INTRODUCTION TO ABELIAN VARIETIES AND THE AX–LINDEMANN–WEIERSTRASS THEOREM

MARTIN ORR

1. Introduction

This paper surveys some aspects of the theory of abelian varieties relevant to the Pila–Zannier proof of the Manin–Mumford conjecture and to the André–Oort conjecture. An abelian variety is a complete algebraic variety with a group law. The geometry of abelian varieties is tightly constrained and well-behaved, and they are important tools in algebraic geometry. Abelian varieties defined over number fields pose interesting arithmetic problems, for example concerning their rational points and associated Galois representations.

The paper is in three parts:

(1) an introduction to abelian varieties;
(2) an outline of moduli spaces of principally polarised abelian varieties, which are the fundamental examples of Shimura varieties;
(3) a detailed proof of the Ax–Lindemann–Weierstrass theorem for abelian varieties, following a method using o-minimal geometry due to Pila, Ullmo and Yafaev.

The first part assumes only an elementary knowledge of algebraic varieties and complex analytic geometry. The second part makes heavier use of algebraic geometry, but still at the level of varieties, and a little algebraic number theory. Like the first part, the algebraic geometry in the third part is elementary; the third part also assumes familiarity with the concept of semialgebraic sets, and uses cell decomposition for semialgebraic sets and the Pila–Wilkie theorem as black boxes. The second and third parts are independent of each other, so the reader interested primarily in the Ax–Lindemann–Weierstrass theorem may skip the second part (sections 4 to 6).

In the first part of the paper (sections 2 and 3), we introduce abelian varieties over fields of characteristic zero, and especially over the complex numbers. The theory of abelian varieties over fields of positive characteristic introduces additional complications which we will not discuss. Our choice of topics is driven by Pila and Zannier's proof of the Manin–Mumford conjecture using o-minimal geometry. We will not discuss the Manin–Mumford conjecture or its proofs directly in this paper; aspects of the proof, and its generalisation to Shimura varieties, are discussed in other papers in this volume.

In this part of the article, we omit proofs of most statements. Most of the material is covered in Milne's online notes [Mil08] (which are a revised version of [Mil86]) and in Birkenhake and Lange's book [BL92] (over the complex numbers only). Another standard reference is Mumford's book [Mum70], which contains some proofs which are omitted in [Mil08] but deals only with algebraically closed fields (of any characteristic) and is a more difficult read. We have not attempted to cite original sources for theorems, but simply refer to whichever of these books offers the most convenient proof for each result.

The second part of the paper (sections 4 to 6) outlines the definition and analytic construction of the moduli spaces of principally polarised abelian varieties. These are algebraic varieties whose points parametrise abelian varieties. The purpose of this part is to give some concrete examples of Shimura varieties to complement to the more abstract discussion in Daw's paper on the André–Oort conjecture. These sections require a higher level of algebraic geometry than the rest of the paper, and section 6 on complex multiplication also uses algebraic number fields.

Again in these sections we omit proofs. Proofs for section 4 can be found in the same books mentioned above. For section 5, see chapter 7 of [Mum65]; there is also a quick sketch in [Mil08] III, section 7. Details of the theory of complex multiplication (section 6) may be found in the online notes [Mil06]; for a gentler introduction, you may restrict your attention to the case of elliptic curves which is discussed in [Sil94] chapter 2.

In the third part of the paper we discuss the Ax–Lindemann–Weierstrass theorem for abelian varieties. This theorem has its roots in transcendence theory and is one of the ingredients in the Pila–Zannier approach to the Manin–Mumford conjecture.

Theorem 1.1. *Let*

- *A be an abelian variety of dimension g over $\mathbb{C}$,*
- *$\pi\colon \mathbb{C}^g \to A$ be the exponential map,*
- *V be a complex algebraic subvariety of A, and*
- *Y be a maximal irreducible complex algebraic subvariety contained in $\pi^{-1}(V)$.*

Then $\pi(Y)$ is a translate of an abelian subvariety of A.

Several proofs of this theorem are known. A proof using the Pila–Wilkie theorem was given by Pila in [Pil11]. This proof forms the basis for subsequent proofs of analogous theorems for Shimura varieties. The proof we will describe is based on Pila's ideas, but incorporating part of Ullmo and Yafaev's proof of the Ax–Lindemann–Weierstrass theorem for compact Shimura varieties [UY13] in order to simplify the proof.

2. Abelian varieties

In this section we will define abelian varieties and their morphisms and state their basic properties, and those of their torsion points. We work over an arbitrary base field, although some of the theorems will include a condition on the characteristic and our main interest is in base fields of characteristic zero.

2.1. Definitions and basic properties.

Definition. An **abelian variety** is a complete group variety.

Let us define the terms which appear in the above definition.

Definition. A **group variety** is an algebraic variety G together with morphisms of varieties $m\colon G \times G \to G$ (multiplication) and $i\colon G \to G$ (inverse) and a point $e \in G$ (the identity element) which satisfy the axioms for a group.

When we say that a group variety is defined over a field k, we mean that the variety G, the morphisms m and i and the point e are all defined over k.

Definition. An algebraic variety X over an algebraically closed field is **complete** if, for every variety Y, the projection $X \times Y \to Y$ is a closed map with respect to the Zariski topologies on $X \times Y$ and Y. (If the base field is not algebraically closed, then we should make the same definition using schemes instead of varieties.)

The property of completeness is the analogue in algebraic geometry of compactness in topology. The fundamental examples of complete varieties are **projective varieties**, that is, closed subvarieties of projective space. In general, it is not true that all complete varieties are projective. In the case of abelian varieties, it turns out that all abelian varieties are projective but this is a far-from-trivial theorem. Indeed, Weil did not know this theorem when he first developed the theory of abelian varieties.

Theorem 2.1 ([Mil08] I, section 6)**.** *Every abelian variety is projective.*

Another fundamental geometric property of abelian varieties, which they share with all group varieties, is that they are smooth.

The twin conditions of being complete and having a group law imply that abelian varieties are topologically and group-theoretically boring; in this they differ from compact Lie groups. At the heart of this is the following rigidity result, which follows from the definition of completeness together with the fact that every morphism from a complete variety to an affine variety is constant.

Theorem 2.2 ([Mil08] I, Theorem 1.1)**.** *Let X, Y and Z be complete varieties and $f\colon X \times Y \to Z$ a morphism. Suppose that there exist points $x \in X$ and $y \in Y$ such that the restrictions $f_{|\{x\} \times Y}$ and $f_{|X \times \{y\}}$ are constant.*

Then f is constant.

By applying Theorem 2.2 to the conjugator map

$$(x, y) \mapsto xyx^{-1}y^{-1} : A \times A \to A$$

we deduce the following corollary.

Corollary 2.3. *The group law on an abelian variety is commutative.*

As a result of this, we shall henceforth write the group law on an abelian variety additively.

We can also show that the group law on an abelian variety over an algebraically closed field of characteristic zero is divisible. Hence apart from describing the torsion, which we shall do later, and rationality questions over non-algebraically closed fields, there is nothing to say about the group theory of abelian varieties.

2.2. **Elliptic curves.** The simplest examples of abelian varieties are **elliptic curves**, which by definition are abelian varieties of dimension 1.

Elliptic curves are commonly described as curves of the form

$$y^2 = x^3 + ax + b$$

for constants a and b satisfying $4a^3 + 27b^2 \neq 0$, or more correctly as the closures of such curves in $\mathbb{P}^2$. The closure of such a curve consists of the affine curve together with the single point $[0 : 1 : 0]$ at infinity. The condition $4a^3 + 27b^2 \neq 0$, or equivalently that the cubic $x^3 + ax + b$ has distinct roots, ensures that the curve is smooth.

Given such a curve, we can define a group law using the classical chord-and-tangent construction, with the point at infinity as the identity element. Thus we get an abelian variety of dimension 1.

Conversely, every abelian variety of dimension 1 is isomorphic to one of the above form.

2.3. **Morphisms and isogenies.** A **morphism** of abelian varieties is a morphism of algebraic varieties which is also a group homomorphism.

