Algebraic Geometry

Sample exam solutions 2018

Question 1.

Sub-question 1(a). *Define an **affine algebraic set**.*

(3 marks)

An **affine algebraic set** is a subset $V \subseteq \mathbb{A}^n$ which consists of the common zeros of some finite set of polynomials $f_1, \ldots, f_m$ with coefficients in k.

Alternatively: an **affine algebraic set** is a set of the form

$$V = \{(x_1, \ldots, x_n) \in \mathbb{A}^n : f_1(x_1, \ldots, x_n) = \cdots = f_m(x_1, \ldots, x_n) = 0\}$$

for some polynomials $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$.

Sub-question 1(b). *Prove that if V is an affine algebraic set, then V is a union of finitely many irreducible closed subsets. You may use Hilbert's Basis Theorem provided you state it clearly and correctly.*

(8 marks)

Suppose for contradiction that V is an affine algebraic set which cannot be written as a union of finitely many irreducible closed subsets.

V is not itself irreducible, so $V = V_1 \cup W_1$ where V_1 and W_1 are proper closed subsets of V.

If each of V_1 and W_1 is a union of finitely many irreducible closed subsets, then V is also a union of finitely many irreducible closed subsets. So at least one of V_1 and W_1 is not a union of finitely many irreducible closed subsets. Without loss of generality, we may suppose that V_1 is not a union of finitely many irreducible closed subsets.

We repeat the argument: V_1 is not reducible, so $V_1 = V_2 \cup W_2$ where V_2 and W_2 are proper closed subsets of V. At least one of V_2 and W_2 is not a union of finitely many irreducible closed subsets; suppose that V_2 is not a union of finitely many irreducible closed subsets.

Repeating this, we build up a descending chain of closed subsets

$$V \supset V_1 \supset V_2 \supset \cdots$$

where all the inclusions are strict.

Hence we get an ascending chain of ideals in $k[X_1, \ldots, X_n]$:

$$\mathbb{I}(V) \subseteq \mathbb{I}(V_1) \subseteq \mathbb{I}(V_2) \subseteq \cdots .$$

Since $\mathbb{V}(\mathbb{I}(W)) = W$ for every affine algebraic set W, the strictness of the inclusions between the V_i implies that the inclusions between the ideals $\mathbb{I}(V_i)$ are strict.

Recall the statement of the Hilbert Basis Theorem.

Theorem. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an ascending chain of ideals in $k[X_1, \ldots, X_n]$. Then there is some N such that $I_n = I_N$ for every $n > N$.

This contradicts the existence of the strictly ascending chain of ideals

$$\mathbb{I}(V) \subset \mathbb{I}(V_1) \subset \mathbb{I}(V_2) \subset \cdots .$$

Sub-question 1(c). *Find the irreducible components of the affine algebraic set*

$$V = \{(x, y, z) \in \mathbb{A}^3 : y^2 = xz, z^2 = y^3\}.$$

For each irreducible component C of V:

(i) Write down polynomials which define C as a subset of $\mathbb{A}^3$.

(ii) Prove that C is isomorphic to $\mathbb{A}^1$.

(9 marks)

Consider $(x, y, z) \in V$. The equation $y^2 = xz$ implies that $y^3 = xyz$. Substituting in $z^2 = y^3$ gives $z^2 = xyz$ and so either $z = 0$ or $z = xy$.

If $z = 0$ then the equations for V become $y^2 = 0$, $y^3 = 0$. Hence V contains the line $y = z = 0$.

If $z = xy$, then we can substitute this into the equations for V to get $y^2 = x^2y$ and $x^2y^2 = y^3$. Hence $y = 0$ or $y = x^2$.

If $y = 0$ then $z^2 = y^3$ implies that $z = 0$, so we are back in the case considered previously.

If $y = x^2$, then we have $z = x^3$. We can see that (x, x^2, x^3) satisfies the equations for V.

Hence

$$V = \{(x, 0, 0) : x \in k\} \cup \{(x, x^2, x^3) : x \in k\}.$$

(i) The component $\{(x, 0, 0)\}$ is defined by the equations $y = z = 0$.
The component $\{(x, x^2, x^3)\}$ is defined by the equations $y = x^2, z = xy$.

(ii) The component $\{(x, 0, 0)\}$ is isomorphic to $\mathbb{A}^1$ via the inverse pair of regular maps $x \mapsto (x, 0, 0)$ and $(x, y, z) \mapsto x$.
The component $\{(x, x^2, x^3)\}$ is isomorphic to $\mathbb{A}^1$ via the inverse pair of regular maps $x \mapsto (x, x^2, x^3)$ and $(x, y, z) \mapsto x$.

