SINGULARITY AND STRATIFICATION THEORY APPLIED TO DYNAMICAL SYSTEMS

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Abstract. We outline the theory of equivariant transversality for maps equivariant with respect to a compact Lie group. We indicate some applications to generic equivariant bifurcation theory.

1. Introduction

Transversality theory is a basic technical and theoretical tool in the study of smooth mappings, dynamical systems and bifurcation theory. In this paper we describe a version of transversality theory applicable to the study of maps and vector fields which are equivariant under the smooth action of a compact Lie group G. From a local point of view, we will be outlining a theory for the analysis of solutions of symmetric equations. From the global point of view, we will be describing an intersection theory for G-manifolds.

The foundational theory of transversality for G-manifolds was developed in the mid 1970's by Bierstone [1] and the author [10]. The focus of Bierstone's work was on extending Mather's theory of stable mappings to smooth equivariant maps. As part of that program, Bierstone extended Thom's jet transversality theorem to equivariant maps [2]. On the other hand, Field's motivation was to extend parts of the Smale program to equivariant dynamical systems [11]. Much later it turned out that techniques of equivariant transversality had powerful applications to equivariant bifurcation theory [16, 12, 13, 14]. Very recently there have also been applications to equivariant reversible systems [8] and equivariant Hamiltonian systems [6]. In this paper, we emphasise applications of equivariant transversality to the bifurcation theory of equivariant vector fields. For a more comprehensive and careful introduction to the theory we refer the reader to the forthcoming monograph [15] which includes the general theory of equivariant transversality and jet transversality as well as applications to equivariant and reversible equivariant dynamical systems and relative equilibria. Part of the motivation for writing this paper was to provide an introduction to some of the main results described in [15] as they apply to bifurcation theory.

1.1. **Equivariant transversality.** Let G be a compact Lie group of transformations acting smoothly (that is, C^∞) on connected differential manifolds M and N. A map $f : M \to N$ is G-*equivariant* if $f(gx) = gf(x)$, for all $g \in G$, $x \in M$. Suppose that Y is a closed G-invariant submanifold of N. We want to describe the 'generic' intersection

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$f^{-1}(Y)$. If there is no symmetry, then the intersection is generic if f is *transverse* to Y – in symbols $f \pitchfork Y$. That is, $\forall x \in M$, either $f(x) \notin Y$ or $T_x f(T_x M) + T_{f(x)} Y = T_{f(x)} N$. It is easy to describe the local structure of $f^{-1}(Y)$. Every $x \in f^{-1}(Y)$ has an open neighbourhood in $f^{-1}(Y)$ diffeomorphic to an open disk in $\mathbb{R}^n$, where $n = \dim(M) - \dim(N) + \dim(Y)$. In particular,

(a) If $\dim(M) < \dim(N) - \dim(Y)$, then $f^{-1}(Y) = \emptyset$.
(b) $f^{-1}(Y)$ is nonsingular and the local topological type of $f^{-1}(Y)$ is constant.

Neither of these statements need hold in the equivariant context.

Example 1.1. Let $G = \mathbb{Z}_2$ act linearly on $\mathbb{R}^2$ by $(x,t) \mapsto (\pm x, t)$ and on $\mathbb{R}$ by $y \mapsto \pm y$. Let $Y = \{0\} \subset \mathbb{R}$. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth $\mathbb{Z}_2$-equivariant map

$$f(-x,t) = -f(x,t), \quad (x,t) \in \mathbb{R}^2.$$

Since $f(0,t) \equiv 0$, we may write $f(x,t) = xg(x,t)$, where g is smooth and even in

