrix} \begin{pmatrix} y_0 & \cdots & y_n \end{pmatrix}.$$

This matrix has rank 1.

Let

$$\Sigma_{m,n} = \{[z_{00} : \cdots : z_{mn}] \in \mathbb{P}^N : \text{the matrix } (z_{ij}) \text{ has rank } 1\}.$$

Some linear algebra shows that we can describe $\Sigma_{m,n}$ as the subset of $\mathbb{P}^N$ where all 2×2 submatrices of the matrix (z_{ij}) have zero determinant. Thus $\Sigma_{m,n}$ is a projective algebraic set, defined by the equations

$$z_{ij} z_{k\ell} = z_{kj} z_{i\ell} \text{ for } 0 \leq i, k \leq m,\ 0 \leq j, \ell \leq n.$$

Lemma 16.3. $\sigma_{m,n}$ is a bijection from $\mathbb{P}^m \times \mathbb{P}^n$ to $\Sigma_{m,n}$.

Proof. (This proof is not part of the course.)

We can define an inverse to $\sigma_{m,n}$ as follows:

Let $a \in \Sigma_{m,n}$, and let A be a matrix giving homogeneous coordinates for a. A is not the zero matrix (because it is a set of homogeneous coordinates), so we can pick j such that the j-th column of A contains a non-zero entry. Define $\pi_1(a) \in \mathbb{P}^m$ to be the point with homogeneous coordinates given by the j-th column of A, that is,

$$\pi_1(a) = [A_{1j} : \cdots : A_{mj}].$$

This is independent of the choice of j because the matrix has rank 1 (every non-zero column is a multiple of every other non-zero column).

Similarly we can pick i such that the i-th row of A contains a non-zero entry, and define $\pi_2(a) \in \mathbb{P}^n$ to be the point with homogeneous coordinates given by the i-th row of A. Again this is independent of the choice of i.

Now $(\pi_1, \pi_2) \colon \Sigma_{m,n} \to \mathbb{P}^m \times \mathbb{P}^n$ is an inverse to $\sigma_{m,n}$. □

This construction shows that the projections $\pi_1 \colon \Sigma_{m,n} \to \mathbb{P}^m$ and $\pi_2 \colon \Sigma_{m,n} \to \mathbb{P}^n$ are regular maps (each column of the matrix is non-zero on a Zariski open subset of $\Sigma_{m,n}$).

The map $\sigma_{m,n} \colon \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$ is called the **Segre embedding** and its image $\Sigma_{m,n} \subseteq \mathbb{P}^N$ is called the **Segre variety**.

17. Completeness of projective varieties

Closed subsets of the Segre variety.

Example. When $m = n = 1$, $N = 3$. The Segre variety $\Sigma_{m,n} \subseteq \mathbb{P}^3$ is defined by the single equation

$$\det \begin{pmatrix} z_{00} & z_{01} \\ z_{10} & z_{11} \end{pmatrix} = z_{00}z_{11} - z_{10}z_{01} = 0.$$

The Segre embedding is given by

$$\sigma_{m,n}([x_0 : x_1], [y_0 : y_1]) = [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1].$$

We see that $\Sigma_{m,n}$ is an irreducible quadric hypersurface in $\mathbb{P}^3$. Therefore by problem sheet 3, question 4, it is birational to $\mathbb{P}^2$. This is not surprising, because of course $\mathbb{P}^1 \times \mathbb{P}^1$ should have an open subset isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$, which in turn is an open subset of $\mathbb{P}^2$..

We gave an informal argument last time that $\mathbb{P}^1 \times \mathbb{P}^1$ is not be isomorphic to $\mathbb{P}^2$. A rigorous proof for this will be on the next problem sheet.

The Zariski topology on $\mathbb{P}^N$ induces a subspace topology on $\Sigma_{m,n}$. One can check that this topology is the same as what we expect, namely:

Lemma 17.1. Let $V \subseteq \mathbb{P}^m \times \mathbb{P}^n$. Then $\sigma_{m,n}(V) \subseteq \Sigma_{m,n}$ is closed if and only if

$$V = \{([x_0 : \dots : x_m], [y_0 : \dots : y_n]) : f_i(x_0, \dots, x_m, y_0, \dots, y_n = 0) \text{ for } 1 \leq i \leq s\}$$

where $f_1, \dots, f_s \in k[X_0, \dots, X_m, Y_0, \dots, Y_n]$ are bihomogeneous polynomials.

Proof. (not part of the course)

Suppose that $\sigma_{m,n}(V)$ is Zariski closed in $\mathbb{P}^N$. Then it is defined by some homogeneous polynomials $g_r(z_{00}, \dots, z_{mn})$. Making the substitutions $z_{ij} = x_iy_j$ (as in the definition of $\sigma_{m,n}$), we get a finite set of polynomials which define V. If g_r is homogeneous in z_{ij} of degree d_r, then $g_r \circ \sigma_{m,n}$ is bihomogeneous of degree (d_r, d_r).

It is easy to see that if V is defined by polynomials f_r, where f_r is bihomogeneous of degree (d_r, d_r), then we can reverse this process to get homogeneous polynomials in z_{ij} which define $\sigma_{m,n}(V)$.

But what if the defining polynomials for V include some f which is bihomogeneous of degree (d, e), where $d \neq e$? Without loss of generality, suppose that $d > e$. Then $f = 0$ is equivalent to the system of equations

$$x_0^{d-e} f = 0, \; \dots, \; x_m^{d-e} f = 0$$

and these equations are bihomogeneous of degree (d, d). □

If $V \subseteq \mathbb{P}^m$ and $W \subseteq \mathbb{P}^n$ are projective algebraic sets, then $V \times W \subseteq \mathbb{P}^m \times \mathbb{P}^n$ is Zariski closed: the homogeneous polynomials defining V become bihomogeneous polynomials of bidegree $(d, 0)$ while those defining W become bihomogeneous polyomials of bidegree $(0, e)$.

Similarly, if $V \subseteq \mathbb{P}^m$ and $W \subseteq \mathbb{P}^n$ are quasi-projective algebraic sets, then the product $V \times W$ is also quasi-projective (it is the intersection of an open subset and a closed subset in $\mathbb{P}^m \times \mathbb{P}^n$, and therefore in $\mathbb{P}^N$ via the Segre embedding).

Graphs of regular functions.

Example. One useful example of a subvariety of a product is the graph of a regular function.

Let $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ be quasi-projective algebraic sets, and let $\varphi\colon V \to W$ be a regular map. The **graph** of φ is

$$\Gamma = \{(x, y) \in V \times W : y = \varphi(x)\}.$$

To check that this is closed in $V \times W$, observe that Γ is the preimage of the diagonal $\Delta \subseteq \mathbb{P}^m \times \mathbb{P}^m$ under the regular map

$$(\iota \circ \varphi, \iota)\colon V \times W \to \mathbb{P}^m \times \mathbb{P}^m$$

where ι denotes the inclusion map $W \to \mathbb{P}^m$. Since $(\iota \circ \varphi, \iota)$ is a regular map, it is continuous. Therefore it suffices to check that the diagonal is a Zariski closed subset of $\mathbb{P}^m \times \mathbb{P}^m$. This is true because we can describe the diagonal by bihomogeneous equations as follows:

$$\Delta = \{([x_0 : \cdots : x_m], [y_0 : \cdots : y_m]) : x_i y_j = x_j y_i \text{ for all } i, j\}.$$

Images of projective varieties.

The following is a key property of projective algebraic varieties, which is analogous to "compactness" in for Hausdorff topological spaces.

Theorem 17.2. Let V be a projective variety. Let $\varphi\colon V \to W$ be a regular map into any quasi-projective variety. Then the image of φ is Zariski closed.

Clearly the theorem is false if V is not projective: consider the projection of the hyperbola $\{(x, y) : xy = 1\}$ onto one of the axes.