Question 2. *In this question, you may use any theorems from the course provided you state them clearly and correctly.*

Let

$$\Sigma_{1,1} = \{[w : x : y : z] \in \mathbb{P}^3 : wz = xy\}.$$

Sub-question 2(a). *Define a* ***regular map*** *between projective algebraic sets.*

(3 marks)

Let $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ be projective algebraic sets.

A **regular map** $\varphi\colon V \to W$ is a function $V \to W$ such that for every point $x \in V$, there exist a Zariski open set $U \subseteq V$ containing x and a sequence of polynomials $f_0, \dots, f_m \in k[X_0, \dots, X_n]$ such that:

(i) $f_0, \dots, f_m$ are homogeneous of the same degree;
(ii) for every $y \in U$, $f_0, \dots, f_m$ are not all zero at y;
(iii) for every $y = [y_0 : \cdots : y_n] \in U$, $\varphi(y) = [f_0(y_0, \dots, y_n) : \cdots : f_m(y_0, \dots, y_n)]$.

Sub-question 2(b). *Write down the Segre embedding $\sigma_{1,1}\colon \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$. (You should write it down in such a way that the image of $\sigma_{1,1}$ is equal to $\Sigma_{1,1}$, but you are not required to prove this.)*

(2 marks)

$$\sigma_{1,1}([s : t], [u : v]) = [su : sv : tu : tv].$$

(Note: other permutations of the coordinates are also possible, but only some of them satisfy the equation for $\Sigma_{1,1}$.)

Sub-question 2(c). *Write down two regular maps $p_1\colon \Sigma_{1,1} \to \mathbb{P}^1$, $p_2\colon \Sigma_{1,1} \to \mathbb{P}^1$ such that (p_1, p_2) is the inverse of $\sigma_{1,1}$. (You are not required to prove that (p_1, p_2) is inverse to $\sigma_{1,1}$.)*

(2 marks)

$$p_1([w : x : y : z]) = [w : y] \text{ if } (w, y) \neq (0, 0),$$
$$p_1([w : x : y : z]) = [x : z] \text{ if } (x, z) \neq (0, 0).$$

$$p_2([w : x : y : z]) = [w : x] \text{ if } (w, x) \neq (0, 0),$$
$$p_2([w : x : y : z]) = [y : z] \text{ if } (y, z) \neq (0, 0).$$

Observe that these definitions are well-defined everywhere on $\Sigma_{1,1}$.

(A different permutation of coordinates for $\sigma_{1,1}$ in (b) will require a different permutation of coordinates here as well.)

Sub-question 2(d). *Prove that $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are birational but not isomorphic.*

(7 marks)

$\mathbb{A}^1 \times \mathbb{A}^1$ is an open subset of $\mathbb{P}^1 \times \mathbb{P}^1$, and so $\mathbb{A}^1 \times \mathbb{A}^1$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$. Trivially, $\mathbb{A}^1 \times \mathbb{A}^1$ is isomorphic, and hence birational, to $\mathbb{A}^2$. $\mathbb{A}^2$ is an open subset of $\mathbb{P}^2$, and so $\mathbb{A}^2$ is birational to $\mathbb{P}^2$. Composing these birational equivalences, we conclude that $\mathbb{P}^1 \times \mathbb{P}^1$ is birational to $\mathbb{P}^2$.

Now we prove that $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$. (There are a variety of ways to prove this, here's one.)

Pick two points $a, b \in \mathbb{P}^1$. Then $\{a\} \times \mathbb{P}^1$ and $\{b\} \times \mathbb{P}^1$ are disjoint irreducible closed subsets of $\mathbb{P}^1 \times \mathbb{P}^1$, each of dimension 1.

We will prove that $\mathbb{P}^2$ cannot contain two disjoint irreducible closed subsets of dimension 1. Assume for contradiction that $V, W \subseteq \mathbb{P}^2$ are disjoint irreducible closed subsets of dimension 1.

Let $f \in k[X_0, X_1, X_2]$ be a homogeneous polynomial which vanishes on V.

If f is reducible, say $f = f_1 f_2$, then we can write V as the union of the closed subsets $V_1 = \{x \in V : f_1(x) = 0\}$ and $V_2 = \{x \in V : f_2(x) = 0\}$. Because V is irreducible, either $V = V_1$ or $V = V_2$. Then we can replace f by either f_1 or f_2, and it will still vanish on V. Hence we can assume that f is irreducible.

Let Z denote the zero set of f. Then Z is a hypersurface in $\mathbb{P}^2$. By a theorem from lectures, the dimension of a hypersurface in $\mathbb{P}^2$ is $2 - 1 = 1$.

So $\dim Z = \dim V$, V is a closed subset of Z and Z is irreducible. By a theorem from lectures, these conditions imply that $V = Z$.