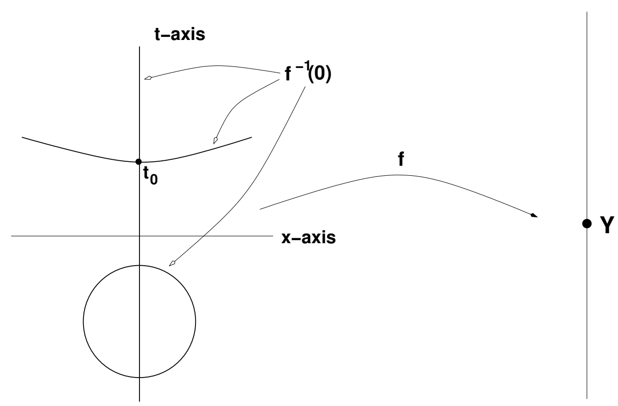

FIGURE 1. An example of a generic $\mathbb{Z}_2$-equivariant intersection

x. Clearly $f^{-1}(Y) \supset \{0\} \times \mathbb{R}$. The map f will be transverse to Y at $(0,t)$ if and only if $g(0,t) \neq 0$. Suppose that $g(0,t_0) = 0$. It follows from the implicit function theorem that if $\frac{\partial g}{\partial t}(0,t_0) \neq 0$, then there will be a curve of solutions to $g(x,t) = 0$ passing through $(0,t_0)$ and distinct from $x = 0$ (in fact perpendicular to $x = 0$ at $(0,t_0)$ – see Figure 1). Consequently, $f^{-1}(Y)$ will be singular at $(0,t_0)$. The singularity cannot be removed by (small) perturbations of f. Indeed, it is easy to construct examples on compact $\mathbb{Z}_2$-manifolds where the singularities in the intersection cannot be removed by any $\mathbb{Z}_2$-equivariant deformation of f.

Unlike what happens when there is no symmetry, it is not easy to give a simple geometric description of what it means for a map $f : M \to N$ to be G-transverse to a G-invariant submanifold Y of N. However, the problem is certainly local and using slice theory we may reduce to the case where M and N are G-representations and Y is the origin of N. We give the local definition of equivariant transversality in section 3 of this paper. It follows from the definition that if f is G-transversal to Y (we write this $f \pitchfork_G Y$), then the local topological type of $Z = f^{-1}(Y)$ is that of a real algebraic variety. Although the local topological type of Z will be locally constant on an open and dense subset Z_0 of Z, it is not (yet) known whether the local topological type is constant on Z_0 (it is for families of equivariant vector fields). The usual openness and density and isotopy theorems hold for G-transversality. Specifically, if $Y \subset N$ is a closed G-invariant submanifold then the set of maps $f : M \to N$ such that $f \pitchfork_G Y$ is an open and dense subset of the space of all smooth G-equivariant maps from M to N (Whitney C^∞-topology). The definition of G-transversal is open in the sense that if f is G-transversal to Y at $x \in M$ then f will be G-transversal to Y at x' for all x' in some neighbourhood of x in M. Finally, there is an equivariant version of Thom's isotopy theorem. However, isotopies will in general *not* be smooth. (We refer to [1, 10, 15] for precise statements and proofs.)

1.2. **Applications to vector fields and bifurcation theory.** We are interested in studying properties of diffeomorphisms and, in particular, vector fields which are symmetric or *equivariant* with respect to G. Noting that the G action on M extends in the obvious way to a smooth G-action on the tangent bundle TM of M, we say that a vector field $X : M \to TM$ is G-equivariant if

$$X(gx) = gX(x),\ (x \in M, g \in G).$$

For simplicity assume for the remainder of the introduction that M is compact and G is finite. Give the space $\mathcal{X} = C^\infty_G(TM)$ of smooth G-equivariant vector fields on M the C^r topology, where $1 \leq r \leq \infty$. It is not hard to show (see [11, 15] for details) that

(1) There is a C^1 open and dense subset $\mathcal{X}_1$ of $C^\infty_G(TM)$ consisting of vector fields all of whose equilibria are hyperbolic.
(2) Given $T > 0$, there is a C^1 open and dense subset $\mathcal{X}_2(T)$ consisting of vector fields with all periodic orbits, period $\leq T$, hyperbolic.
(3) There is a residual subset $\mathcal{X}_3$ of $\cap_{T>0}\mathcal{X}_2(T)$ (C^∞-topology) consisting of vector fields such that all invariant manifolds of equilibria and limit cycles meet G-transversally (equivariant version of the Kupka-Smale theorem).

There is an analogous result when G is compact but not finite. In this case, 'equilibria' (resp. 'periodic orbits') is replaced by 'relative equilibria' (resp. 'relative periodic orbits') and 'hyperbolic' by 'normally hyperbolic'.