Before proving Theorem 17.6, we shall give some important corollaries.

Corollary 17.3. Every regular function on an irreducible projective variety is constant.

Proof. Let V be an irreducible projective variety and $\varphi\colon V \to \mathbb{A}^1$ a regular function. Let $\iota\colon \mathbb{A}^1 \to \mathbb{P}^1$ be the natural inclusion.

Then $\iota \circ \varphi\colon V \to \mathbb{P}^1$ is a regular map, so by Theorem 17.2, its image is a closed subset of $\mathbb{P}^1$. But the image of $\iota \circ \varphi$ is contained in $\mathbb{A}^1$, so it cannot be all of $\mathbb{P}^1$. Therefore the image of φ is finite.

Since V is irreducible, its image is also irreducible and therefore consists of a single point. □

Thus projective algebraic sets are essentially "opposite" to affine ones, since an affine algebraic set is determined by its ring of regular functions while a projective algebraic set has no regular functions except constants.

Corollary 17.4. The image of a regular map from an irreducible projective variety to an affine variety is a point.

Proof. Suppose we have a regular map $\varphi\colon V \to W$, where V is projective and irreducible and W is affine. We can suppose that $W \subseteq \mathbb{A}^m$, and let $X_1, \dots, X_m$ denote the coordinate functions on W. Then $X_1 \circ \varphi, \dots, X_m \circ \varphi$ are all constant by Corollary 17.3, and so φ is constant. □

Lemma 17.5. Let $V \subseteq \mathbb{P}^n$ be an infinite projective algebraic set and let $H \subseteq \mathbb{P}^n$ be a hyperplane. Then the intersection $V \cap H$ is non-empty.

Proof. Suppose for contradiction that $V \cap H = \emptyset$. Then $V \subseteq \mathbb{P}^n \setminus H$, which is isomorphic to $\mathbb{A}^n$. Hence we get an injective regular map $\iota\colon V \to \mathbb{A}^n$.

By Corollary 17.4, ι is constant on each irreducible component of V. Since ι is injective, each irreducible component of V is a point.

But V has only finitely many irreducible components, so this contradicts the hypothesis that V is infinite. □

Definition of completeness.

Theorem 17.2 is equivalent to the following theorem, which will be a more convenient statement to prove.

Theorem 17.6. Let V be a projective variety. For any quasi-projective variety W, the second projection $p_2\colon V \times W \to W$ maps closed sets to closed sets.

Again, we can see that Theorem 17.6 does not apply when V is not projective by taking $V = W = \mathbb{A}^1$ and taking the hyperbola as a closed subset of $V \times W$.

Theorem 17.6 ⇒ Theorem 17.2: Apply Theorem 17.6 to the graph Γ of $\varphi\colon V \to W$, using that $\mathrm{im}(\varphi) = p_2(\Gamma)$.

Theorem 17.2 ⇒ Theorem 17.6: Apply Theorem 17.2 it to $\pi_2 \circ \iota\colon Z \to W$ where ι is the inclusion map $Z \to V \times W$.

Definition. A variety V is **complete** if it satisfies the conclusion of Theorem 17.6. In other words, for every quasi-projective variety W, the second projection $p_2\colon V \times W \to W$ maps closed sets to closed sets.

For quasi-projective varieties, complete is equivalent to projective, but if we go beyond the world of quasi-projective varieties (we have not defined non-quasi-projective varieties at all in this course) then it is possible to find algebraic varieties which are complete but not projective.

Completeness is the natural analogue in algebraic geometry for compactness in topology; this is justified by the following result from topology. (In the lecture, I included the word "Hausdorff" in this lemma, but it is not needed.)

Lemma 17.7. Let S be a topological space. S is compact if and only if, for every topological space T, the second projection map $S \times T \to T$ maps closed sets to closed sets.

We remark that, over the complex numbers, an algebraic variety is complete if and only if it is compact for the analytic topology (this is hard to prove).

18. Proof of completeness

Affine open covers.

By definition, every quasi-projective algebraic set is contained in a projective algebraic set. We can use this to reduce some proofs for quasi-projective algebraic sets to the projective case (proving from the outside in). On the other hand, it is often useful to know that we can find affine varieties as open sets *inside* each quasi-projective algebraic set. This can be used to reduce some proofs to the affine case (proving from the inside out).

Lemma 18.1. Let V be a quasi-projective variety. For every point $x \in V$, there exists an open set $U \subseteq V$ which contains x and is (isomorphic to) an affine variety.

Proof. Write $V = V_0 \cap U_0$ where $V_0 \subseteq \mathbb{P}^n$ is closed and $U_0 \subseteq \mathbb{P}^n$ is open.

Given a point $x \in V$, we may assume that x is in $\mathbb{A}^n \subseteq \mathbb{P}^n$ (embedded by setting $X_0 = 1$ – we can achieve this by changing the coordinate system if necessary).

Since $\mathbb{P}^n \setminus U_0$ is a projective algebraic set which does not contain x, there is some homogeneous polynomial f which vanishes on $\mathbb{P}^n \setminus U_0$ but not at x. Then x is contained in the set

$$U = V_0 \cap D(f) = V \cap D(f)$$

where $D(f) = \{y \in \mathbb{A}^n : f(1, y_1, \dots, y_n) \neq 0\}$. (We have $V_0 \cap D(f) = V \cap D(f)$ because $D(f) \subseteq U_0$.)

U is an open subset of V because $D(f)$ is an open subset of $\mathbb{P}^n$.

U is a closed subset of $D(f)$, so in order to show that U is an affine variety, it suffices to show that $D(f)$ is an affine variety. We can prove this using the "hyperbola trick": consider the set

$$E(f) = \{(y_1, \dots, y_n, z) \in \mathbb{A}^{n+1} : z.f(y_1, \dots, y_n) = 0\}.$$

$E(f)$ is an affine algebraic set in $\mathbb{A}^{n+1}$, and projection onto the first n coordinates gives an isomorphism between $E(f)$ and $D(f)$. □

Proof of completeness.

We will now prove the completeness of projective varieties, in the form of Theorem 17.6 (which we recall for convenience).

Theorem 18.2. Let V be a projective variety. For any quasi-projective variety W, the second projection map $p_2 \colon V \times W \to W$ maps closed sets to closed sets.

Let Z be a closed subset of $V \times W$.

By Lemma 18.1, we may cover W by open sets U_α such that each U_α is an affine variety. According to the topological fact from the proof of Lemma 14.1, in order to show that $p_2(Z)$ is closed in W, it suffices to show that $p_2(Z) \cap U_\alpha$ is closed in U_α for every α. In other words (replacing W by U_α), it suffices to prove Theorem 18.2 for the case where W is affine.

Then we can replace $V \subseteq \mathbb{P}^m$ by $\mathbb{P}^m$ and $W \subseteq \mathbb{A}^n$ by $\mathbb{A}^n$ ($Z \subseteq V \times W$ is closed in $\mathbb{P}^m \times \mathbb{A}^n$). The benefit of doing this is that it simplifies the algebra when we change everything into coordinates. Thus it suffices to prove the following special case of Theorem 18.2.

Theorem 18.3. The second projection map $p_2 \colon \mathbb{P}^m \times \mathbb{A}^n \to \mathbb{A}^n$ maps closed sets to closed sets.

Proof. We can describe a Zariski closed subset $Z \subseteq \mathbb{P}^m \times \mathbb{A}^n$ as the zero set of some polynomials $f_1, \dots, f_r \in k[X_0, \dots, X_m, Y_1, \dots, Y_n]$ which are homogeneous with respect to $X_0, \dots, X_m$. ($Y_1, \dots, Y_n$ are affine coordinages, so there is no homogeneity condition with respect to them.)