An **isogeny** is a morphism of abelian varieties which is surjective and has finite kernel. Note that the domain and codomain of an isogeny always have the same dimension.

The **degree** of an isogeny $f\colon A \to B$ is its degree as a morphism of algebraic varieties, or in other words the degree of the associated extension of fields of rational functions $[k(A) : k(B)]$. Over an algebraically closed field of characteristic zero, this is the same as the number of points in the kernel of the isogeny.

Examples of isogenies are the multiplication-by-n maps from an abelian variety to itself. We will see later that this fact is easy to prove analytically when A is defined over $\mathbb{C}$. A general proof is much harder.

Proposition 2.4 ([Mil08] I, Theorem 7.2)**.** *Let A be an abelian variety and N a non-zero integer. Let $[N]\colon A \to A$ be the morphism which sends x to Nx in the group law. Then $[N]$ is an isogeny of degree N^{2g}.*

We say that two abelian varieties A and B are **isogenous** if there exists an isogeny $A \to B$. The relationship of being isogenous is an equivalence relation – the hard part of this to prove is that if there is an isogeny $A \to B$ then there is also an isogeny $B \to A$.

2.4. **Abelian subvarieties.** Let A be an abelian variety. An **abelian subvariety** of A is an algebraic subvariety which is also a subgroup.

Given any abelian subvariety $B \subset A$, we can construct a quotient abelian variety A/B. It is not necessarily true that A is isomorphic to the direct product $B \times A/B$, only that A is isogenous to $B \times A/B$. If we attempt to decompose A as an internal direct product, we get the following theorem called the Poincaré Reducibility Theorem.

Theorem 2.5 ([Mil08] proof of I, Proposition 10.1)**.** *Let A be an abelian variety and $B \subset A$ an abelian subvariety. Then there exists an abelian subvariety $C \subset A$ such that B and C together generate A and $B \cap C$ is finite (but $B \cap C$ might not be trivial).*

A **simple abelian variety** is an abelian variety A whose only abelian subvarieties are A itself and $\{0\}$. The Poincaré Reducibility Theorem implies that every abelian variety is isogenous (but not necessarily isomorphic) to a direct product of simple abelian varieties.

2.5. **Torsion points.** Let A be an abelian variety of dimension g and let N be a positive integer. We write $A[N]$ for the set of N-torsion points, that is, points $x \in A$ such that $Nx = 0$ in the group law on A.

When talking about N-torsion points, we should assume that the characteristic of the base field does not divide N – otherwise the kernel of the multiplication map $[N]$ is a non-reduced group scheme and not just a set of points. If we make this assumption on the characteristic, and also suppose that the base field is algebraically closed, then the following description of the group of N-torsion points is equivalent to Proposition 2.4.

Proposition 2.6. *Let A be an abelian variety of dimension g defined over an algebraically closed field whose characteristic does not divide N. Then $A[N]$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^{2g}$ as a group.*

Suppose that A is defined over an arbitrary field k whose characteristic does not divide N. If P is a $\bar{k}$-point of $A[N]$, then the $\mathrm{Gal}(\bar{k}/k)$-conjugates of P are also N-torsion points of A. Hence $\mathrm{Gal}(\bar{k}/k)$ acts on $A[N](\bar{k})$. These Galois actions are much studied, especially in the cases where k is a number field, local field or function field.

Let $P \in A(\bar{k})$ be a torsion point of order N. Then the degree $[k(P) : k]$ of the field of definition of P is bounded above by $\#A[N](\bar{k}) = N^{2g}$. We can improve this upper bound slightly by excluding the points of $A[N](\bar{k})$ whose order is smaller than N, but this only gives lower order terms in the bound.

When k is a number field, we can also prove a lower bound for $[k(P) : k]$. This generalises the bound for torsion points on elliptic curves discussed in Habegger's paper in this volume.

Theorem 2.7 ([Mas84])**.** *Let A be an abelian variety of dimension g defined over a number field k. There exist effective constants c, depending on A and k, and ρ, depending only on g, such that for all torsion points $P \in A(\bar{k})$ of order N,*

$$[k(P) : k] \geq cN^{\rho}.$$

3. Complex tori

There is a very simple analytic description of abelian varieties over the complex numbers, as complex tori. This analytic description is essential to the Pila–Zannier proof of the Manin–Mumford conjecture, and is needed for the statement of the Ax–Lindemann–Weierstrass theorem.

Let V be a finite-dimensional complex vector space, which we consider as a group under addition. A **lattice** in V is a discrete subgroup $\Lambda \subset V$ such that the quotient V/Λ is compact. Observe that if Λ is a lattice, then Λ is isomorphic as a group to $\mathbb{Z}^{2g}$ (where $g = \dim_{\mathbb{C}} V$) and that the $\mathbb{R}$-span of Λ is V.

A complex manifold of the form V/Λ, where V is a complex vector space and $\Lambda \subset V$ is a lattice, is called a **complex torus**. Note that the word "torus" is confusingly over-used in the world of group varieties. In this case, it is used because V/Λ is diffeomorphic to $\mathbb{R}^{2g}/\mathbb{Z}^{2g} \cong (S^1)^{2g}$; so in particular, if $g = 1$ then V/Λ is diffeomorphic to the classical torus $\mathbb{R}^2/\mathbb{Z}^2$.

Theorem 3.1. *Every abelian variety over $\mathbb{C}$ is isomorphic as a complex Lie group to a complex torus.*

Proof. Let A be an abelian variety over $\mathbb{C}$ and let V be its tangent space at the identity. Because $A(\mathbb{C})$ is a complex Lie group, there is a holomorphic exponential map $\exp \colon V \to A(\mathbb{C})$. Because A is commutative, exp is a surjective group homomorphism. Its kernel Λ is a lattice, and $A(\mathbb{C}) \cong V/\Lambda$. □

Note that the converse is false – not every complex torus is isomorphic to an abelian variety. This will be discussed in section 4.

3.1. **Morphisms of complex tori.**

Let A and A' be complex abelian varieties, isomorphic to the complex tori V/Λ and V'/Λ' respectively. If $f \colon A \to A'$ is a morphism of abelian varieties, then it lifts to a $\mathbb{C}$-linear map $\tilde{f} \colon V \to V'$ such that $\tilde{f}(\Lambda) \subset \Lambda'$.

Conversely, any $\mathbb{C}$-linear map $V \to V'$ which maps Λ into Λ' descends to a holomorphic map $V/\Lambda \to V'/\Lambda'$, and by Chow's theorem this is a morphism of algebraic varieties $A \to A'$. Hence there is a canonical bijection

$$\left\{ \begin{matrix} \text{morphisms of abelian} \\ \text{varieties } A \to A' \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \mathbb{C}\text{-linear maps } V \to V' \\ \text{mapping } \Lambda \text{ into } \Lambda' \end{matrix} \right\}$$

3.2. **Abelian subvarieties of complex tori.** Let A be a complex abelian variety isomorphic to V/Λ. Suppose that A' is an abelian subvariety of A. Then $A' = V'/\Lambda'$ where V' is some complex vector subspace of V and $\Lambda' = V' \cap \Lambda$.

Conversely, let V' be a complex vector subspace of V. We say that V' is a **full subspace** if $V' \cap \Lambda$ is a lattice in V'. Note that most subspaces are not full – indeed for a generic subspace V', $V' \cap \Lambda = \{0\}$.

Clearly V' being full is a necessary condition for $V'/(V' \cap \Lambda)$ to be an abelian variety. If V' is full, then Chow's theorem implies that $V'/(V' \cap \Lambda)$ is in fact an abelian subvariety of A.

Hence we get a canonical bijection

$$\{\text{abelian subvarieties of } A\} \longleftrightarrow \{\text{full } \mathbb{C}\text{-vector subspaces of } (V, \Lambda)\}.$$

3.3. **Torsion points in complex tori.** The N-torsion points of the complex torus V/Λ are $\frac{1}{N}\Lambda/\Lambda$. Since $\Lambda \cong \mathbb{Z}^{2g}$, where $g = \dim_{\mathbb{C}} V$, there is a group isomorphism $\frac{1}{N}\Lambda/\Lambda \cong (\mathbb{Z}/N\mathbb{Z})^{2g}$. This gives a quick proof of Proposition 2.4 for abelian varieties defined over $\mathbb{C}$.