Suppose that $X : M \times \mathbb{R} \to M$ is a smooth 1-parameter family of equivariant vector fields on M. For $\lambda \in \mathbb{R}$, set $X_\lambda(x) = X(x, \lambda)$, so that $X_\lambda \in \mathcal{X}$, all $\lambda \in \mathbb{R}$. It follows from (1) that for generic families $X_\lambda \in \mathcal{X}_1$ except for λ lying in a discrete subset $\mathcal{B}(X) \subset \mathbb{R}$. If $\lambda_0 \in \mathcal{B}(X)$ there exists at least one equilibrium x_{λ_0} for X_{λ_0} which is not hyperbolic. We want to describe the typical bifurcation behavior of the family X_λ near the bifurcation point $(x_{\lambda_0}, \lambda_0)$. That is, the typical local structure of the germ of $X^{-1}(0)$ at $(x_{\lambda_0}, \lambda_0)$. In the case where there is no group action, it is well-known that generically $X^{-1}(0)$ is a non-singular curve. The only generic bifurcation of equilibria that we see in 1-parameter families is the *saddle-node* bifurcation and this corresponds to a change of stability (index) along the curve of equilibria. Bifurcations of saddle-node type can occur in families of equivariant vector fields. However, these bifurcations are well-understood and involve little in the way of new ideas – essentially everything is reduced via slices to the non-equivariant case. Our focus will be on investigating bifurcations where the symmetry plays an essential role. Typically in these bifurcations we see the appearance of new branches at the bifurcation point that have different (less) symmetry: a symmetry breaking bifurcation. Furthermore, the germ of $X^{-1}(0)$ will generally be singular (this is the case in almost all known examples). We describe some of the basic ideas and indicate the proof of a characteristic genericity and determinacy theorem in section 4. All of what we describe works also for general compact groups G, relative equilibria and (relative) periodic orbits. There is also a theory for equivariant maps (see [14, 15]).

We start with a section covering basic definitions and results on smooth G-actions, equivariant maps, stratifications and semialgebraic sets. Much of this section is directed towards experts in dynamical systems who are not familiar with singularity theory and the geometry of stratified sets.

2. Preliminaries and notation

2.1. **Smooth G-actions.** We start by reviewing some facts about smooth actions by compact Lie groups. Proofs and more details may be found in the text by Bredon [7, Chapter VI].

Let G be a compact Lie group acting smoothly on the connected differential manifold M. If $x \in M$, let $Gx = \{gx \mid g \in G\}$ denote the G-orbit through x and $G_x = \{g \in G \mid gx = x\}$ denote the isotropy subgroup of (the action of) G at x. Each isotropy group G_x is a closed (therefore Lie) subgroup of G and Gx is (G-equivariantly) diffeomorphic to the compact homogeneous space G/G_x.

Points $x, y \in M$ have the same *isotropy type* if G_x, G_y are conjugate subgroups of G. If $y = gx$, then $G_y = gG_xg^{-1}$ and so all points on the same G-orbit have the same isotropy type. Denote the set of isotropy types for the action of G on M by $\mathcal{O} = \mathcal{O}(M, G)$. If M is compact or G is a linear action on a finite dimensional vector space, then $\mathcal{O}$ is finite.

Given $x \in M$, let $\iota(x) \in \mathcal{O}$ denote the isotropy type of x. If $\tau \in \mathcal{O}$, define $M_\tau = \{x \in M \mid \iota(x) = \tau\}$. In this way we define a partition $\mathcal{M} = \{M_\tau \mid \tau \in \mathcal{O}\}$ into points of the same isotropy type. We refer to $\mathcal{M}$ as the stratification of M by isotropy type or the *orbit stratification* of M. Using slices (see Bredon [7, Chapter VI]), it follows easily that each stratum M_τ is a smooth G-invariant submanifold of M and that $\mathcal{M}$ is a Whitney stratification of M (see subsection 2.5).

We define a partial order $<$ on $\mathcal{O}$ by $\tau < \mu$ if $\exists H \in \tau$, $\exists J \in \mu$ such that $H \subsetneq J$. We remark that this condition holds if $\partial M_\tau \cap M_\mu \neq \emptyset$. The converse is true for linear actions.