For each point $(y_1, \dots, y_n) \in \mathbb{A}^n$, we can substitute the values $(y_1, \dots, y_n)$ into these polynomials and get a projective algebraic set

$$Z_{\underline{y}} = \{[x_0 : \cdots : x_m] \in \mathbb{P}^n : f_i(\underline{x}, \underline{y}) = 0 \text{ for all } i\}.$$

Observe that $\underline{y} \in p_2(Z)$ if and only if $Z_{\underline{y}}$ is non-empty.

Let $I_{\underline{y}}$ denote the ideal in $k[X_0, \dots, X_m]$ generated by the polynomials

$$f_0(X_0, \dots, X_m, \underline{y}), \dots, f_r(X_0, \dots, X_m, \underline{y}).$$

By the Projective Nullstellensatz, $Z_{\underline{y}}$ is non-empty if and only if $\operatorname{rad} I_{\underline{y}}$ is not equal to either the full ring $k[X_0, \dots, X_m]$ or to the ideal $(X_0, \dots, X_m)$. It is easy to see that this is equivalent to: $I_{\underline{y}}$ does not contain S_d for any positive integer d, where S_d denotes the set of all homogeneous polynomials of degree d in $k[X_0, \dots, X_m]$.

For each positive integer d, write

$$W_d = \{(y_1, \dots, y_n) \in \mathbb{A}^n : I_{\underline{y}} \not\supseteq S_d\}.$$

We have shown that $p_2(Z) = \bigcap_{d \in \mathbb{N}} W_d$.

Let the polynomials $f_0, \dots, f_r$ have degrees $d_0, \dots, d_r$ with respect to the X variables. We shall show that W_d is closed for $d \geq \max(d_0, \dots, d_r)$. Since the W_d are a descending chain of sets, this is sufficient to show that $p_2(Z)$ is closed.

If $g \in S_d$, then $g \in I_{\underline{y}}$ if and only if we can write

$$g(X_0, \dots, X_m) = \sum_{i=1}^{r} f_i(X_0, \dots, X_m, \underline{y})\, h_i(X_0, \dots, X_m)$$

for some homogeneous polynomials $h_1, \dots, h_r$, where $\deg h_i = d - d_i$. Hence $S_d \cap I_{\underline{y}}$ is the image of the linear map $\alpha_{d,\underline{y}} \colon \bigoplus_{i=1}^r S_{d-d_i} \to S_d$ given by

$$\alpha_{d,\underline{y}}(h_1, \dots, h_r) = \sum_{i=1}^{r} f_i(X_0, \dots, X_m, \underline{y})\, h_i(X_0, \dots, X_m).$$

Therefore

$$\begin{aligned} W_d &= \{\underline{y} \in \mathbb{A}^n : \alpha_{d,\underline{y}} \text{ is not surjective}\} \\ &= \{\underline{y} \in \mathbb{A}^n : \operatorname{rk} \alpha_{d,\underline{y}} < \dim S_d\} \\ &= \{\underline{y} \in \mathbb{A}^n : \text{all } (\dim S_d \times \dim S_d) \text{ submatrices of } \alpha_{d,\underline{y}} \text{ have determinant } 0\} \end{aligned}$$

(where we fix bases for S_d and $\bigoplus_i S_{d-d_i}$ and use these to write $\alpha_{d,\underline{y}}$ as a matrix). The determinants of these submatrices are polynomials in $y_1, \dots, y_n$, proving that W_d is Zariski closed in $\mathbb{A}^n$. ☐

19. Definition of dimension

Dimension and transcendence degree.

We want to define the dimension of algebraic varieties. There are several different definitions, all equivalent but each being useful in different situations. Note of these definitions is particularly obvious, so we begin by listing some properties that the "dimension" of an irreducible quasi-projective variety V ought to have. (We only consider irreducible varieties here, because a reducible variety might have components of different dimensions so it is harder to be confident about what properties the dimension of a reducible variety should have.)

(1) $\dim V$ is a nonnegative integer.
(2) $\dim V = 0$ if and only if V is a point (remember that we are assuming that V is irreducible).
(3) $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$.
(4) If U is an open subset of V, then $\dim U = \dim V$ (note that this holds for manifolds in differential geometry).
(5) If V and W are birational, then $\dim V = \dim W$ (this follows from (5)).

We can generalise property (5) to generically finite rational maps, which are defined as follows.

Definition. Let V and W be irreducible quasi-projective varieties. A dominant rational map $\varphi\colon V \dashrightarrow W$ is **generically finite** if there is a non-empty open set $U \subseteq W$ such that $\varphi^{-1}(x)$ is finite for every $x \in U$.

Note: there is more than one possible definition of "generically finite" for non-dominant rational maps. I shall avoid the issue by only using the words "generically finite" when the map is dominant.

Now we expect that

(6) If there exists a generically finite dominant rational map $\varphi\colon V \dashrightarrow W$, then $\dim V = \dim W$.

It turns out that these properties are enough to tell us the dimension of every irreducible quasi-projective variety, thanks to the following lemma.

Lemma 19.1. Let V be an irreducible quasi-projective variety. Then there exists a generically finite dominant rational map $V \dashrightarrow \mathbb{P}^d$ for some d.

Proof. By Lemma 18.1, V has a non-empty affine open subset $U \subseteq V$. By Corollary 11.5 (which we used in the proof of the Nullstellensatz), U is birational to a hypersurface H in some affine space $\mathbb{A}^n$. Taking the projective closure $\overline{H}$ of H in $\mathbb{P}^n$, we conclude that V is birational to $\overline{H}$.

Now projection from any point $p \in \mathbb{P}^n \setminus \overline{H}$ gives a generically finite dominant rational map $\overline{H} \dashrightarrow \mathbb{P}^{n-1}$. □

Using properties (3) and (6), we can calculate $\dim V$ by finding a generically finite dominant rational map $V \dashrightarrow \mathbb{P}^d$ and then saying that $\dim V = d$. There is one problem with this definition: maybe we can find generically finite dominant rational maps from V to two different projective spaces, giving two values for $\dim V$.

Fortunately this cannot happen, which is proved using the notion of transcendence degree from algebra (note that we already made use of the idea of transcendence degree in the proof of Corollary 11.5, even if we did not prove it).

Definition. Let k and K be fields, with $k \subseteq K$. The **transcendence degree** of K over k is the size of a maximal k-algebraically independent set in K. (By an algebraic theorem, all maximal k-algebraically independent sets in K have the same size, so this is well-defined.)

Definition. The **dimension** of an irreducible quasi-projective variety V is the transcendence degree (over k) of the field of rational functions $k(V)$.

This definition satisfies property (3) above: $k(\mathbb{P}^n) = k(\mathbb{A}^n) = k(X_1, \dots, X_n)$ has transcendence degree n because $\{X_1, \dots, X_n\}$ is a maximal algebraically independent set. It clearly also satisfies property (5). We need to prove that it satisfies property (7); it will then be easy to deduce the rest of the properties listed above.

Lemma 19.2. Let V and W be irreducible quasi-projective varieties. If $\varphi \colon V \dashrightarrow W$ is a generically finite dominant rational map, then

$$\operatorname{trdeg}(k(V)/k) = \operatorname{trdeg}(k(W)/k).$$

Proof. We can replace V by the open subset $\operatorname{dom} \varphi$, so that φ becomes a regular map. Using Lemma 18.1, we can replace V and W by affine open subsets, and then replace V by the graph of φ in $V \times W$. Hence it suffices to assume that $\varphi = p_{2|V}$, where p_2 is the projection $\mathbb{A}^{m+n} \to \mathbb{A}^n$ and V is a closed subset of $\mathbb{A}^{m+n}$.

By breaking up φ into projections $\mathbb{A}^{m+n} \to \mathbb{A}^{m+n-1} \to \cdots \to \mathbb{A}^{1+n} \to \mathbb{A}^n$, we may reduce to the case $m = 1$.