4. Riemann forms and polarisations

In the previous section, we saw that every complex abelian variety is a complex torus. The question remains of which complex tori are abelian varieties – in other words (given Theorem 2.1), when can the complex manifold V/Λ be embedded as a closed subvariety of some projective space? The answer to this question is provided by Riemann forms.

In this section, we will define Riemann forms, then briefly discuss dual abelian varieties and polarisations. Riemann forms are defined complex analytically; polarisations provide a substitute which can be generalised to arbitrary base fields. We will use polarisations as a technical tool when defining moduli spaces.

4.1. **Riemann forms.** The definition of Riemann forms is rather opaque; they are simply what is needed to answer the question of which complex tori are abelian varieties.

Let V be a complex vector space and Λ a lattice in V. A **Riemann form** on (V, Λ) is a symplectic bilinear form $\psi \colon \Lambda \times \Lambda \to \mathbb{Z}$ such that:

(1) $\psi_{\mathbb{R}}(iu, iv) = \psi(u, v)$ for all $u, v \in V$; and
(2) $\psi_{\mathbb{R}}(iv, v) > 0$ for all $v \in V - \{0\}$.

Here, $\psi_{\mathbb{R}} \colon V \times V \to \mathbb{R}$ means the $\mathbb{R}$-bilinear extension of ψ (note that this is not $\mathbb{C}$-bilinear – indeed it cannot be $\mathbb{C}$-bilinear because its image is contained in $\mathbb{R}$). Condition (2) implies that ψ is nondegenerate.

There is a second perspective on Riemann forms, using Hermitian forms on V instead of symplectic forms on Λ.

Lemma 4.1. *The map sending a Hermitian form to its imaginary part is a bijection*

$$\left\{ \begin{array}{c} \text{positive definite Hermitian forms} \\ H\colon V \times V \to \mathbb{C} \text{ s.t. } \operatorname{Im} H(\Lambda \times \Lambda) \subset \mathbb{Z} \end{array} \right\} \to \left\{ \begin{array}{c} \text{Riemann forms} \\ \text{on } (V, \Lambda) \end{array} \right\}$$

whose inverse sends a Riemann form ψ to the Hermitian form

$$H(u, v) = \psi_{\mathbb{R}}(iu, v) + i\psi_{\mathbb{R}}(u, v).$$

The importance of Riemann forms is due to the following theorem.

Theorem 4.2 ([Mum70] section 3, Corollary)**.** *A complex torus V/Λ is isomorphic to an abelian variety if and only if there exists a Riemann form on (V, Λ).*

In fact this theorem can be strengthened as follows.

Theorem 4.3. *There is a canonical bijection between the set of Riemann forms on (V, Λ) and the set of homological equivalence classes of ample line bundles on V/Λ.*

We can interpret the lattice Λ, together with the action of $\mathbb{C}$ on $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, as a Hodge structure of type $\{(-1, 0), (0, -1)\}$ – for more on this, see Daw's paper. A Riemann form $\psi\colon \Lambda \times \Lambda \to \mathbb{Z}$ is the same as what is known as a polarisation in Hodge theory. It might seem a little confusing that we are about to define a different type of object called a polarisation in the context of abelian varieties; however, because there is a canonical bijection between Riemann forms and polarisations as we will define them below they are really two different ways of looking at the same thing.

4.2. **Polarisations over $\mathbb{C}$.** If $A \cong V/\Lambda$ is an abelian variety over $\mathbb{C}$, then for each Riemann form on (V, Λ) we can construct an isogeny from A to the so-called dual abelian variety. The isogenies constructed in this way are called polarisations. Because polarisations are objects of algebraic rather than analytic geometry, they can be generalised to abelian varieties over arbitrary base fields.

Let V be a complex vector space and Λ a lattice in V. Let $\bar{V}^\vee$ be the conjugate-dual space of V, that is

$$\bar{V}^\vee = \{f\colon V \to \mathbb{C} \mid f \text{ is additive and } f(av) = \bar{a}f(v) \text{ for all } a \in \mathbb{C}, v \in V\}.$$

Let $\bar{\Lambda}^\vee$ be the following lattice in $\bar{V}^\vee$.

$$\bar{\Lambda}^\vee = \{f\colon V \to \mathbb{C} \mid \operatorname{Im} f(\Lambda) \subset \mathbb{Z}\}$$

(where Im means the imaginary part).

Suppose that V/Λ is isomorphic to an abelian variety A. Then there exists a Riemann form $\psi\colon \Lambda \times \Lambda \to \mathbb{Z}$. We can use ψ to construct a Riemann form on

$(\bar{V}^\vee, \bar{\Lambda}^\vee)$. Hence $\bar{V}^\vee/\bar{\Lambda}^\vee$ is isomorphic to an abelian variety, which we denote $A^\vee$ and call the **dual abelian variety** of A.

Let $H \colon V \times V \to \mathbb{C}$ be the Hermitian form associated with the Riemann form ψ as in Lemma 4.1. Then

$$v \mapsto H(v, -)$$

is a $\mathbb{C}$-linear isomorphism $V \to \bar{V}^\vee$ which maps Λ into $\bar{\Lambda}^\vee$, and hence induces a morphism of abelian varieties $\lambda\colon A \to A^\vee$. The fact that ψ is non-degenerate implies that λ is an isogeny.

Any isogeny $A \to A^\vee$ which is induced by a Riemann form according to the above recipe is called a **polarisation** of A.

4.3. **Polarisations over arbitrary fields.** The details of how to define dual abelian varieties and polarisations over arbitrary fields are beyond the scope of this paper. We merely remark that, if A is an abelian variety defined over a field k, then it is possible to define the dual abelian variety $A^\vee$ over k, and that a polarisation of A is an isogeny $A \to A^\vee$ which is induced according to a certain recipe by an ample line bundle on A (Theorem 4.3 implies that this is equivalent to the previous definition when $k = \mathbb{C}$). Because every abelian variety is projective, there always exists at least one polarisation on A.

4.4. **Principal polarisations.** A polarisation is said to be **principal** if it has degree 1 as an isogeny, or in other words, if it is an isomorphism between A and $A^\vee$. Not all abelian varieties possess principal polarisations, but the following results mean that it is often possible to reduce a question about general abelian varieties to a question about abelian varieties with principal polarisations:

Proposition 4.4 ([Mil08] I, Corollary 13.10, Theorem 13.12)**.**

(1) Over a field of characteristic zero, every abelian variety is isogenous to an abelian variety with a principal polarisation.

(2) ("Zarhin's Trick") Over any base field, if A is any abelian variety, then $(A \times A^\vee)^4$ has a principal polarisation.

4.5. **Polarisations for elliptic curves.** In one dimension, every lattice $\Lambda \subset \mathbb{C}$ has a Riemann form and so $\mathbb{C}/\Lambda$ is an elliptic curve. To write down the Riemann form explicitly, rescale Λ so that it has a $\mathbb{Z}$-basis $\{1, \tau\}$ with $\operatorname{Im}\tau > 0$. Then

$$(u, v) \mapsto \frac{\operatorname{Im}(u\bar{v})}{\operatorname{Im}\tau}$$

is a Riemann form on $(\mathbb{C}, \Lambda)$.

The above Riemann form defines a principal polarisation of the corresponding elliptic curve E. In fact this is the unique principal polarisation of E so it gives a canonical isomorphism $E \to E^\vee$. Hence it is rarely necessary to explicitly talk about polarisations and dual varieties when working with elliptic curves.

5. The moduli space of principally polarised abelian varieties

The moduli space of principally polarised abelian varieties of dimension g is an algebraic variety $\mathcal{A}_g$ equipped with a bijection between its points (over an algebraically closed field) and the set of isomorphism classes of principally polarised abelian varieties of dimension g, satisfying a sort of weakened universal property which we shall discuss below.