Thus V is a hypersurface in $\mathbb{P}^2$. By a theorem from lectures, a hypersurface and a positive-dimensional closed subset of $\mathbb{P}^2$ must have non-empty intersection. We conclude that $V \cap W$ is non-empty, contradicting the assumption that V and W are disjoint..

Thus $\mathbb{P}^2$ cannot contain two disjoint irreducible closed subsets of dimension 1, but $\mathbb{P}^1 \times \mathbb{P}^1$ does contain such subsets, so they are not isomorphic.

Sub-question 2(e). *Let*

$$\Gamma = \{([a_0 : \cdots : a_n], [b_0 : \cdots : b_n]) \in \mathbb{P}^n \times \mathbb{P}^n : \sum_{i=0}^{n} a_i b_i = 0\}.$$

Use the Segre embedding to prove that $(\mathbb{P}^n \times \mathbb{P}^n) \setminus \Gamma$ *is isomorphic to an affine algebraic set.*

(6 marks)

Let $N = (n+1)^2 - 1 = b^2 + 2n$. Label the homogeneous coordinates on $\mathbb{P}^N$ as $[(z_{ij})]$ for $0 \leq i, j \leq n$.

The Segre embedding of $\mathbb{P}^n \times \mathbb{P}^n$ is a function $\sigma_{n,n} \colon \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^N$ which sends $([a_0 : \ldots : a_n], [by_0 : \ldots : b_n])$ to a point in $\mathbb{P}^N$ with coordinates given by

$$z_{ij} = a_i b_j.$$

By a theorem from lectures, the image of the Segre embedding is a Zariski closed set $\Sigma_{n,n} \subseteq \mathbb{P}^N$.

Using the above explicit description of $\sigma_{n,n}$, we can describe the image of Γ as

$$\{[(z_{ij})] \in \Sigma_{n,n} : \sum_{i=0}^{n} z_{ii} = 0\}.$$

In other words, $\sigma_{n,n}(\Gamma) = \Sigma_{n,n} \cap H$ where H is the hyperplane in $\mathbb{P}^N$ defined by the equation

$$\sum_{i=0}^{n} z_{ii} = 0.$$

By definition, the structure of algebraic variety on $\mathbb{P}^n \times \mathbb{P}^n$ is the same as that on $\Sigma_{n,n}$ via $\sigma_{n,n}$. Hence $(\mathbb{P}^n \times \mathbb{P}^n) \setminus \Gamma$ is isomorphic to $\Sigma_{n,n} \setminus \sigma_{n,n}(\Gamma)$. By the above, this is equal to

$$\Sigma_{n,n} \cap (\mathbb{P}^N \setminus H).$$

Since $\Sigma_{n,n}$ is closed in $\mathbb{P}^N$, we conclude that $(\mathbb{P}^n \times \mathbb{P}^n) \setminus \Gamma$ is isomorphic to a closed subset of $\mathbb{P}^N \setminus H$.

Since H is a hyperplane, $\mathbb{P}^N \setminus H$ is isomorphic to $\mathbb{A}^N$. Thus $(\mathbb{P}^n \times \mathbb{P}^n) \setminus \Gamma$ is isomorphic to a closed subset of $\mathbb{A}^N$, that is, to an affine algebraic set.

Question 3.

Sub-question 3(a). *Let V be an affine algebraic set and let x be a point in V. Define the* ***tangent space*** *of V at x.*

(4 marks)
For each $f \in \mathbb{I}(V)$, let $df_x \colon k^n \to k$ denote the linear map

$$(a_1, \dots, a_n) \mapsto \sum_{i=1}^{n} \left. \frac{\partial f}{\partial X_i} \right|_x a_i.$$

The **tangent space** of V at x is the intersection

$$\bigcap_{f \in \mathbb{I}(V)} \ker df_x.$$

Sub-question 3(b). *Let V be the algebraic set in $\mathbb{A}^3$ defined by the polynomial*

$$X^2Y^2 + Y^2Z^2 + X^2Z^2 - XYZ.$$

You may assume that this polynomial is irreducible.

For each point $x \in V$, determine the dimension of T_xV. What is the singular locus of V?

(6 marks)
Let $f(X, Y, Z) = X^2Y^2 + Y^2Z^2 + X^2Z^2 - XYZ$. Since f is irreducible, it generates $\mathbb{I}(V)$ and so the tangent space to V at p is $\ker df_p$.

df_p is a linear map $k^3 \to k$, so its kernel has dimension 3 if it is the zero map and dimension 2 otherwise.