There exists a unique minimal isotropy type ν and M_ν is an open and dense subset of M. In the sequel we refer to ν as the *principal* isotropy type and M_ν as the principal stratum. If τ is a *maximal* isotropy type, M_τ is always a closed submanifold of M. Linear actions have a unique maximal isotropy type (G). Nonlinear actions may have many maximal isotropy types.

If H is a subset of G, let $M^H = \{x \in M \mid Hx = x\}$ (in the bifurcation literature, this subspace is often denoted by $\mathrm{Fix}(H)$). The fixed point space M^H is a closed submanifold of M and $M^H = M^{\langle H \rangle}$ ($\langle H \rangle$ is the closure of the subgroup of G generated by H). If $H \in \tau \in \mathcal{O}$, then $M_\tau^H \subset M^H$. The inclusion will be strict unless τ is a maximal isotropy type. We have

$$M_\tau = \cup_{H \in \tau} M_\tau^H.$$

Suppose N is a G-manifold and $f : M \to N$ is G-equivariant. For all $x \in M$, $G_{f(x)} \supset G_x$. It follows that given $H \subset G$ we have

$$f(M^H) \subset N^H.$$

If f is 1:1 then $\mathcal{O}(M) \subset \mathcal{O}(N)$ and f preserves isotropy type.

$$f(M_\tau) \subset N_\tau, \text{ for all } \tau \in \mathcal{O}(M). \tag{2.1}$$

If f is a diffeomorphism we have equality in (2.1).

2.2. **Equivariant vector fields.** Suppose that X is an equivariant vector field on M. We list some simple consequences of equivariance and (2.1).

(1) If $X(x) = 0$, then $X(gx) = 0$, all $g \in G$. (Equilibria occur in group orbits.)
(2) The flow $\phi_t^X = \phi_t$ of X is G-equivariant: $\phi(gx, t) = \phi_t(gx) = g\phi_t(x)$, all $x \in M$, $g \in G$ (we assume flows are complete – defined for all time. This is so if M is compact and can be achieved by time rescaling if M is not compact).
(3) The flow respects the orbit stratification $\mathcal{M}$ and X is tangent to each orbit stratum.
(4) The G-orbit Gx is a *relative equilibrium* of X if X is tangent to Gx (by equivariance, tangent at one point will suffice). Equilibria are always relative equilibria. The converse is only true if G is finite. What we discuss applies to relative equilibria and non-finite groups – however, we only describe results for equilibria.

2.3. **Representations.** Let V be a real finite dimensional inner product space. An orthogonal representation (V, G) of the compact Lie group G on V is a homomorphism $\rho : G \to \mathrm{O}(V)$. We have a corresponding action of G on V by orthogonal transformations. The action is *trivial* if $G\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$ (that is $V_{(G)} = V$) and *irreducible* if there are no proper G-invariant linear subspaces of V.

In this paper we assume representations are defined over $\mathbb{R}$. Let $L_G(V, V)$ denote the space of linear G-equivariant maps. If (V, G) is an irreducible representation then it follows from Frobenius' theorem that $L_G(V, V)$ is isomorphic (as a division algebra) to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$ (the quaternions).

We shall only consider *absolutely irreducible* representations where $L_G(V, V) \approx \mathbb{R}$ (we refer to [13, 15] for the general theory).

Example 2.1. Let $\mathbf{D}_n \subset \mathrm{O}(2)$ denote the group of isometries of the regular n-gon. The induced action of $\mathbf{D}_n$ on $\mathbb{R}^2$ is absolutely irreducible for all $n \geq 3$. Similarly, the symmetry groups of the platonic solids and SO(3), O(3) define absolutely irreducible representations on $\mathbb{R}^3$. In Figure 2 we show the orbit stratification of $\mathbb{R}^2$ for the standard action of $\mathbf{D}_4$ on $\mathbb{R}^2$. Note that non-zero points on the diagonals $x^2 = y^2$ and axes $xy = 0$ have isotropy isomorphic to $\mathbb{Z}_2$. However, these isotropy groups are not conjugate within $\mathbf{D}_4$ and so define *different* isotropy types.