We will begin by explaining how this universal property is made precise, giving a definition of the moduli space as a variety over any perfect field. We then outline Siegel's analytic construction of the moduli space over the complex numbers. This is a prototype for the definition of a Shimura variety as a quotient of a Hermitian symmetric domain by the action of an arithmetic group. Historically, the analytic construction came first, and Satake and Baily proved that the complex analytic moduli space is an algebraic variety defined over $\mathbb{Q}$ before the algebraic theory was developed by Mumford.

5.1. **Principally polarised abelian varieties.** There is no "moduli space of abelian varieties of dimension g": in order to get a moduli space it is necessary to include polarisations.

A **principally polarised abelian variety** is a pair (A, λ) where A is an abelian variety and $\lambda \colon A \to A^\vee$ is a principal polarisation. An **isomorphism** between principally polarised abelian varieties (A, λ) and (B, μ) is an isomorphism $f \colon A \to B$ of abelian varieties such that

$$\lambda = f^\vee \circ \mu \circ f$$

where $f^\vee \colon B^\vee \to A^\vee$ is the dual morphism of f.

Theorem 5.1 ([Mil08] I, Theorem 15.1)**.** *Let A be an abelian variety. There are finitely many isomorphism classes of principally polarised abelian varieties* (A, λ).

Note that the theorem does not assert that A has finitely many principal polarisations, because there may be distinct principal polarisations λ and λ' such that (A, λ) and (A, λ') are isomorphic as principally polarised abelian varieties.

5.2. **Universal property definition of the moduli space.** The idea behind the universal property definition of the moduli space $\mathcal{A}_g$ is that it should be the base variety for a "universal principally polarised family of abelian varieties." The problem is that no such universal family exists. The best we can do is to require that $\mathcal{A}_g$ satisfies the property that it would satisfy if a universal family did exist (property (a) below). Since this is not enough to determine $\mathcal{A}_g$ up to isomorphism, we add an additional condition that $\mathcal{A}_g$ is universal among varieties satisfying (a). A variety satisfying properties with the form of (a) and (b) is called a **coarse moduli space**.

We will first work over an algebraically closed field k.

A **family of abelian varieties** is a morphism of algebraic varieties $\pi\colon A \to S$ together with morphisms $m\colon A \times A \to A$, $i\colon A \to A$ and $e\colon S \to A$ such that:

(i) π is proper; and
(ii) for each point $s \in S(k)$, the fibre A_s together with the restrictions $m_{|A_s \times A_s}$ and $i_{|A_s}$ and the point $e(s)$ form an abelian variety.

It is possible to define a notion of principally polarised family of abelian varieties.

The **moduli space of principally polarised abelian varieties of dimension** g over k is an algebraic variety $\mathcal{A}_g$ defined over k equipped with a bijection

$$J_g\colon \left\{ \begin{array}{c} \text{isomorphism classes of principally polarised} \\ \text{abelian varieties of dimension } g \text{ defined over } k \end{array} \right\} \longrightarrow \mathcal{A}_g(k)$$

satisfying the following conditions:

(a) for every algebraic variety T defined over k and every principally polarised family of abelian varieties $B \to T$ of relative dimension g, if φ denotes the map $T(k) \to \mathcal{A}_g(k)$ which sends $t \in T(k)$ to $J_g(B_t) \in \mathcal{A}_g(k)$, then φ is a morphism of algebraic varieties;
(b) for every algebraic variety $\mathcal{A}'_g$ over k equipped with a bijection J'_g as above, if $(\mathcal{A}'_g, J'_g)$ satisfies (a), then $J'_g \circ J_g^{-1}\colon \mathcal{A}_g(k) \to \mathcal{A}'_g(k)$ is a morphism of algebraic varieties.

These properties are sufficient to characterise the variety $\mathcal{A}_g$, together with the bijection J_g, up to unique isomorphism.

Over a non-algebraically closed perfect field k, we define the moduli space of principally polarised abelian varieties to be an algebraic variety defined over k together with a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of principally polarised} \\ \text{abelian varieties of dimension } g \text{ defined over } \bar{k} \end{array} \right\} \longrightarrow \mathcal{A}_g(\bar{k})$$

satisfying the properties (a) and (b) over the algebraic closure $\bar{k}$, and such that the bijection commutes with the natural actions of $\mathrm{Gal}(\bar{k}/k)$ on either side. When k is non-algebraically closed, there is still a map from isomorphism classes of principally polarised abelian varieties defined over k to the k-points of $\mathcal{A}_g$, but it is usually neither injective nor surjective.

Theorem 5.2 ([Mum65] chapter 7)**.** *The moduli space of principally polarised abelian varieties of dimension g exists over any base field k.*

5.3. **Analytic construction of the moduli space.** Over the complex numbers, we can construct the moduli space of principally polarised abelian varieties analytically as a quotient

$$\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g.$$

Here $\mathcal{H}_g$ is the **Siegel upper half-space**

$$\mathcal{H}_g = \{ Z \in \mathrm{M}_{g\times g}(\mathbb{C}) \mid Z \text{ is symmetric and } \operatorname{Im} Z \text{ is positive definite} \}$$

and $\mathrm{Sp}_{2g}(\mathbb{Z})$ is the symplectic group

$$\mathrm{Sp}_{2g}(\mathbb{Z}) = \{M \in \mathrm{GL}_{2g}(\mathbb{Z}) \mid M^t J M = J\} \text{ where } J = \left(\begin{smallmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{smallmatrix}\right).$$

The action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{H}_g$ is given by

$$\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) Z = (AZ + B)(CZ + D)^{-1} \text{ where } A, B, C, D \in \mathrm{M}_g(\mathbb{Z}),$$

generalising the standard action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane by Möbius transformations.

Theorem 5.3. *The quotient* $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$ *is the complex analytic space underlying a quasi-projective variety defined over* $\mathbb{Q}$*. It is the same as the moduli space of principally polarised abelian varieties over* $\mathbb{C}$ *defined in the previous section.*

We shall explain how the bijection between $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$ and the isomorphism classes of principally polarised abelian varieties over $\mathbb{C}$ is defined. Let (A, λ) be a principally polarised abelian variety of dimension g over $\mathbb{C}$, and let V/Λ be a complex torus isomorphic to A. The polarisation λ corresponds to a Riemann form ψ on (V, Λ).

Choose a symplectic basis $\{e_1, \dots, e_g, f_1, \dots, f_g\}$ for (Λ, ψ), that is, a $\mathbb{Z}$-basis for Λ satisfying

$$\psi(e_j, f_j) = 1 \text{ for all } j, \text{ and}$$
$$\psi(e_j, e_k) = \psi(f_j, f_k) = \psi(e_j, f_k) = 0 \text{ for all } j \neq k.$$

The conditions for ψ to be a Riemann form imply that $e_1, \dots, e_g$ form a $\mathbb{C}$-basis for V, so we can form a $g \times g$ complex matrix consisting of the coordinates of $f_1, \dots, f_g$ with respect to $e_1, \dots, e_g$. This matrix is called a **period matrix** for (A, λ), and the definition a Riemann form implies that it is in $\mathcal{H}_g$.

Different choices of symplectic basis for Λ may give rise to different period matrices for the same principally polarised abelian variety; this ambiguity is precisely the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathcal{H}_g$.

5.4. **The moduli space of elliptic curves.** We have already seen that every elliptic curve has a unique principal polarisation, so we can describe $\mathcal{A}_1$ as the moduli space of elliptic curves without mentioning polarisations.

The Siegel upper half-space $\mathcal{H}_1$ is simply the upper half-plane consisting of complex numbers with positive imaginary part, and the symplectic group $\mathrm{Sp}_2(\mathbb{Z})$ is equal to $\mathrm{SL}_2(\mathbb{Z})$. Hence the moduli space of elliptic curves is given analytically by the well-known quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}_1$, which is isomorphic to the affine line $\mathbb{C}$.