Since φ is a dominant rational map, it induces an injection of fields $\varphi^* \colon k(W) \to k(V)$. We have to prove that the resulting field extension $k(V)/\varphi^*(k(W))$ is algebraic, and hence that the transcendence degrees are the same.

Look at the coordinate function X_1 on V. Because φ is a projection, $X_2, \dots, X_{1+n}$ on V are all in $\varphi^*(k(W))$ and so the field $k(V)$ is generated by $\varphi^* k(W)$ and X_1.

Since φ is generically finite, V is strictly contained in $W \times \mathbb{A}^1$. Hence there is a non-zero polynomial $f \in k[W][X_{n+1}]$ which vanishes on V. This gives an $k(W)$-algebraic relation satisfied by $X_{1|V}$ in $k(V)$. Now $k(V)$ is generated by $k(W)$ and $X_{1|V}$, so $k(V)$ is algebraic over $k(W)$ as required. ☐

20. Dimension of intersection with a hyperplane

Facts about dimension.

Let V, W be irreducible quasi-projective algebraic varieties.

(1) If $\varphi\colon V \to W$ is a dominant rational map, then $\dim W \leq \dim V$. This follows from the fact that φ^* is an injection $k(W) \to k(V)$.

(2) $\dim(V \times W) = \dim V + \dim W$. This holds because if $\varphi\colon V \dashrightarrow \mathbb{A}^d$ and $\psi\colon W \dashrightarrow \mathbb{A}^e$ are generically finite dominant rational maps, then $(\varphi, \psi)\colon V \times W \dashrightarrow \mathbb{A}^{d+e}$ is a generically finite dominant rational map.

Lemma 20.1. Let V be an irreducible quasi-projective variety and let W be an irreducible closed subset of V. Then $\dim W \leq \dim V$.

Proof. It suffices to prove the lemma for irreducible V and W. Using Lemma 18.1, we may assume that V and W are affine algebraic sets in some affine space $\mathbb{A}^n$.

Let $d = \dim V$. Then any $d+1$ of the coordinate functions are algebraically dependent in $k(V)$. In other words, there exists a polynomial $f \in k[T_1, \dots, T_{d+1}]$ such that $f(X_{i_1|V}, \dots, X_{i_{d+1}|V}) = 0$ in $k(V)$. Since $W \subseteq V$, this relation still holds after restricting to W:

$$f(X_{i_1|W}, \dots, X_{i_{d+1}|W}) = 0 \text{ in } k[W].$$

But the field of functions $k(W)$ is generated (as a k-field) by $X_{1|W}, \dots, X_{n|W}$, so this establishes that $\operatorname{trdeg}(k(W)/k) \leq d$. □

We will later show that equality can only happen in Lemma 20.1 if $W = V$. (We could prove this algebraically now, but instead we will end the algebraic proofs using transcendence degree here and prove everything else geometrically. This means that we will need several steps before improving Lemma 20.1 to a strict inequality.)

Dimension of a reducible variety.

So far we have defined the dimension of an irreducible quasi-projective variety. The **dimension** of a reducible variety is defined to be the maximum of the dimensions of the irreducible components.

To explain why this is a sensible definition (why not minimum for example?), note that if $V = V_1 \cup \dots \cup V_r$ are the irreducible components of V, then $V_i \subseteq V$ so we should have $\dim V_i \leq \dim V$ for each i.

Intersection with a hyperplane.

We begin by studying intersections between a projective algebraic set and hypersurfaces (for today, just hyperplanes). This is much simpler for projective varieties than for quasi-projective varieties, because then we know that there can be no intersections "hiding at infinity." The expectation is that, if V is an algebraic set and H is a hypersurface, then $\dim(V \cap H)$ should usually be $\dim V - 1$ (because it is just adding one more equation to the equations defining V).

Before proving this, we need a couple of lemmas.

Firstly, there is no room between a hypersurface and $\mathbb{P}^n$ to squeeze in another (irreducible) algebraic set.

Lemma 20.2. Let $H \subseteq \mathbb{P}^n$ be a hyperplane (or more generally a hypersurface). Let $V \subseteq \mathbb{P}^n$ be an irreducible projective algebraic set.

If $V \neq \mathbb{P}^n$ and $V \neq H$, then $H \not\subseteq V$.

Proof. Look at ideals of polynomials which vanish on H and V. □

Secondly, we need to know how projection interacts with dimension.

Lemma 20.3. Let $V \subseteq \mathbb{P}^n$ be an irreducible projective algebraic set.

Let $p \in \mathbb{P}^n$ and let $Z \subseteq \mathbb{P}^n$ be a hyperplane such that $p \notin Z$. Let $\pi \colon \mathbb{P}^n \dashrightarrow Z$ be the projection from p onto Z.

If $p \notin V$, then $\pi(V)$ is a Zariski closed subset of Z and $\pi_{|V} \colon V \to \pi(V)$ is generically finite, so $\dim \pi(V) = \dim V$.

Proof. The projection π is regular on $\mathbb{P}^n \setminus \{p\}$, and in particular it is regular on V. Since V is complete, $\pi(V)$ is a closed subset of Z.

Now $\pi_{|V} \colon V \to \pi(V)$ is certainly dominant (indeed it is surjective). In order to show that $\dim \pi(V) = \dim V$, it suffices to show that $\pi_{|V}$ is generically finite.

Consider a point $y \in \pi(V)$. The preimage of y under $\pi_{|V}$ is the intersection $V \cap L_{py}$, where L_{py} is the line through p and y. Now $V \cap L_{py}$ is a closed subset of L_{px}. Furthermore $V \cap L_{py} \neq L_{py}$ because $p \notin V$. Because $L_{py} \cong \mathbb{P}^1$, we conclude that $V \cap L_{py}$ must be finite. In other words $\pi_{|V}^{-1}(y)$ is finite for all $y \in \pi(V)$, and so $\pi_{|V}$ is generically finite. □

Now we are ready to prove the result on the dimension of intersection with a hyperplane. Note the exceptional cases: if H contains a component of V of maximum dimension, then $\dim(V \cap H) = \dim V$, while if $\dim V = 0$ then $V \cap H$ might be empty. (If $\dim V > 0$, then $V \cap H \neq \emptyset$ by Lemma 17.5.)

Proposition 20.4. Let $V \subseteq \mathbb{P}^n$ be a projective algebraic set. Let $H \subseteq \mathbb{P}^n$ be a hyperplane which does not contain any irreducible component of V.

If $\dim V > 0$, then $V \cap H$ is non-empty and $\dim(V \cap H) = \dim V - 1$.

Proof. First replace V by an irreducible component V_1 such that $\dim V_1 = \dim V$. Thus we may assume that V is irreducible.

The proof is by induction on n, the dimension of the ambient space.

The base case of the induction is when $V = \mathbb{P}^n$ (remember we are inducting on n, not $\dim V$). This is trivial: $V \cap H = H \cong \mathbb{P}^{n-1}$ certainly has dimension $n-1$.

Otherwise, $V \neq \mathbb{P}^n$. We will project into $\mathbb{P}^{n-1}$. In order to use Lemma 20.3, we need to project from a point $p \notin V$. In order for the projection to interact nicely with H, we need $p \in H$. Fortunately, we can use Lemma 20.2 to show that a suitable p exists.

We are assuming $V \neq \mathbb{P}^n$ while the hypothesis of the proposition tells us that $V \not\subseteq H$, so $V \neq H$. Therefore by Lemma 20.2, $H \not\subseteq V$ (the opposite way round to our hypothesis!), so we can select a point $p \in H$ such that $p \notin V$.

Choose a hyperplane $Z \subseteq \mathbb{P}^n$ such that $p \notin Z$ (it doesn't matter which we choose). Let $\pi \colon \mathbb{P}^n \to Z$ be the projection from p onto Z.