We calculate

$$\frac{\partial f}{\partial X} = 2XY^2 + 2XZ^2 - YZ,$$
$$\frac{\partial f}{\partial Y} = 2YX^2 + 2YZ^2 - XZ,$$
$$\frac{\partial f}{\partial Z} = 2ZX^2 + 2ZY^2 - XY.$$

Let $p = (x, y, z)$ be a point where $\ker df_p = k^3$. Then

$$0 = 2f(x, y, z) - x \left. \frac{\partial f}{\partial X} \right|_p = 2x^2z^2 - xyz.$$

Hence either $xz = 0$ or $2xz = y$.

Similarly, we get

$$xy = 0 \text{ or } 2xy = z$$

and

$$yz = 0 \text{ or } 2yz = x.$$

If $x = 0$, then the equation $2yz = x$ forces either y or z to be zero. By similar reasoning for the other variables, we conclude that if any one of x, y, z is zero, then at least two of them are zero.

If two out of x, y, z are zero, then every term in f and $\frac{\partial f}{\partial X}$, $\frac{\partial f}{\partial Y}$, $\frac{\partial f}{\partial Z}$ is zero at (x, y, z). Hence $p \in V$ and $df_p = 0$, so $\dim T_pV = 3$.

Otherwise, all of x, y, z are non-zero. Then we must have

$$2xz = y, \ 2xy = z, \ 2yz = x.$$

Substituting the first equation in the second, we get

$$2x(2xz) = z.$$

Since $z \neq 0$, this implies that $4x^2 = 1$ and so

$$x = \pm\tfrac{1}{2}.$$

Similarly

$$y = \pm\tfrac{1}{2}, \quad z = \pm\tfrac{1}{2}.$$

But now $x^2y^2 + y^2z^2 + z^2x^2 = \frac{3}{16}$ and $xyz = \pm\frac{1}{8}$ so $f(x, y, z) \neq 0$. Hence this point is not in V.

In conclusion: the singular locus of V is

$$\operatorname{Sing} V = \{(0, 0, z) : t \in k\} \cup \{(0, y, 0) : y \in k\} \cup \{(x, 0, 0) : x \in k\}.$$

For $p \in V$, we have $\dim T_pV = 3$ if $p \in \operatorname{Sing} V$ and 2 if $p \notin \operatorname{Sing} V$.

Sub-question 3(c). *Let f be a non-zero polynomial in $k[X_1, \dots, X_n]$ with no repeated factors. Let $V \subseteq \mathbb{A}^n$ be the hypersurface defined by f. Let x be a point in V and let u be a vector in k^n.*

Prove that the polynomial

$$f(x + Tu) \in k[T]$$

has a repeated root at $T = 0$ if and only if $u \in T_xV$.

(5 marks)

Since f has no repeated factors, f generates $\mathbb{I}(V)$. Hence T_xV is the kernel of the linear map

$$df_x \colon (a_1, \dots, a_n) \mapsto \sum_{i=1}^{n} \left.\frac{\partial f}{\partial X_i}\right|_x a_i.$$

Since $x \in V$, $f(x + Tu)$ has a root at $T = 0$. So it has a repeated root at $T = 0$ if and only if

$$\left.\frac{d}{dT} f(x + Tu)\right|_{T=0} = 0.$$

By the chain rule for partial derivatives,

$$\left.\frac{d}{dT} f(x + Tu)\right|_{T=0} = \sum_{i=1}^{n} \left.\frac{\partial f}{\partial X_i}\right|_x u_i.$$

Hence $u \in T_xV = \ker df_x$ if and only if $f(x + Tu)$ has a repeated root at $T = 0$.

Sub-question 3(d). *Let $V \subseteq \mathbb{A}^n$ be a hypersurface defined by a polynomial of degree 2. Prove that if x is a singular point of V and $L \subseteq \mathbb{A}^n$ through x, then either $L \subseteq V$ or $L \cap V = \{x\}$.*

(5 marks)

Let f be a polynomial which generates $\mathbb{I}(V)$.

Because x is a singular point of V, $\dim T_xV > \dim V$. Because V is a hypersurface in $\mathbb{A}^n$, $\dim V = n - 1$. But $T_xV \subseteq k^n$. So we must have $\dim T_xV = n$ and $T_xV = k^n$.

Let u be a vector in k^n such that $L = \{x + tu : t \in k\}$. Since $T_x V = k^n$, we get $u \in T_x V$.

By (c), the polynomial $f(x+Tu)$ has a repeated root at $T = 0$. But f has degree 2, so $f(x + Tu)$ has degree ≤ 2. So in order to have a repeated root at 0, we must have

$$f(x + Tu) = aT^2$$

for some $a \in k$ (a may be zero or non-zero).

Now $L \cap V = \{x + tv : t \in k, f(x + tu) = 0\}$. So if $f(x + Tu) = aT^2$ with $a \neq 0$, then $L \cap V = \{x\}$. On the other hand if $f(x + Tu) = 0$ then $L \subseteq V$.