2.4. **Smooth invariant theory.** Let (V, G), (W, G) be G-representations. Let $P(V)^G$ denote the $\mathbb{R}$-algebra of G-invariant polynomials on V and $P_G(V, W)$ denote the $P(V)^G$-module of G-equivariant polynomial maps from V to W. It follows from the Hilbert basis theorem (using Haar integration) that $P(V)^G$ is finitely generated as an $\mathbb{R}$-algebra and $P_G(V, W)$ is finitely generated as a $P(V)^G$-module (see [24, 19]).

Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a minimal set of homogeneous generators for the $P(V)^G$-module $P_G(V, W)$. Let $\deg(F_j) = d_j$ and label the generators so that $0 \leq d_1 \leq \ldots \leq d_k$. It follows easily from minimality and the homogeneity of the F_j that the number of generators k and the degrees $d_1, \ldots, d_k$ depend only on the isomorphism class of the representations (V, G), (W, G) (see also remarks 3.3).

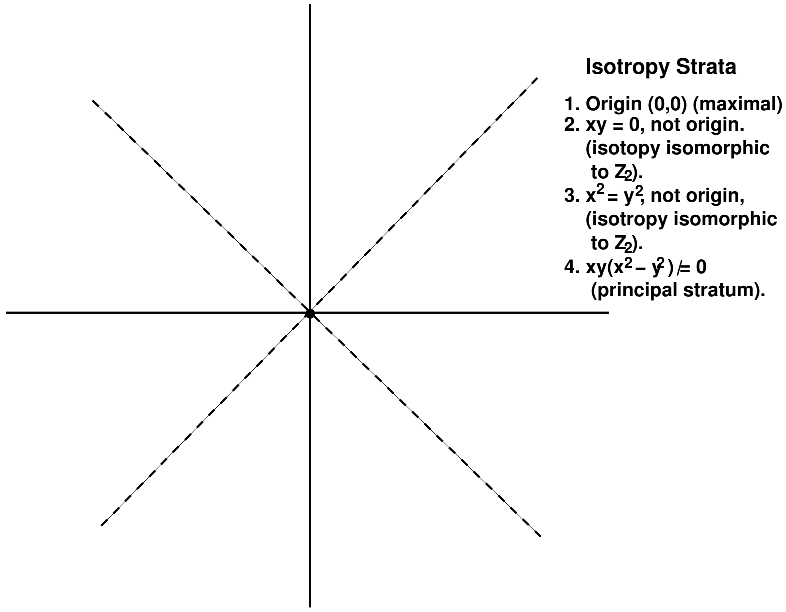

FIGURE 2. Orbit stratification of $\mathbb{R}^2$ when $G = \mathbf{D}_4$

Let $C^\infty(V)^G$ denote the $\mathbb{R}$-algebra of G-invariant smooth functions on V and $C^\infty_G(V, W)$ denote the $C^\infty(V)^G$-module of G-equivariant smooth maps from V to W. It follows from the equivariant version of Stone-Weierstrass approximation theorem that the $C^\infty(V)^G$-submodule of $C^\infty_G(V, W)$ generated by $\mathcal{F}$ is a dense subset of $C^\infty_G(V, W)$ (here, as elsewhere in this section, we always take the C^∞-topology). Since $\mathcal{F}$ consists of a finite set of polynomials, it follows by results of Malgrange [21, 27] on closed ideals of differentiable functions that the $C^\infty(V)$-submodule of $C^\infty(V, W)$ generated by $\mathcal{F}$ is closed in the C^∞-topology. Averaging over G using Haar measure, it follows that the $C^\infty(V)^G$-submodule of $C^\infty_G(V, W)$ generated by $\mathcal{F}$ equals $C^\infty_G(V, W)$. That is, every $f \in C^\infty_G(V, W)$ may be written

$$f(x) = \sum_{j=1}^{k} f_j(x) F_j(x),$$

where $f_j \in C^\infty(V)^G$. The coefficient functions f_j will not generally be unique.