Because $p \in H$, all lines through p and a point of H lie entirely in H. Therefore

$$x \in H \setminus \{p\} \Leftrightarrow \pi(x) \in H \cap Z$$

and consequently
$$\pi(V \cap H) = \pi(V) \cap (H \cap Z).$$
This implies that $\pi(V) \not\subseteq H \cap Z$, because $V \not\subseteq H$.

By completeness, $\pi(V)$ is a closed subset of $Z \cong \mathbb{P}^{n-1}$, while $H \cap Z$ is a hyperplane in Z. Furthermore, $\pi(V)$ is irreducible and we have shown that $\pi(V) \not\subseteq H \cap Z$. Therefore, by induction, we have
$$\dim(\pi(V) \cap (H \cap Z)) = \dim \pi(V) - 1.$$
We conclude by using Lemma 20.3, which tells us that
$$\dim \pi(V) = \dim V \quad \text{and} \quad \dim \pi(V \cap H) = \dim(\pi(V) \cap (H \cap Z)). \qquad \square$$

21. Dimension and equations

Veronese embedding.

In order to generalise Proposition 20.4 from intersections with hyperplanes to intersections with hypersurfaces, we use the Veronese embedding. This is defined as follows.

Let d and n be positive integers and let $N = \binom{n+d}{d} - 1$. There are $N+\ 1$ monomials of degree d in variables $X_0, \dots, X_n$ (expressions of the form $X_0^{a_0} X_1^{a_1} \cdots X_n^{a_n}$ where $a_0, \dots, a_n \in \mathbb{Z}_{\geq 0}$ with $a_0 + \cdots + a_n = d$). We define a regular map $\nu_{n,d} \colon \mathbb{P}^n \to \mathbb{P}^N$ by writing down all these monomials of degree d (in some order). For example, for $n = d = 2$ we get $N = 5$ and

$$\nu_{2,2}([X_0 : X_1 : X_2]) = [X_0^2 : X_1^2 : X_2^2 : X_0X_1 : X_1X_2 : X_0X_2].$$

This is called the **degree d Veronese embedding** of $\mathbb{P}^n$.

By completeness, the image of $\nu_{n,d}$ is a projective algebraic set $V_{n,d} \subseteq \mathbb{P}^N$. One can write down explicit polynomials defining this algebraic set (they are determinants of 2×2 matrices). Importantly, $\nu_{n,d}$ is an isomorphism $\mathbb{P}^n \to V_{n,d}$ (proving this is elementary but the notation gets pretty complicated).

The benefit of doing all this is that, if $H \subseteq \mathbb{P}^n$ is a hypersurface defined by some homogeneous polynomial $f = \sum_I a_I \underline{X}^I$ of degree d, then because the monomials of degree d become individual homogeneous coordinates via the Veronese embedding, the equation for $\nu_{n,d}(H)$ is a linear equation $\sum_I a_I Z_I = 0$. Thus $\nu_{n,d}(H) = V_{n,d} \cap Z$ for some hyperplane $Z \subseteq \mathbb{P}^N$.

Therefore, instead of studying the intersection between $V \subseteq \mathbb{P}^n$ and a hypersurface $H \subseteq \mathbb{P}^n$, we can instead study the intersection between $\nu_{n,d}(V) \subseteq V_{n,d} \subseteq \mathbb{P}^N$ and a hyperplane $Z \subseteq \mathbb{P}^N$. Because $\nu_{n,d}$ is an isomorphism, we can use Proposition 20.4 to deduce the same result for intersections with hypersurfaces:

Theorem 21.1. Let $V \subseteq \mathbb{P}^n$ be a projective algebraic set. Let $H \subseteq \mathbb{P}^n$ be a hypersurface which does not contain any irreducible component of V.

If $\dim V > 0$, then $V \cap H$ is non-empty and $\dim(V \cap H) = \dim V - 1$.

Dimension of proper closed subsets.

I mentioned last time that we can strengthen Lemma 20.1 to a strict inequality, as long as $W \neq V$. In this lemma, it is essential that V is irreducible, whereas in Lemma 20.1, that condition is not necessary.

Lemma 21.2. Let V be an *irreducible* quasi-projective variety and let W be a closed subset of V. If $W \neq V$, then $\dim V < \dim W$.

Proof. Suppose that V is a quasi-projective algebraic set in $\mathbb{P}^n$. Let $\overline{V}$ and $\overline{W}$ denote the closures of V and W respectively in $\mathbb{P}^n$. Because W is closed in V and not equal to V, $\overline{V} \neq \overline{W}$.

So we can pick a homogeneous polynomial $f \in k[X_0, \dots, X_n]$ which vanishes on $\overline{W}$ but not on $\overline{V}$. Let H be the hypersurface defined by f. Then $\overline{W} \subseteq \overline{V} \cap H$ so Theorem 21.1 implies that

$$\dim \overline{W} \leq \dim(\overline{V} \cap H) = \dim \overline{V} - 1.$$

Since V is open in $\overline{V}$, $\dim V = \dim \overline{V}$ and similarly $\dim W = \dim \overline{W}$ which completes the proof. □

Dimension and equations.

What is the dimension of a subset of $\mathbb{P}^n$ defined by r homogeneous polynomial equations? We can try to work this out by applying Theorem 21.1 repeatedly.

The zero set of a single homogeneous polynomial f_1 is a hypersurface H_1, which we know has dimension $n-1$. The zero set of two homogeneous polynomials f_1, f_2 is an intersection $H_1 \cap H_2$ of two hypersurfaces. If f_1 and f_2 have no common factor, then H_2 does not contain any irreducible component of H_1 and so Theorem 21.1 tells us that $\dim(H_1 \cap H_2) = n - 2$.

But once we look at three homogeneous polynomials f_1, f_2, f_3, we try to apply Theorem 21.1 to $V = H_1 \cap H_2$ so we have to ask whether H_3 contains any irreducible component of $H_1 \cap H_2$. There is no easy condition to tell whether this is true (consider the examples from problem sheets 1 and 2: there were algebraic sets defined by two polynomials with no common factors; working out the irreducible components of the intersection was hard work). The best we can say is $\dim(H_1 \cap H_2 \cap H_3) = n - 2$ or $n - 3$.

As we repeat the process, controlling the irreducible components only gets harder. All we can say is that for each extra equation, the dimension goes down by either 0 or 1. By induction, we get is the following inequality.

Proposition 21.3. Let $f_1, \dots, f_r \in k[X_0, \dots, X_n]$ be homogeneous polynomials and let $V \subseteq \mathbb{P}^n$ be the zero set of these polynomials. If $r \leq n$, then

$$V \neq \emptyset \text{ and } \dim V \geq n - r.$$

Proof. Let H_i be the hypersurface defined by the equation $f_i = 0$. By Theorem 21.1, if H_i does not contain any irreducible component of $H_1 \cap \dots \cap H_{i-1}$, then

$$\dim(H_1 \cap \dots \cap H_i) = \dim(H_1 \cap \dots \cap H_{i-1}) - 1.$$

On the other hand, if H_i does contain an irreducible component of $H_1 \cap \dots \cap H_{i-1}$, then the dimension might not go down at all. In any case,

$$\dim(H_1 \cap \dots \cap H_i) \geq \dim(H_1 \cap \dots \cap H_{i-1}) - 1.$$

Iterating this proves the corollary. □

Complete intersections.

In reverse, we can ask: if $V \subseteq \mathbb{P}^n$ is a projective algebraic set of dimension $n-r$, do there exist r homogeneous polynomials which define V? Answer: not always.

For example,take the two planes in $\mathbb{P}^4$:

$$P_1 = \{x \in \mathbb{P}^4 : x_1 = x_2 = 0\}, \; P_2 = \{x \in \mathbb{P}^4 : x_3 = x_4 = 0\}.$$

These intersect in only one point, namely $[1 : 0 : 0 : 0 : 0]$. The union $P_1 \cup P_2$ has dimension 2 but it needs 4 equations to define it. (One can also find examples of *irreducible* 2-dimensional algebraic sets of $\mathbb{P}^4$ which are not set-theoretic complete intersections, with a singularity which looks like the intersection point of the two planes in $P_1 \cup P_2$.)