Over any field, the moduli space of elliptic curves $\mathcal{A}_1$ is isomorphic to the affine line $\mathbb{A}^1$. The point of $\mathbb{A}^1$ associated with an isomorphism class of elliptic curves is given by the j-invariant. For an elliptic curve E in Weierstrass form $y^2 = x^3 + ax + b$, this is given by

$$j(E) = \frac{1728 \cdot 4a^3}{4a^3 + 27b^2}.$$

5.5. **Level structures.** The moduli spaces $\mathcal{A}_g$ have several deficiencies:

(i) In property (a) in the definition of the moduli space, we have to take principally polarised abelian varieties over an open cover of $\mathcal{A}_g$ rather than a single principally polarised family of abelian varieties over all of $\mathcal{A}_g$.
(ii) The map $\mathcal{H}_g \to \mathcal{A}_g(\mathbb{C})$ is ramified.
(iii) For $g \geq 2$, $\mathcal{A}_g$ is not smooth.

All of these deficiencies are related to the fact that $\mathrm{Sp}_{2g}(\mathbb{Z})$ contains torsion elements, and to the fact that a principally polarised abelian variety can have non-trivial automorphisms. To avoid these problems, we introduce level structures.

Let N be a positive integer. Recall that the N-torsion of an abelian variety A of dimension g is isomorphic as a group to $(\mathbb{Z}/N\mathbb{Z})^{2g}$. A principal polarisation $\lambda\colon A \to A^\vee$ induces a symplectic pairing

$$e_N\colon A[N] \times A[N] \to \mu_N$$

where μ_N is the multiplicative group of roots of unity.

Fix a group isomorphism $\iota\colon \mu_N \to \mathbb{Z}/N\mathbb{Z}$. A **level-$N$ structure** on (A, λ) is a symplectic basis for $A[N]$ as a $\mathbb{Z}/N\mathbb{Z}$-module with respect to the symplectic pairing $\iota \circ e_N$.

For any N, we can define the **moduli space of principally polarised abelian varieties with level-N structure** in a similar way to the definition of the moduli space of principally polarised abelian varieties. This is an algebraic variety defined over any field which contains μ_N. It is denoted $\mathcal{A}_g(N)$ or $\mathcal{A}_{g,1,N}$ (in the second notation, the 1 indicates that we are dealing with polarisations of degree 1).

For $N \geq 4$, there is a universal principally polarised family of abelian varieties of dimension g with level-N structure on the base $\mathcal{A}_g(N)$ – in other words, property (a) holds for the open cover consisting of the single open set $\mathcal{A}_g(N)$. Furthermore $\mathcal{H}_g \to \mathcal{A}_g(N)$ is the topological universal cover and $\mathcal{A}_g(N)$ is smooth. All of these properties are related to the fact a principally polarised abelian variety with level-N structure has no non-trivial automorphisms preserving the level structure when $N \geq 4$.

Complex analytically, $\mathcal{A}_g(N)$ is given by $\Gamma(N)\backslash\mathcal{H}_g$ where $\Gamma(N)$ is the group of matrices in $\mathrm{Sp}_{2g}(\mathbb{Z})$ which are congruent to the identity modulo N.

Whenever M divides N, there is a morphism $\mathcal{A}_g(N) \to \mathcal{A}_g(M)$ which sends a principally polarised abelian variety with level-N structure $(A, \lambda, \{e_1, \dots, e_{2g}\})$ to the same principally polarised abelian variety with the level-M structure

$$\{(N/M)e_1, \dots, (N/M)e_{2g}\}.$$

5.6. **The moduli space as a Shimura variety.** We will not define any of the concepts related to Shimura varieties, but to aid the reader we will explain how the objects defined in this section relate to the definitions in Daw's paper.

The Siegel upper half-space $\mathcal{H}_g$ is an example of a Hermitian symmetric domain. Its group of holomorphic automorphisms is $\mathrm{Sp}_{2g}(\mathbb{R})/\{\pm 1\}$, which is the neutral component of the group of real points of the adjoint algebraic group PGSp_{2g}.

When setting up a Shimura datum, it is convenient to work with GSp_{2g} (the group of matrices which preserve a symplectic form up to a multiplication by a scalar) instead of PGSp_{2g} (the quotient of GSp_{2g} by its centre). This is because the standard $2g$-dimensional representation of GSp_{2g} does not factor through PGSp_{2g}

There is an equivariant bijection between a certain $\mathrm{GSp}_{2g}(\mathbb{R})$-conjugacy class of morphisms $\mathbb{S} \to \mathrm{GSp}_{2g}$ and the union of the Siegel upper and lower half-spaces $\mathcal{H}_g^{\pm}$; this can be made explicit by means of period matrices. Hence $(\mathrm{GSp}_{2g}, \mathcal{H}_g^{\pm})$ is an example of a Shimura datum.

The groups $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\Gamma(N)$ are congruence subgroups of $\mathrm{GSp}_{2g}(\mathbb{Q})$ and so the quotients

$$\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g = \mathcal{A}_g(\mathbb{C}), \quad \Gamma(N)\backslash\mathcal{H}_g = \mathcal{A}_g(N)(\mathbb{C})$$

are connected components of Shimura varieties. In fact $\mathcal{A}_g$ is itself a Shimura variety with a single connected component but $\mathcal{A}_g(N)$ is a component of a Shimura variety which may have multiple connected components.

The fact that $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$ is the complex analytic space of an algebraic variety defined over $\mathbb{Q}$ (Theorem 5.3) is an example of a general theorem on Shimura varieties, the existence of canonical models.

5.7. **Definability of theta functions.** We have mentioned that

$$\mathcal{A}_g(\mathbb{C}) \cong \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathcal{H}_g$$

is a quasi-projective variety. We can realise it as a subset of projective space by using theta functions, which are holomorphic functions $\mathcal{H}_g \to \mathbb{C}$ satisfying suitable transformation rules with respect to the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$. (When $g = 1$, the classical j-function is an example of such a function.)

There is a standard fundamental set in $\mathcal{H}_g$ for the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$, called the **Siegel fundamental set** and denoted $\mathcal{F}_g$. This is a semialgebraic set and by a theorem of Peterzil and Starchenko [PS13], the restrictions of theta functions to $\mathcal{F}_g$ are definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$ generated by restricted analytic functions and the real exponential function. This is the first ingredient required to use o-minimality to study subvarieties of $\mathcal{A}_g$.

6. Complex multiplication

An abelian variety is said to have complex multiplication if its endomorphism algebra is as large as possible. Abelian varieties with complex multiplication are important because they have special arithmetic properties. These arithmetic properties are used in the definition of canonical models of Shimura varieties. From the point of view of the André–Oort conjecture, the special points on the moduli space of principally polarised abelian varieties are precisely the points which correspond to abelian varieties with complex multiplication.

6.1. **Definition of complex multiplication.** Let A be an abelian variety over an algebraically closed field of characteristic zero. An **endomorphism** of A means a morphism $A \to A$, and the endomorphisms of A form a ring $\operatorname{End} A$.

Suppose that A is simple. There is a classification of the possible endomorphism rings of A, due to Albert. In particular $\operatorname{End} A$ has rank at most $2 \dim A$ as a $\mathbb{Z}$-module.

A simple abelian variety A is said to **have complex multiplication** (or **CM**) if $\operatorname{End} A$ is commutative and has rank equal to $2 \dim A$. (A non-simple abelian variety is said to have CM if it is isogenous to a product of simple abelian varieties with CM.)

If A is a simple abelian variety with complex multiplication, then $\operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Q}$ is a **CM field**. This means a totally complex number field F which contains a totally real subfield F_0 such that $[F : F_0] = 2$; equivalently it is a number field F such that the automorphism of F induced by complex conjugation is non-trivial and independent of the choice of embedding $F \hookrightarrow \mathbb{C}$.

6.2. **Construction of CM abelian varieties.** Let F be a number field. An **order** in F is a subring R of F which is finitely-generated as a $\mathbb{Z}$-module and such that $\mathbb{Q}R = F$. For example, if $F = \mathbb{Q}(i)$, then the ring of integers $\mathbb{Z}[i]$ is an order but there are also smaller orders $\mathbb{Z}[Ni]$ for any positive integer N.