Remarks 2.2. (1) Although we will not need it here, we recall the basic result on smooth invariants proved by Schwarz [26]. This states if $p_1, \dots, p_\ell$ is a set of polynomial generators for the $\mathbb{R}$-algebra $P(V)^G$, then every smooth invariant may be written as a smooth function of $p_1, \dots, p_\ell$. If we write $P = (p_1, \dots, p_\ell) : V \to \mathbb{R}^\ell$ and let $P^\star : C^\infty(\mathbb{R}^\ell) \to C^\infty(V)^G$ denote the mapping defined by composition with P, then Schwarz's result amounts to showing that $P^\star(C^\infty(\mathbb{R}^\ell)) \subset C^\infty(V)^G$ is a *closed* subspace of $C^\infty(V)$ in the C^∞-topology. ($P^\star(C^\infty(\mathbb{R}^\ell))$ is dense in $C^\infty(V)^G$ by the Weierstrass approximation theorem). Schwarz's original proof used properties of the G-action. Subsequently, it has been shown that $P^\star(C^\infty(\mathbb{R}^\ell))$ is a closed linear subspace of $C^\infty(V)$ with closed complement whenever P is a proper polynomial map (see for example [5, 4, 3] and note that rather simple proofs of Mather's extension [23] showing that $P^\star$ has a continuous linear section can be given based on results of Vogt and Wagner [28, 29]).

(2) Schwarz's result on smooth invariants can be used to give an alternative proof of the previous result on smooth equivariants. The method depends on an observation of Malgrange and may be found in [24] or [1, §3].
(3) We have stated our results with the domain of functions and maps equal to V. Similar results hold if we replace V by any nonempty G-invariant open subset of V. It is also not necessary (or always desirable) to assume that polynomial generators are homogeneous.

2.5. **Semialgebraic sets and stratifications.** We start with generalities about semialgebraic sets and their stratifications and conclude by describing the *canonical* "minimum" stratification of a semialgebraic set. We refer the reader to Coste [9], Mather [22], Gibson et al. [18] or Risler [25] for proofs and further details about semialgebraic sets.

Definition 2.3. A semialgebraic subset X of $\mathbb{R}^n$ is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid p_i(x) = 0,\ q_j(x) > 0\},$$

where $p_i, q_j : \mathbb{R}^n \to \mathbb{R}$ are a finite set of polynomials.

(P0) *The collection of semialgebraic subsets of $\mathbb{R}^n$ is closed under finite union, intersection and complementation.*

(P1) *The closure, interior and frontier of a semialgebraic set $X \subset \mathbb{R}^n$ are semialgebraic. The frontier ∂X of X is of dimension strictly less than that of X.*

(P2) *A semialgebraic subset has finitely many connected components.*

(P3) [Tarski-Seidenberg theorem] *If $P : \mathbb{R}^n \to \mathbb{R}^m$ is a polynomial and $X \subset \mathbb{R}^n$ is semialgebraic, then $P(X)$ is a semialgebraic subset of $\mathbb{R}^m$.*

Stratifications. Recall that a *stratification* $\mathcal{S}$ of a subset X of $\mathbb{R}^n$ is a locally finite partition of X into smooth and connected submanifolds of $\mathbb{R}^n$ called *strata*. We denote the union of the i-dimensional strata by $\mathcal{S}_i$, $i \geq 0$. By abuse of notation, we also regard $\mathcal{S}_i$ as the set of i-dimensional strata. If X is semialgebraic, we say that $\mathcal{S}$ is a *semialgebraic stratification* if each stratum is semialgebraic.

In order to obtain a satisfactory definition of transversality to a stratified set we need to recall the recall some facts about the Whitney regularity conditions.

Definition 2.4. A stratification $\mathcal{S}$ of a set $X \subset \mathbb{R}^n$ satisfies Whitney's condition (b) if given any pair $U, V \in \mathcal{S}$ then for all $u \in U \cap \overline{V}$ and sequences $(u_i) \subset U$, $(v_i) \subset V$ such that

1. $u_i \to u$ and $v_i \to u$,
2. the line joining u_i to v_i converges (in $P^{n-1}(\mathbb{R})$) to a line L, and
3. the family of tangent planes $T_{v_i}Q$ converges in the Grassmannian of $\dim(V)$ planes to a plane P,

we have $P \supset L$.
If the stratification satisfies Whitney's condition (b), we refer to $\mathcal{S}$ as a Whitney stratification.

Remark 2.5. If $\mathcal{S}$ satisfies Whitney's condition (b), it follows easily that $\mathcal{S}$ satisfies Whitney's condition (a). That is, given a pair $U, V \in \mathcal{S}$, $u \in U \cap \overline{V}$ and sequence $(v_i) \subset V$ such that $v_i \to u$ and $T_{v_i}V \to P$, we have $P \supset T_u U$.