There are two relevant definitions. The first one is more in the style of this course, but the second one turns out to be more natural because it gives more algebraic information.

Definition. Let $V \subseteq \mathbb{P}^n$ be an algebraic set of dimension $n - r$.

V is a **set-theoretic complete intersection** if there exist r homogeneous polynomials such that V is the zero set of these polynomials.

V is a **complete intersection** if there exist r homogeneous polynomials which generate the ideal of V.

Being a complete intersection is a stronger property than being a set-theoretic complete intersection.

For example, a set of three non-collinear points in $\mathbb{P}^2$ is a set-theoretic complete intersection but not a complete intersection: there exist 2 polynomials defining this set, but you need 3 polynomials to generate its ideal.

We had to go to $\mathbb{P}^4$ to give explicit examples of non-set-theoretic complete intersections. It is an open question whether every irreducible algebraic set in $\mathbb{P}^3$ is a set-theoretic complete intersection.

Generalising to quasi-projective varieties.

Theorem 21.1 applies to irreducible quasi-projective algebraic sets $V \subseteq \mathbb{P}^n$ as well as projective algebraic sets, except that for a quasi-projective algebraic set it can happen that $V \cap H = \emptyset$ (Lemma 17.5 applies only to projective algebraic sets). The precise statement is as follows:

Theorem 21.4. Let $V \subseteq \mathbb{P}^n$ be an irreducible quasi-projective algebraic set.

Let $H \subseteq \mathbb{P}^n$ be a hypersurface which does not contain V.

If $V \cap H \neq \emptyset$, then

$$\dim(V \cap H) = \dim V - 1.$$

This is much harder to prove than Theorem 21.1, so we will omit the proof. One might attempt to prove Theorem 21.4 by writing V as $\overline{V} \cap U$, where $\overline{V}$ is the closure of V in $\mathbb{P}^n$ and U is an open set and then applying Theorem 21.1 to $\overline{V}$. The problem with this is that the maximum-dimension components of $\overline{V} \cap H$ might be contained in the closed set which is the complement of U, and then $V \cap H$ would have dimension less than $\dim(\overline{V} \cap H) = \dim V - 1$. Actually this can't happen: with harder work we can show that *every* irreducible component of $\overline{V} \cap H$ has dimension equal to $\dim V - 1$. You can do this either geometrically or using an algebraic result called Krull's Hauptidealsatz.

Of course the non-emptiness part of Proposition 21.3 does not generalise to affine sets, but the dimension inequality does provided we assume that the set is non-empty.

22. Topological definition of dimension and fibre dimension

Topological definition of dimension.

Our previous definition of dimension was algebraic. We can also describe the dimension of a variety in terms of its topology.

Theorem 22.1. Let V be a projective variety.

The dimension of V is the maximum integer d such that there exists a chain of irreducible closed subsets

$$V \supseteq V_d \supsetneq V_{d-1} \supsetneq \cdots \supsetneq V_0 \supsetneq \emptyset.$$

Some care is required in the statement of this theorem to get the numbering right! The point is that $\dim V_i = i$, so V_0 is still non-empty. Note that $V = V_d$ if and only if V is irreducible; all the other inclusions must be strict. In Theorem 22.1, it is essential to require all the V_i to be irreducible. Otherwise we could make the chain arbitrarily long by inserting reducible sets with more and more components, all of dimension i, in between V_i and V_{i+1}.

Proof. First we prove that such a sequence with $d = \dim V$ exists.

Choose V_d to be an irreducible component of V whose dimension is equal to $\dim V$. Choose H as in Proposition 20.4 applied to V_d. Let V_{d-1} be an irreducible component in $V_d \cap H$ such that

$$\dim V_{d-1} = \dim(V_d \cap H) = \dim V - 1.$$

We can repeat this procedure, getting $V_i \subsetneq V_{i+1}$ with $\dim V_i = i$ until we get to V_0 with $\dim V_0 = 0$.

In the other direction, to show that there is no such sequence with $d > \dim V$, this follows immediately from the fact that $\dim V_i < \dim V_{i+1}$ (Lemma 20.1). □

Just like Theorem 21.1, Theorem 22.1 generalises to quasi-projective varieties.

However, Theorem 22.1 is not really strong enough to be useful. For example, in $\mathbb{P}^n$, we can write down a chain of closed subsets

$$\mathbb{P}^n \supsetneq \mathbb{P}^{n-1} \supsetneq \mathbb{P}^{n-2} \supsetneq \cdots \supsetneq \mathbb{P}^1 \supsetneq \{\mathrm{pt}\} \supsetneq \emptyset.$$

This chain is *maximal* – we cannot insert another irreducible closed subset anywhere in the middle of it. But just exhibiting this chain is not enough to prove that $\dim \mathbb{P}^n = n$ – maybe there is a completely different chain which is longer.

It turns out that that can't happen: every maximal chain of irreducible closed subsets in an irreducible quasi-projective variety V has length equal to $\dim V$. This is another hard theorem, requiring the same work as proving Theorem 21.4 (about the intersection of a quasi-projective algebraic set with a hypersurface).

Fibre dimension theorem.

We have now seen several definitions of dimension. None of these is easy to compute for specific examples, except in simple cases. When we want to calculate the dimension of a particular variety, we often use the following powerful theorem.

Theorem 22.2. Let V, W be irreducible quasi-projective varieties and let $\varphi\colon V \to W$ be a surjective regular map. Then:

(i) For every $w \in W$, $\dim \varphi^{-1}(w) \geq \dim V - \dim W$.
(ii) There exists a non-empty open subset $U \subseteq W$ such that $\dim \varphi^{-1}(w) = \dim V - \dim W$ for all $w \in U$.

(The sets $\varphi^{-1}(w)$ for $w \in W$ are called the **fibres** of φ.)

For example, consider the projection from $\mathbb{A}^{n+m}$ to $\mathbb{A}^n$: all the fibres are copies of $\mathbb{A}^m$, which has dimension equal to $\dim \mathbb{A}^{n+m} - \dim \mathbb{A}^n$. Part (ii) of Theorem 22.2 tells us that *most* fibres have the "expected" dimension, as in this example, but there might be a proper closed subset of exceptions. Part (i) of Theorem 22.2 tells us that for the exceptional fibres, the dimension is always *bigger* than expected.

It is complicated to write down examples of surjective maps where there is a non-empty exceptional set using equations. So I shall cheat and give an example which is not surjective, only dominant (so the theorem does not actually apply to this example, but it still illustrates the idea that fibre dimension gets bigger over a closed subset). Consider $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ given by

$$\varphi(x, y) = (x, xy).$$

Consider the vertical line $L_x = \{(x, y) : y \in k\}$. When $x \neq 0$, φ restricts to an isomorphism $L_x \to L_x$. But when $x = 0$, φ maps all of L_0 down to $(0, 0)$. Hence the image of φ is $\big(\mathbb{A}^2 \setminus \{(0, y)\}\big) \cup \{(0, 0)\}$.

We see that, above the open set $\{(x, y) : x \neq 0\}$, the fibres of φ are single points i.e. with dimension $2 - 2 = 0$. On the other hand, above the point $(0, 0)$, the fibre $\varphi^{-1}((0, 0))$ is a line, so has dimension $1 \geq 2 - 2$.

We will not prove Theorem 22.2. The proof uses similar methods to Theorem 21.4, plus an induction.

We generally use this theorem in situations where we know the dimension of either V or W and want to work out the other. If we can work out $\dim \varphi^{-1}(w)$ for just a single $w \in W$, then we get an inequality. If we can work out $\dim \varphi^{-1}(w)$ for w in some open set then we can work out the desired dimension exactly.