If F is a CM field of degree $2g$, we define a **CM type** Φ for F to be a set of g embeddings $F \hookrightarrow \mathbb{C}$ containing one embedding from each conjugate pair. Given a CM type Φ, an order $R \subset F$ and an ideal $I \subset R$, we can define a lattice $\Lambda \subset \mathbb{C}^g$ as the image of I under the map

$$(\phi_1, \dots, \phi_g) \colon F \to \mathbb{C}^g$$

where the CM type Φ is $\{\phi_1, \dots, \phi_g\}$. Then $\mathbb{C}^g/\Lambda$ is an abelian variety with complex multiplication, whose endomorphism ring is equal to R.

Conversely, every abelian variety with complex multiplication has the above form for some order R, CM type Φ and ideal I.

It follows that if we fix the order R, there are finitely many isomorphism classes of abelian varieties with complex multiplication whose endomorphism ring is isomorphic to R, because there are finitely many CM types for the field $R \otimes_{\mathbb{Z}} \mathbb{Q}$ and the ideal class group of R is finite. Since there are countably many CM fields and each contains countably many orders this implies that there are countably many isomorphism classes of abelian varieties with complex multiplication.

6.3. **Arithmetic properties of CM abelian varieties.** Every abelian variety with complex multiplication is defined over a number field. Note that this field of definition is not equal to the CM field F.

The arithmetic of the CM field F is closely related to the arithmetic of abelian varieties with complex multiplication by F. In particular, the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the isomorphism classes of abelian varieties with complex multiplication by F, and on their torsion points, can be described in terms of the class field theory of F.

6.4. **Complex multiplication and transcendence.** Because an abelian variety A with complex multiplication is isomorphic to $\mathbb{C}^g/\Lambda$ where Λ, as described above, consists of points whose coordinates algebraic numbers, any period matrix for A has entries in $\bar{\mathbb{Q}}$. In fact the property that both A and its period matrices are defined over $\bar{\mathbb{Q}}$ characterises abelian varieties with complex multiplication; this generalises Schneider's theorem on the transcendence of the j-invariant.

Theorem 6.1 ([Shi92]). *Let A be a complex abelian variety and Z a period matrix for A.*

A has complex multiplication if and only if A and Z are both defined over $\bar{\mathbb{Q}}$.

7. The Ax–Lindemann–Weierstrass theorem for abelian varieties

In the final part of these notes, we discuss the Ax–Lindemann–Weierstrass theorem for abelian varieties, and give a proof of this theorem using o-minimal geometry and combining ideas of Pila, Ullmo and Yafaev. We recall the statement of this theorem.

Theorem 7.1. *Let*

- *A be an abelian variety of dimension g over $\mathbb{C}$,*
- *$\pi\colon \mathbb{C}^g \to A$ be the exponential map,*
- *V be a complex algebraic subvariety of A, and*
- *Y be a maximal irreducible complex algebraic subvariety contained in $\pi^{-1}(V)$.*

Then $\pi(Y)$ is a translate of an abelian subvariety of A.

In the statement of the theorem, when we say that Y is a **maximal** irreducible algebraic subvariety in $\pi^{-1}(V)$, we mean that there is no irreducible algebraic subvariety Y' of $\mathbb{C}^g$ such that $Y \subset Y' \subset \pi^{-1}(V)$ and $Y \neq Y'$.

The same theorem holds with "abelian variety" replaced by "torus" (meaning $(\mathbb{C}^\times)^n$)) or indeed "commutative algebraic group over $\mathbb{C}$", and $\pi\colon \mathbb{C}^g \to A$ by the appropriate exponential map. These results are implied by (and are weaker than) the so-called Ax–Schanuel theorem and its generalisation to commutative algebraic groups, proved by differential-algebraic methods in [Ax71] and [Ax72]. For example, Theorem 7.1 is equivalent to the following transcendence statement, which can be deduced from Theorem 3 of [Ax72].

Theorem 7.2. *Let A be an abelian variety of dimension g over $\mathbb{C}$ and let $\pi\colon \mathbb{C}^g \to A$ be the exponential map. Let $f_1, \dots, f_n\colon A \to \mathbb{C} \cup \{\infty\}$ be meromorphic functions on A such that $[f_1 : \cdots : f_n]$ defines a projective embedding of A.*

Let Y be an irreducible complex algebraic variety and let $y_1, \dots, y_g \in \mathbb{C}[Y]$ be regular functions on Y. Suppose that the image of $\pi \circ (y_1, \dots, y_g) \colon Y \to A$ is not contained in any translate of a proper abelian subvariety of A.

Then the set of meromorphic functions

$$\{f_j \circ \pi \circ (y_1, \dots, y_g) \colon Y \to \mathbb{C} \cup \{\infty\} \mid j = 1, \dots, n\}$$

has transcendence degree g over $\mathbb{C}$.

Theorem 7.1 is an ingredient in the proof of the Pila–Zannier proof of the Manin–Mumford conjecture. The Pila–Zannier approach to the André–Oort conjecture requires an analogous theorem for Shimura varieties. Such a theorem was proved for products of modular curves by Pila [Pil11], for compact Shimura varieties by Ullmo and Yafaev [UY13], for the moduli space of principally polarised abelian varieties by Pila and Tsimerman [PT12], and finally for general Shimura varieties by Klingler, Ullmo and Yafaev [KUY13].

7.1. **Outline of proof.** Let $\Lambda = \ker \pi$, which is a lattice in $\mathbb{C}^g$, and let $\mathcal{F} \subset \mathbb{C}^g$ be the interior of a fundamental parallelepiped for Λ. We will apply the Pila–Wilkie theorem to the set

$$\Sigma = \{x \in \mathbb{C}^g \mid (Y + x) \cap \mathcal{F} \neq \emptyset \text{ and } Y + x \subset \pi^{-1}(V)\}.$$

Note that in most papers on this subject, Σ is defined as

$$\{x \in \mathbb{C}^g \mid \dim((Y + x) \cap \mathcal{F} \cap \pi^{-1}(V)) = \dim Y\},$$

which is the same set as the previous definition.

Of the two conditions in the definition of Σ, the important one for proving Theorem 7.1 is the second condition,

$$Y + x \subset \pi^{-1}(V).$$

If W is an irreducible semialgebraic set of points satisfying this condition and containing 0, then the maximality of Y implies that $Y + W = Y$. So if there exists such a W with positive dimension then we can make the key conclusion that the stabiliser of Y has positive dimension.

However the set

$$\{x \in \mathbb{C}^g \mid Y + x \subset \pi^{-1}(V)\}$$

is usually not definable in an o-minimal structure because it has infinitely many connected components. Adding the condition $(Y + x) \cap \mathcal{F} \neq \emptyset$ ensures that Σ is definable in the o-minimal structure $\mathbb{R}_{\text{an}}$, and so we can apply the Pila–Wilkie theorem to Σ.

The proof of Theorem 7.1 has the following steps:

(1) Show that the number of points of $\Sigma \cap \Lambda$ of height up to T grows at least linearly with T.
(2) Apply the Pila–Wilkie theorem to deduce that Σ contains a semialgebraic set of positive dimension.

(3) Deduce that the stabiliser of Y has positive dimension, and show that the image of this stabiliser under $\pi \colon \mathbb{C}^g \to A$ is equal to the stabiliser of the Zariski closure of $\pi(Y)$ (this step uses the maximality of Y).
(4) Quotient out by the stabiliser of Y and complete the proof of Theorem 7.1 by applying the argument again to the quotient.

The proofs of the Ax–Lindemann–Weierstrass theorem for Shimura varieties all follow the same structure as above. Step 1 is much harder for Shimura varieties than for abelian varieties, steps 2 and 3 are essentially the same and step 4 must be replaced by more a more complicated argument using monodromy.