(P4) [Frontier condition] *If $\mathcal{S}$ is a Whitney stratification of a (semialgebraic) subset of $\mathbb{R}^n$ then the frontier of every stratum is a union of lower dimensional strata.*

If $\mathcal{S}$ is a Whitney stratification of a semialgebraic subset of $\mathbb{R}^n$ then a stratum S is 'top-dimensional' if S is not contained in the union of frontiers of other strata.

Filtrations. Let $\mathcal{S}$ be a Whitney stratification of a semialgebraic set $X \subset \mathbb{R}^n$. We define the associated filtration of X by dimension to be the filtration (X_i) of X obtained by taking X_i to be the union of all strata of dimension $\leq i$. If $\mathcal{T}$ is another Whitney semialgebraic stratification of X, we write $\mathcal{S} < \mathcal{T}$ if there exists an index j such that $X_j \supsetneq T_j$ and $X_i = T_i$, for $i > j$. We say $\mathcal{S}$ is minimal if $\mathcal{S} < \mathcal{T}$ for all Whitney semialgebraic stratifications $\mathcal{T} \neq \mathcal{S}$ of X.

(P5) [22] *Every semialgebraic subset X of $\mathbb{R}^n$ has a canonical minimal stratification.*
(Mather shows that the canonical minimal semialgebraic stratification of a semialgebraic set is minimal amongst all stratifications by smooth manifolds – not just semialgebraic (or semianalytic) stratifications.)

Henceforth, we refer to the stratification of X given by (P5) as the *canonical stratification* of X.

Transversality to stratified sets. Let $\mathcal{S}$ be a Whitney stratification of the closed subset $X \subset \mathbb{R}^n$, M be a differential manifold and $f : M \to \mathbb{R}^n$ be a smooth map. Given $x \in M$, f is transverse to $\mathcal{S}$ at x if either $f(x) \notin X$ or $f(x) \in U \in \mathcal{S}$ and f is transverse to $U \subset \mathbb{R}^n$ at x. It follows from Whitney regularity (in fact (a)-regularity) that if f is transverse to $\mathcal{S}$ at x then f will be transverse to $\mathcal{S}$ at all points in some neighbourhood of x in M.

We say f is transverse to $\mathcal{S}$ if f is transverse to $\mathcal{S}$ at all points of M. In case $\mathcal{S}$ is the canonical stratification of the semialgebraic set X, we often just say f is transverse to X and write $f \pitchfork X$.

Using (b)-regularity it may be shown that if f is transverse to a Whitney stratification $\mathcal{S}$ then an isotopy theorem holds (though isotopies will in general only be continuous). For further details on all of this we refer to [22]. We caution that although the theory works well for the canonical stratification of a semialgebraic set, it is well known that the canonical stratification may sometimes not be the most natural Whitney stratification.

3. Local theory of equivariant transversality

Let $f : M \to N$ be a G-equivariant diffeomorphism and P be a G-invariant closed submanifold of N. Just as in standard transversality theory, it is easy to give a local description of the intersection $f^{-1}(P)$ in terms of solutions to equivariant equations defined on a representation (see [1, 10, 15] for details). In what follows we assume this reduction and focus on the issue of finding generic conditions for solutions of equivariant equations.

Suppose then that (V, G), (W, G) are finite dimensional real G-representations. Following 2.4, let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a minimal set of homogenous generators for the $P(V)^G$-module $P_G(V, W)$ and set degree$(F_j) = d_j$, labelling generators so that $0 \leq d_1 \leq d_2 \leq \ldots \leq d_k$.

Lemma 3.1. *Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a minimal set of homogenous generators for $P_G(V, V)$. Then any relation of the form*

$$\sum_{j=1}^{k} p_j F_j = 0, \ (p_j \in P(V)^G),$$

implies that $p_j(0) = 0$, $1 \leq j \leq k$. The same result holds if we allow $p_j \in C^\infty(V)^G$.

