Algebraic Geometry

Problem Sheet 4 – Feedback from marking

Mark scheme.

(1) 5 marks
  (a) 2 (b) 2 (c) 1
(2) 4 marks
  (a) 2 (b) 2
(3) 3 marks
  (a) $1\frac{1}{2}$ (b) $1\frac{1}{2}$
(4) 2 marks
(5) 6 marks
  (a) 1 (b) 1 (c) 2 (d) 2

Question 1. For question 1(b), most people forgot to prove that φ is not regular at the points $[0:0:1]$, $[0:1:0]$, $[1:0:0]$ (by checking other representatives for φ). Otherwise this question was done well.

Question 2. To my surprise, 2(a) turned out to be the one of the hardest questions on the sheet, perhaps because there is no "standard approach". There are a number of ways to do it using regular maps and completeness – see the official solutions for a quick way.

Some people talked about "closed sets" without making it clear "closed in what?" For example, "V is a closed subset of $\mathbb{P}^1 \times \mathbb{A}^1$" and "$V$ is a closed subset of $\mathbb{P}^1 \times \mathbb{P}^1$" do not mean the same thing.

Question 3. Many people had difficulty with the notion of "regular function on a projective algebraic set" for 3(a). By definition, a regular function on a $V \subseteq \mathbb{P}^n$ is a regular map $V \to \mathbb{A}^1$. In other words, it is a regular map $V \to \mathbb{P}^1$ whose image happens to land inside $\mathbb{A}^1 \subseteq \mathbb{P}^1$. Thus, we can represent $X_i X_j^{d-1}/f$ as the regular map

$$[f : X_i X_j^{d-1}] \colon V \to \mathbb{P}^1.$$

Note that f and $X_i X_j^{d-1}$ are not themselves regular maps, because they are not well-defined as functions on V (for example, $f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n) \neq f(x_0, \dots, x_n)$. The point is that by looking at the ratio $X_i X_j^{d-1}/f$, because both have the same degree, we do get a well-defined regular map.

Question 4. Solutions to question 4 were mostly good, though many people forgot to consider the case $x = y$. The official solution avoids needing to do this by doing everything in terms of the "affine cone" $C(L)$. However, if you wrote down $z \in L_{xy}$ means $z = [s\underline{x} + t\underline{y}]$ or similar, then this only works when $x \neq y$ (otherwise there is no unique line L_{xy}) so you have to deal with this case separately.

Question 5. I thought question 5 was quite hard, but people did it very well – indicating good familiarity with lines in projective space, homogeneous coordinates and the Segre embedding.