8. Relationship between semialgebraic and complex algebraic sets

Before we prove Theorem 7.1, we first need a lemma saying that a maximal complex algebraic variety contained in a complex analytic set Z is in fact maximal among connected irreducible semialgebraic subsets of Z.

In the course of the proof of Theorem 7.1, we will construct a semialgebraic set Y' such that $Y \subset Y' \subset \pi^{-1}(V)$. We wish to use the maximality of Y to deduce that $Y = Y'$, but our hypothesis says that Y is maximal among complex algebraic subvarieties contained in $\pi^{-1}(V)$ and Y' is only semialgebraic. To following lemma deals with this problem.

The lemma as we give it below is Lemma 4.1 of [PT13] which seems to be the most elegant statement of this type; a weaker version, sufficient for the applications to André–Oort, was Lemma 2.1 in [PZ08].

Lemma 8.1. *Let Z be a complex analytic set in $\mathbb{C}^g$ and $X \subset Z$ a connected irreducible real semialgebraic subset. Then there is a complex algebraic variety X' such that $X \subset X' \subset Z$.*

In the lemma, we identify $\mathbb{C}^g$ with $\mathbb{R}^{2g}$ using the real and imaginary parts of the coordinates. We say that a semialgebraic set $X \subset \mathbb{R}^{2g}$ is **irreducible** if it cannot be written as a union $X = X_1 \cup X_2$ of two proper subsets which are closed in the topology induced on X by the Zariski topology of real algebraic sets in $\mathbb{R}^{2g}$.

Proof. Let S be the real Zariski closure of X in $\mathbb{R}^{2g}$. Any cell of maximum dimension in X is Zariski dense in S. This implies that S is geometrically irreducible (that is, S is irreducible over as an algebraic set over $\mathbb{C}$) and that $\dim S = \dim X$.

Define $f \colon \mathbb{C}^{2g} \to \mathbb{C}^g$ by

$$f(x_1, y_1, \dots, x_g, y_g) = (x_1 + iy_1, \dots, x_g + iy_g)$$

and let ι be the inclusion $\mathbb{R}^{2g} \to \mathbb{C}^{2g}$. Then the composite $f \circ \iota$ is the isomorphism we are using to identify $\mathbb{R}^{2g}$ with $\mathbb{C}^g$.

Let S_1 be the closure of $\iota(S)$ in the complex Zariski topology on $\mathbb{C}^{2g}$ (S_1 is the "extension of scalars" of S from $\mathbb{R}$ to $\mathbb{C}$). Let $S_2 = f(S_1)$.

We have the following diagram:

$$\begin{array}{ccc} S & \subset & \mathbb{R}^{2g} \\ \big\downarrow & & \big\downarrow \iota \\ S_1 & \subset & \mathbb{C}^{2g} \\ \big\downarrow & & \big\downarrow f \\ S_2 & \subset & \mathbb{C}^g \end{array} \qquad (\mathrm{id}: \mathbb{R}^{2g} \to \mathbb{C}^g)$$

By Chevalley's theorem, S_2 is constructible in the complex Zariski topology on $\mathbb{C}^g$. Hence the closure of S_2 in the Euclidean topology is a complex algebraic set. We shall denote this closure of S_2 by X', and show that it has the required property $X \subset X' \subset Z$.

It is easy to show that $X \subset X'$: we have

$$X \subset S \subset S_2 \subset X'.$$

To show that $X' \subset Z$, we shall apply the following claim to $f^{-1}(Z) \cap S_1$. This contains $\iota(X)$ by the hypothesis $X \subset Z$ and by the definition of S_1. Hence by the claim, $f^{-1}(Z) \cap S_1 = S_1$ and so $f(S_1) \subset Z$. Since Z is closed and $f(S_1)$ is dense in X' for the Euclidean topology, we can conclude that $X' \subset Z$.

Claim. *There are no proper closed analytic subsets of S_1 containing $\iota(X)$.*

To prove the claim, let W be the smallest analytic subset of S_1 containing $\iota(X)$.

Let x be a point in a cell of maximum dimension in X such that W is smooth at $x_1 = \iota(x)$. To see that such a point exists, observe that the smooth points of W form a non-empty open subset of W. By the minimality of W, $\iota(X) \not\subset W^{\text{sing}}$ so the set of points $x \in X$ such that $\iota(x)$ is a smooth point of W is a non-empty open subset of X. In particular, this set intersects at least one cell of maximum dimension in X.

Since x is in a maximum dimensional cell of X, locally near x, S and X coincide. Since $\iota(X) \subset W$, we deduce that

$$T_{x_1}\iota(S) \subset T_{x_1}W.$$

Here $\iota(S)$ is a real algebraic set and W is complex analytic, so $T_{x_1}\iota(S)$ is a real vector space and $T_{x_1}W$ is a complex vector space. Thus $T_{x_1}W$ contains the complex vector space $\mathbb{C}T_{x_1}\iota(S)$ generated by $T_{x_1}\iota(S)$.

By the definition of S_1,

$$\mathbb{C}T_{x_1}\iota(S) = T_{x_1}S_1$$

so we have

$$T_{x_1}S_1 \subset T_{x_1}W.$$

Since W is smooth at x_1, $\dim T_{x_1}W = \dim W$. So we get

$$\dim T_{x_1}S_1 \leq \dim T_{x_1}W = \dim_{x_1} W \leq \dim S_1 \leq \dim T_{x_1}S_1.$$

These inequalities must be equalities so

$$\dim_{x_1} W = \dim S_1.$$

Combined with the fact that $W \subset S_1$ and that S_1 is irreducible, we deduce that $W = S_1$. □

9. Proof of the Ax–Lindemann–Weierstrass theorem for abelian varieties

In this section we prove Theorem 7.1. We will follow the outline of the proof and use the notation from section 7.

We suppose that $\dim Y > 0$ – otherwise the theorem is trivial because any point in A is translate of the abelian subvariety $\{0\}$. We also assume that V is the Zariski closure in A of $\pi(Y)$ – replacing a larger subvariety V by the Zariski closure of $\pi(Y)$ will not change the fact that Y is a maximal complex algebraic subvariety in $\pi^{-1}(V)$.

9.1. **Lattice points in Σ.** Recall that Λ denotes the kernel of $\pi \colon \mathbb{C}^g \to A$, which is a lattice in $\mathbb{C}^g$. Fix a $\mathbb{Z}$-basis $e_1, \dots, e_{2g}$ for Λ and use this to define the height of elements of Λ:

$$H(a_1 e_1 + \cdots + a_{2g} e_{2g}) = \max(|a_1|, \dots, |a_{2g}|).$$

Observe that all points $x \in \Lambda$ satisfy $Y + x \subset \pi^{-1}(V)$ because $\pi^{-1}(V)$ is Λ-invariant. Hence

$$\Sigma \cap \Lambda = \{x \in \Lambda \mid (Y + x) \cap \mathcal{F} \neq \emptyset\}$$

and in order to count lattice points in Σ, it suffices to count points in the latter set.

Lemma 9.1. *There exists $T_0 \in \mathbb{R}$ such that for all $T > T_0$,*

$$\#\{x \in \Sigma \cap \Lambda \mid H(x) \leq T\} \geq T/2.$$

Proof. Since Y is an irreducible affine algebraic variety, it is path-connected and unbounded with respect to the usual norm on $\mathbb{C}^g$. Hence we can find a continuous function $\gamma \colon [0, \infty) \to Y$ whose image is unbounded.

Each time the image of γ crosses a boundary from one fundamental domain $\mathcal{F} - x$ to another $\mathcal{F} - x'$ (with $x, x' \in \Lambda$), the heights of x and x' differ by at most 1. So the heights of points in

$$\Lambda_\gamma = \{x \in \Lambda \mid \operatorname{im} \gamma \cap (\mathcal{F} - x) \neq \emptyset\}$$

form a set of consecutive integers.

Since γ is unbounded, Λ_γ contains points of arbitarily large height. Hence there is some h_0 such that for every integer $h > h_0$, Λ_γ contains at least one point of height h.