An importantly special case: if there exists w such that $\dim \varphi^{-1}(w) = 0$, $\dim V = \dim W$.

Universal family of hypersurfaces.

The fibre dimension theorem is particularly useful when applied to "families of algebraic varieties" and "parameter spaces." These are a powerful feature of algebraic geometry: often we can consider some collection of algebraic varieties, and construct another algebraic variety which has one point for each variety in the collection. We may also be able to fit all the varieties of the collection together into a single big algebraic variety. This is different form other forms of geometry, where a "family of objects" rarely forms an object of the same type.

Definition. Let B be a quasi-projective variety. A **family of projective algebraic sets over** B is a Zariski closed subset $\mathcal{V} \subseteq B \times \mathbb{P}^n$.

For each $b \in B$, we write $\mathcal{V}_b = \{x \in \mathbb{P}^n : (b, x) \in V\}$ and call this a **fibre** of $\mathcal{V}$.

The set B is called the **base** or **parameter space** of the family.

This definition might seem rather abstract; to give some idea of what is going on, we will look at a simple example (we will see some more complex examples later). A hypersurface of degree d in $\mathbb{P}^n$ means the zero set of a non-zero homogeneous polynomial in $k[X_0, \ldots, X_n]$ of degree d. These polynomials form a vector space of dimension $\binom{n+d}{d}$.

If one homogeneous polynomial is a scalar multiple of another, then they define the same hypersurface. Hence we get a hypersurface $\mathcal{H}_a$ associated with each point $a \in \mathbb{P}^N$, where $N = \binom{n+d}{d} - 1$. (The homogeneous coordinates of a form the coefficients of the polynomial defining $\mathcal{H}_b$.)

These hypersurfaces form a family in $\mathbb{P}^N \times \mathbb{P}^n$ – we will finish this example in the next lecture.

23. Parameter spaces

Universal family of hypersurfaces.

Let's recall what we were doing last time: let $V_{n,d}$ denote the vector space of homogeneous polynomials in $k[X_0, \dots, X_n]$ of degree d. We can count the dimension of this vector space: $\dim V_{n,d} = \binom{n+d}{d}$. Let $P_{n,d}$ denote the projective space associated with $V_{n,d}$ i.e.

$$P_{n,d} = (V_{n,d} \setminus \{0\})/(\text{scalars}) \cong \mathbb{P}^N$$

where $N = \binom{n+d}{d} - 1$.

For a polynomial $f \in V_{n,d}$, let's write $[f]$ for the corresponding point in $P_{n,d}$ Using the basis for $V_{n,d}$ which consists of the monomials $X_0^{i_0} \cdots X_n^{i_n}$ (where $i_0 + \cdots + i_n = d$), we see that the homogeneous coordinates of $[f] \in P_{n,d}$ are given by the coefficients of f.

Each non-zero polynomial $f \in V_{n,d}$ defines a hypersurface $H_f \subseteq \mathbb{P}^n$. If f is a scalar multiple of g, then they define the same hypersurface: $H_f = H_g$. (This is not quite an if and only if, because things can go wrong with polynomials that do not generate a radical ideal. Try to come up with an example.)

Thus, instead of labelling hypersurfaces by polynomials $f \in V_{n,d}$ we can label them instead by points in $P_{n,d}$. This has two benefits:

(1) The association of hypersurfaces with points in $P_{n,d}$ is "almost" injective (it is injective for polynomials f which generate radical ideals – and these form a dense open subset of $P_{n,d}$).
(2) By using the projective base $P_{n,d}$ instead of the affine base $V_{n,d}$, we can take advantage of properties like completeness.

We can fit these hypersurfaces together into a family over the base $P_{n,d}$. In other words, there is a single closed set $\mathcal{H} \subseteq P_{n,d} \times \mathbb{P}^n$ such that the fibre

$$\mathcal{H}_{[f]} = \{x \in \mathbb{P}^n : ([f], x) \in \mathcal{H}\}$$

is the hypersurface defined by the polynomial f. To see that $\mathcal{H}$ is closed, we observe that it is defined by a polynomial equation which is bihomogeneous of degree $(1, d)$:

$$\mathcal{H} = \left\{ ([f], x) \in P_{n,d} \times \mathbb{P}^n : \sum_{\substack{0 \le i_0, \dots, i_n \le d \\ i_0 + \cdots + i_n = d}} f_{i_0 \cdots i_n} X_0^{i_0} \cdots X_n^{i_n} = 0 \right\} \qquad (*)$$

where $f_{i_0 \cdots i_n}$ denote the coefficients of the polynomial $f \in V_{n,d}$.

We call $\mathcal{H}$ the **universal family of hypersurfaces of degree** d **in** $\mathbb{P}^n$. We think of $P_{n,d} \cong \mathbb{P}^N$ as "the parameter space for hypersurfaces of degree d in $\mathbb{P}^n$."

Aside. The word "universal" here is related to the fact that every hypersurface of degree d appears as a fibre in this family, and most of them only appear once (if we work with schemes instead of varieties, then each hypersurface will really appear exactly once). However a rigorous definition of what it means for a family to be "universal" is more subtle than this, and too complicated to define in this course (it involves the notion of a "flat family of schemes").

Subsets of parameter spaces.

One of the benefits of parameter spaces and families of varieties is that they give us a way of talking about all varieties with some particular property at once. If we take a family $\mathcal{V} \subseteq B \times \mathbb{P}^n$ and consider the subset of fibres which satisfy an interesting geometric condition, then very often the corresponding set of points in the parameter space

$$\{b \in B : \mathcal{V}_b \text{ satisfies given condition}\}$$

is an open or closed subset of B.

As a simple example, if we fix a point $x \in \mathbb{P}^n$, then the set

$$\{b \in B : x \in \mathcal{V}_b\}$$

is a closed subset. This is the image of the closed set $(B \times \{x\}) \cap \mathcal{V} \subseteq B \times \mathbb{P}^n$ under the projection $p_1 \colon B \times \mathbb{P}^n \to B$, so it is closed because $\mathbb{P}^n$ is complete (Theorem 17.6).

Another example: the set

$$\{[f] \in P_{n,d} : f \text{ is irreducible}\}$$

is an open set, and so is the set

$$\{[f] \in P_{n,d} : f \text{ generates a radical ideal}\}.$$

– this will be on problem sheet 5.

Dimension counting.

An important use of families of varieties, and the fact that the family is itself a variety, is that we can calculate the dimension of the parameter space, or of interesting subsets of it, using the fibre dimension theorem. By doing this, we can show that certain sets are empty/non-empty/finite/infinite/equal or not equal to the entire parameter space.

Example. Consider the intersection of $n+1$ hypersurfaces in $\mathbb{P}^n$. From our earlier discussions of dimension, we expect that usually such an intersection should be empty (because $n+1 > n$), but of course sometimes it will be non-empty. By counting dimensions of parameter spaces, we can be more specific about how often "sometimes non-empty" occurs: we will prove that the subset of the parameter space where this intersection is non-empty is a closed subset, and then we will compare its dimension with the dimension of the entire parameter space.

What is the appropriate parameter space? We are looking at sequences of $n+1$ hypersurfaces, so the parameter space we need is $(P_{n,d})^{n+1}$. Then we will consider the subset

$$S = \Big\{(a_0, \dots, a_n) \in (P_{n,d})^{n+1} : \bigcap_{i=0}^{n} \mathcal{H}_{a_i} \neq \emptyset\Big\}.$$

The algebraic varieties we are interested in (intersections of $n+1$ hypersurfaces) should form a family over $(P_{n,d})^{n+1}$. More precisely, we want a family of algebraic varieties over $(P_{n,d})^{n+1}$ such that the fibre above $(a_0, \dots, a_n)$ is $\bigcap_{i=0}^{n} \mathcal{H}_{a_i}$. We can define this family by

$$\Sigma = \{(a_0, \dots, a_n, x) \in (P_{n,d})^{n+1} \times \mathbb{P}^n : x \in \mathcal{H}_{a_i} \text{ for all } i\}.$$

For each i, the condition $x \in \mathcal{H}_{a_i}$ is given by a polynomial (*), so Σ is a closed subset of $(P_{n,d})^{n+1} \times \mathbb{P}^n$. Thus it is a "family of algebraic varieties" in the sense we defined in the previous lecture.

