Throughout this paper k is an algebraically closed field.

1. (a) Define an **affine algebraic set**.

   (b) Prove that if V is an affine algebraic set, then V is a union of finitely many irreducible closed subsets. You may use Hilbert's Basis Theorem provided you state it clearly and correctly.

   (c) Find the irreducible components of the affine algebraic set

$$\{(x, y, z) \in \mathbb{A}^3 : y^2 = xz, z^2 = y^3\}.$$

   For each irreducible component C of V:

   (i) Write down polynomials which define C as a subset of $\mathbb{A}^3$.

   (ii) Prove that C is isomorphic to $\mathbb{A}^1$.

2. In this question, you may use any theorems from the course provided you state them clearly and correctly.

   Let

$$\Sigma_{1,1} = \{[w : x : y : z] \in \mathbb{P}^3 : wz = xy\}.$$

   (a) Define a **regular map** between projective algebraic sets.

   (b) Write down the Segre embedding $\sigma_{1,1} \colon \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$. (You should write it down in such a way that the image of $\sigma_{1,1}$ is equal to $\Sigma_{1,1}$, but you are not required to prove this.)

   (c) Write down two regular maps $p_1 \colon \Sigma_{1,1} \to \mathbb{P}^1$, $p_2 \colon \Sigma_{1,1} \to \mathbb{P}^1$ such that (p_1, p_2) is the inverse of $\sigma_{1,1}$. (You are not required to prove that (p_1, p_2) is inverse to $\sigma_{1,1}$.)

   (d) Prove that $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are birational but not isomorphic.

   (e) Let

$$\Gamma = \{([a_0 : \cdots : a_n], [b_0 : \cdots : b_n]) \in \mathbb{P}^n \times \mathbb{P}^n : \sum_{i=0}^{n} a_i b_i = 0\}.$$

   Use the Segre embedding to prove that $(\mathbb{P}^n \times \mathbb{P}^n) \setminus \Gamma$ is isomorphic to an affine algebraic set.

3. In this question, you may use any theorems from the course provided you state them clearly and correctly.

(a) Let V be an affine algebraic set and let x be a point in V. Define the **tangent space** of V at x.

(b) Let V be the algebraic set in $\mathbb{A}^3$ defined by the polynomial

$$X^2Y^2 + Y^2Z^2 + X^2Z^2 - XYZ.$$

You may assume that this polynomial is irreducible.

For each point $x \in V$, determine the dimension of T_xV. What is the singular locus of V?

(c) Let f be a non-zero polynomial in $k[X_1, \ldots, X_n]$ with no repeated factors. Let $V \subseteq \mathbb{A}^n$ be the hypersurface defined by f. Let x be a point in V and let u be a vector in k^n.

Prove that the polynomial

$$f(x + Tu) \in k[T]$$

has a repeated root at $T = 0$ if and only if $u \in T_xV$.

(d) Let $V \subseteq \mathbb{A}^n$ be a hypersurface defined by a polynomial of degree 2. Prove that if x is a singular point of V and $L \subseteq \mathbb{A}^n$ through x, then either $L \subseteq V$ or $L \cap V = \{x\}$.