By the observation above the lemma, $\Lambda_\gamma \subset \Sigma \cap \Lambda$. Hence the lemma is proved with $T_0 = 2h_0$. □

Lemma 9.1 is straightforward to prove, but the analogous statement for Shimura varieties (proved by Klingler, Ullmo and Yafaev) is much more difficult. This is because the arithmetic group acting on the uniformising space of a Shimura variety (the analogue of the lattice Λ) is non-commutative and the heights of elements of this group grow exponentially instead of linearly with respect to the word metric. Hence in order to prove an analogue of Lemma 9.1 it is necessary to show that Y intersects exponentially many fundamental domains (instead of just one) at a given distance from the base point.

9.2. **Applying the Pila–Wilkie theorem.** We will now apply the Pila–Wilkie theorem to deduce that Σ contains a real semialgebraic set of positive dimension.

In order to state the Pila–Wilkie theorem, we make the following definition: if X is a subset of $\mathbb{R}^n$, let X^{alg} denote the union of all connected positive-dimensional real semialgebraic subsets contained in X.

Theorem 9.2 ([PW06] Theorem 1.8)**.** *Let $X \subset \mathbb{R}^n$ be a set definable in $\mathbb{R}_{\mathrm{an}}$ (or any o-minimal structure). Let $\epsilon > 0$.*

There exists a constant c, depending only on X and ϵ, such that for all $T \geq 1$,

$$\#\{x \in X - X^{\mathrm{alg}} \mid x \in \mathbb{Q}^n \text{ and } H(x) \leq T\} \leq cT^{\epsilon}.$$

In applying this theorem, we will identify $\mathbb{C}^g$ with $\mathbb{R}^{2g}$ by identifying the basis $e_1, \ldots, e_{2g}$ which we chose for Λ with the standard basis for $\mathbb{R}^{2g}$. Thus Λ is identified with $\mathbb{Z}^{2g}$ and the height of points in Λ which we defined above is identified with the naive height on $\mathbb{Z}^{2g}$.

In order to apply Theorem 9.2 to Σ, we need to check that Σ is definable in $\mathbb{R}_{\mathrm{an}}$, the o-minimal structure generated by restricted analytic functions.

Lemma 9.3. *The set Σ is definable in $\mathbb{R}_{\mathrm{an}}$.*

Proof. Note that Σ is equal to

$$\{x \in \mathbb{C}^g \mid (Y + x) \cap \mathcal{F} \neq \emptyset \text{ and } (Y + x) \cap \mathcal{F} \subset \pi^{-1}(V) \cap \mathcal{F}\}.$$

This holds because $Y + x$ and $\pi^{-1}(V)$ are both complex analytic sets, with $Y + x$ irreducible, and $(Y + x) \cap \mathcal{F}$ is an open subset of $Y + x$. Hence if $(Y + x) \cap \mathcal{F}$ is non-empty and is contained in $\pi^{-1}(V)$, then by analytic continuation $Y + x$ is contained in $\pi^{-1}(V)$.

The set above is definable in $\mathbb{R}_{\mathrm{an}}$ because $(Y + x) \cap \mathcal{F}$ and $\pi^{-1}(V) \cap \mathcal{F}$ are each defined by analytic functions on $\mathbb{C}^g$ restricted to the parallelepiped $\mathcal{F}$. $\square$

Applying Theorem 9.2 together with the lower bound from Lemma 9.1 shows that $\Sigma^{\mathrm{alg}} \cap \Lambda$ is non-empty. Let W be a connected irreducible positive-dimensional semialgebraic set, such that $W \subset \Sigma$ and W contains some point $w_0 \in \Lambda$.

9.3. **The stabilisers of Y and V.** Let $\Theta \subset \mathbb{C}^g$ denote the stabiliser of Y (which is a vector subspace of $\mathbb{C}^g$), and let $B \subset A$ denote the identity component of the stabiliser of V (which is an abelian subvariety of A). We will show that $W - w_0 \subset \Theta$ and that $\pi(\Theta) = B$. Hence both Θ and B have positive dimension.

Lemma 9.4. *If $W \subset \Sigma$ is a connected irreducible semialgebraic set and $w_0 \in W \cap \Lambda$, then $Y + W - w_0 = Y$.*

Proof. By the definition of Σ, every point $w \in W$ satisfies $Y + w \subset \pi^{-1}(V)$. Hence

$$Y + W \subset \pi^{-1}(V).$$

Since $\pi^{-1}(V)$ is Λ-invariant, we also have

$$Y + W - w_0 \subset \pi^{-1}(V).$$

But $Y + W - w_0$ is a connected irreducible real semialgebraic set so Lemma 8.1 implies that there is some irreducible complex algebraic variety Y' such that

$$Y \subset Y + W - w_0 \subset Y' \subset \pi^{-1}(V).$$

By hypothesis, Y is a maximal algebraic variety contained in $\pi^{-1}(V)$. Hence

$$Y = Y + W - w_0 = Y'.$$

□

Lemma 9.5. $\pi(\Theta) = B$.

Proof. First we show that $\pi(\Theta) \subset B$, using the minimality of V. Suppose that $x \in \Theta$. Then

$$Y + x = Y \subset \pi^{-1}(V)$$

so $Y \subset \pi^{-1}(V) - x$.

Hence

$$\pi(Y) \subset V \cap (V - \pi(x))$$

The set $V \cap (V - \pi(x))$ is an algebraic subvariety of A, so the assumption that V is the Zariski closure of $\pi(Y)$ implies that $V = V - \pi(x)$.

Thus $\pi(\Theta)$ stabilises V. Since Θ is connected in the Euclidean topology, $\pi(\Theta)$ is connected in the Euclidean and hence also in the Zariski topology, so $\pi(\Theta) \subset B$.

Now we show that $B \subset \pi(\Theta)$. Let Θ' be the identity component of $\pi^{-1}(B)$ in the Euclidean topology, and note that the image of $\pi_{\Theta'}$ is an analytic subgroup of B of the same dimension as B, so must be equal to B. Hence it will suffice to show that $\Theta' \subset \Theta$.

Since Θ' stabilises $\pi^{-1}(V)$, we have

$$Y + \Theta' \subset \pi^{-1}(V).$$

But $Y + \Theta'$ is an irreducible complex algebraic set containing Y, so the maximality of Y implies that $Y + \Theta' = Y$ as required. □

9.4. **Conclusion of proof.** We have proved that $\dim B > 0$ where B is the stabiliser of V, which is an abelian subvariety of A.

Let A' denote the abelian subvariety A/B. We have proved that $B = \pi(\Theta)$ so we have a diagram of quotient maps

$$\begin{array}{ccc} \mathbb{C}^g & \xrightarrow{\tilde{q}} & \mathbb{C}^g/\Theta \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ A & \xrightarrow{q} & A' \end{array}$$

Let $V' = q(V)$ and let Y' be the closure of $\tilde{q}(Y)$ in $\mathbb{C}^g/\Theta$. Note that $V = q^{-1}(V')$ and $Y = \tilde{q}^{-1}(Y')$.

Now Y' is a maximal irreducible algebraic subvariety in $\pi'^{-1}(V')$. To prove this, suppose that there were some irreducible algebraic subvariety Z' of $\mathbb{C}^g/\Theta$ such that $Y' \subset Z' \subset \pi'^{-1}(V')$. Then

$$Y \subset \tilde{q}^{-1}(Z') \subset \pi^{-1}(V)$$

and each irreducible component of $\tilde{q}^{-1}(Z)$ is an algebraic subvariety of $\mathbb{C}^g$. So the maximality of Y implies that irreducible component of $\tilde{q}^{-1}(Z)$ containing Y is equal to Y, and hence $Z = Y'$.

If $\dim Y' > 0$ then we can apply the above argument to (A', V', Y') to deduce that the stabiliser of V' in A' has positive dimension. But the preimage of this stabiliser under q stabilises V, so this contradicts the fact that $\ker q$ is equal to the stabiliser of V.

Hence Y' is a point and $\pi(Y)$ is a translate of the abelian subvariety $B \subset A$ as required.

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