24. Dimension counting example

Continuing from last time: we are interested in the set

$$S = \Big\{(a_0, \dots, a_n) \in (P_{n,d})^{n+1} : \bigcap_{i=0}^{n} \mathcal{H}_{a_i} \neq \emptyset\Big\}.$$

To study this, we will use the family

$$\Sigma = \{(a_0, \dots, a_n, x) \in (P_{n,d})^{n+1} \times \mathbb{P}^n : x \in \mathcal{H}_{a_i} \text{ for all } i\}.$$

Σ is a closed subet of $(P_{n,d})^{n+1} \times \mathbb{P}^n$ because each condition $x \in \mathcal{H}_{a_i}$ is given by a polynomial condition in the homogeneous coordinates of $a_i \in P_{n,d}$ (= coefficients of a polynomial f_i such that $a_i = [f_i]$) and of $x \in \mathbb{P}^n$.

Let π_1 denote the projection $\Sigma \subseteq (P_{n,d})^{n+1} \times \mathbb{P}^n \to (P_{n,d})^{n+1}$. By definition, $\pi_1^{-1}(a_0, \dots, a_n) \cap \Sigma \neq \emptyset$ if and only $(a_0, \dots, a_n) \in S$. In other words, $S = \pi_1(\Sigma)$. Therefore, because $\mathbb{P}^n$ is complete, S is a closed subset of $(P_{n,d})^{n+1}$.

(Why are we focussing on the set where $\bigcap_{i=0}^n \mathcal{H}_{a_i}$ is non-empty rather than the set where it is empty? Because closed sets are usually more interesting than open sets, e.g. it makes sense to ask what is the dimension of a closed subset, while the dimension of an open set is always the same as the dimension of the space it is contained in.)

What is $\dim S$?

We can work this out by two applications of the fibre dimension theorem: first we apply it to the projection $\Sigma \to \mathbb{P}^n$ to find $\dim \Sigma$, then we apply it to the projection $\Sigma \to S$ to find $\dim S$. The reason we can do this is that we know the dimension of $\mathbb{P}^n$ and we can work out the dimensions of the fibres of both projections from Σ.

In order to apply the fibre dimension theorem to Σ, we need to know that Σ is irreducible. This is true, and could be proved using tools from this course, but is a little more complicated than we want to do now, so we shall take it for granted.

To compute $\dim \Sigma$, we will apply the fibre dimension theorem to the projection $p\colon \Sigma \to \mathbb{P}^n$. This map is surjective: for any $x \in \mathbb{P}^n$, we can pick $a \in P_{n,d}$ such that $x \in \mathcal{H}_a$ and then $(a, \dots, a, x) \in p^{-1}(x) \subseteq \Sigma$. The fibres are

$$\begin{aligned} p^{-1}(x) &= \{(a_0, \dots, a_n) \in (P_{n,d})^{n+1} : x \in \mathcal{H}_{a_i} \text{ for all } i\} \\ &= \Big(\{a \in P_{n,d} : x \in \mathcal{H}_a\}\Big)^{n+1}. \end{aligned}$$

Thus

$$\dim p^{-1}(x) = (n+1) \dim\{a \in P_{n,d} : x \in \mathcal{H}_a\}.$$

In order to calculate $\dim\{a \in P_{n,d} : x \in \mathcal{H}_a\}$, make a linear change of coordinates so that $x = [0 : \cdots : 0 : 1]$. This change of coordinates won't change the dimension of $\{a \in P_{n,d} : x \in \mathcal{H}_a\}$, so it suffices to work out the dimension for the special case of $[0 : \cdots : 0 : 1]$.

Now $[0 : \cdots : 0 : 1] \in \mathcal{H}_a$ if and only if the homogeneous polynomial f vanishes at $[0 : \cdots : 0 : 1]$, where $a = [f]$. The value of f at $[0 : \cdots : 0 : 1]$ is just the X_n^d

coefficient of f. Thus

$$[0 : \cdots : 0 : 1 \in \mathcal{H}_{[f]} \Leftrightarrow \text{the } X_n^d \text{ coefficient of } f \text{ is zero.}$$

In other words, $\{a \in P_{n,d} : x \in \mathcal{H}_a\}$ is a subspace of $P_{n,d}$ defined by one linear equation, so $\dim\{a \in P_{n,d} : x \in \mathcal{H}_a\} = \dim P_{n,d} - 1 = N - 1$ where $N = \binom{n+d}{d} - 1$. (Alternatively, we could have seen this without reducing to the case $x = [0 : \cdots : 0 : 1]$ by observing that the condition $x \in \mathcal{H}_{[f]}$ is a single linear condition on the coefficients of f, as you see just by expanding out $f(x_0, \dots, x_n) = 0$.)

Therefore, for every $x \in \mathbb{P}^n$,

$$\dim p^{-1}(x) = (n + 1)(N - 1).$$

We can apply the fibre dimension theorem (Theorem 22.2) to get

$$\dim \Sigma = \dim \mathbb{P}^n + \dim p^{-1}(x) = n + (n + 1)(N - 1) = N(n + 1) - 1.$$

(By part (ii) of the fibre dimension theorem, this holds for all x in some non-empty open subset of $\mathbb{P}^n$. It doesn't matter which x we choose because we showed that all the fibres have the same dimension.)

Now to compute $\dim S$, we will apply the fibre dimension theorem to the projection $q \colon \Sigma \to S$. This map is surjective by construction. This time, the fibres do not all have the same dimension, but the minimum dimension of the fibres is zero – to see that there are fibres of dimension zero, observe that for suitable choices of $n + 1$ homogeneous polynomials of degree d, the intersection of the corresponding hypersurfaces is finite and non-empty for example: $f_i = X_i^d$ for $0 \le i \le n - 1$, $f_n = X_0^d$ (repeating f_0) have the unique common solution $[0 : \cdots : 0 : 1]$.

Therefore the fibre dimension theorem implies that

$$\dim S = \dim \Sigma - 0 = N(n + 1) - 1.$$

We recall how this follows from the fibre dimension theorem: by part (i), the fact that there is just a single fibre of dimension 0 implies that $0 \ge \dim \Sigma - \dim S$. By part (ii) of the fibre dimension theorem, there exists an open subset $U \subseteq S$ on which $\dim q^{-1}(s) = \dim \Sigma - \dim S$. But since $q^{-1}(s)$ can never be negative, this forces $0 \le \dim q^{-1}(s) = \dim \Sigma - \dim S$. Combining these gives $\dim S = \dim \Sigma$ as we claimed.

In particular, we have $\dim S = \dim(P_{n,d})^{n+1} - 1$. This means that it is only slightly unusual for $n+1$ hypersurfaces to have non-empty intersection: this subset of the parameter space has dimension only 1 less than the entire parameter space.

As an application of this calculation, we see that S is a hypersurface in $(P_{n,d}^{n+1})$, and therefore it is defined by a single polynomial

$$F \in k[X_{iJ} : 0 \le i \le n, 0 \le I \le N].$$

In other words, there exists some polynomial F such that, when we evaluate it at the coefficients of $n + 1$ homogeneous polynomials $f_0, \dots, f_n$ of degree d, we get zero if and only if the intersection $\bigcap_{i=0}^n \mathcal{H}_{[f_i]}$ is non-empty.