Proof. If $p \in P(V)^G$, let p^ℓ denote the homogeneous part of p of degree ℓ. For $1 \leq i \leq k$ we have

$$-p_i F_i = \sum_{j \neq i} p_j F_j.$$

Taking the homogeneous parts of degree d_i we see that

$$-p_i(0) F_i = \sum_{j \neq i} p_j^{d_i - d_j} F_j.$$

Hence $p_i(0) = 0$ by the minimality of $\mathcal{F}$. If we allow the coefficients p_i to be smooth invariants, the result follows immediately by taking the d_k-jet of $\sum_{j=1}^{k} p_j F_j$ at the origin and applying the result for polynomials. □

Let $\mathfrak{M} = \{p \in P(V)^G \mid p(0) = 0\}$ and $\mathfrak{M}_\infty = \{f \in C^\infty(V)^G \mid f(0) = 0\}$.

Lemma 3.2. (1) *Any minimal set of homogeneous generators for $P_G(V, W)$ maps to a vector space basis of $P_G(V, W)/\mathfrak{M} P_G(V, W)$.*
(2) *$C^\infty_G(V, W)/\mathfrak{M}_\infty C^\infty_G(V, W) \approx P_G(V, W)/\mathfrak{M} P_G(V, W)$ (as vector spaces).*

Proof. (1) follows from Lemma 3.1, (2) from 2.4 (smooth invariant theory). □

Remarks 3.3. (1) It follows from Lemma 3.2 that the number of polynomials in a minimal set of homogeneous generators for $P_G(V, W)$ depends only on the isomorphism class of the representations V and W.
(2) If $\mathcal{F}$ is a minimal set of homogeneous generators for $P_G(V, W)$, then the set of degrees (counting multiplicities) $\{d_1, \dots, d_k\}$ depends only on isomorphism class of the representations V and W.
(3) For our purposes it suffices to restrict attention to homogeneous generators. However, when it comes to proving openness of G-transversality it is necessary to allow for sets of inhomogeneous generators (see [1, 15]).

Set $\mathbb{U} = P_G(V, W)/\mathfrak{M} P_G(V, W)$ and let $\Pi : C^\infty_G(V, W) \to \mathbb{U}$ be the projection given by Lemma 3.2.

It follows from lemma 3.2 that $\mathcal{F}$ determines a vector space isomorphism $I_\mathcal{F}$ between $\mathbb{U}$ and $\mathbb{R}^k$. Set $\gamma = I_\mathcal{F} \Pi : C^\infty_G(V, W) \to \mathbb{R}^k$. If $f = \sum f_j F_j$, then $\gamma(f) = (f_1(0), \dots, f_k(0))$.

Set $d = d_k$. For $f \in C^\infty_G(V, W)$, let $J^d(f)$ denote the d-jet (Taylor polynomial of degree d) of f at the origin. If $J^d(f) = 0$ then $\gamma(f) = 0$. Hence γ factorizes as

$$C^\infty_G(V, W) \xrightarrow{J^d} P^{(d)}_G(V, W) \xrightarrow{\bar{\gamma}} \mathbb{R}^k,$$

where $\bar{\gamma} = \gamma | P^{(d)}_G(V, W)$. It follows that γ is continuous if we give $C^\infty_G(V, W)$ the C^r-topology, $r \geq d$ (Whitney or uniform convergence on compact sets).

Lemma 3.4. *Suppose V, W are G-representations and $\mathbb{R}^s$ is a trivial G-representation. Every minimal set of homogeneous generators $\mathcal{F}$ for $P_G(V,W)$ defines a minimal set of homogeneous generators for $P_G(V\times\mathbb{R}^s, W)$. (Each $F \in \mathcal{F}$ defines a map $F : V\times\mathbb{R}^s \to W$ by $F(x,t) = F(x)$.)*

Proof. We leave this as an easy exercise for the reader. □

Suppose that V, W are G-representations and $\mathbb{R}^s$ is a trivial G-representation. It follows from Lemmas 3.2, 3.4 that we have a linear map $\Pi^s : C^\infty_G(V\times\mathbb{R}^s, W) \to C^\infty(\mathbb{R}^s, \mathbb{U})$ defined by $\Pi^s(f)(t) = \Pi(f_t) \in \mathbb{U}$. Given $f \in C^\infty_G(V\times\mathbb{R}^s, W)$ we may write

$$f(x,t) = \sum_{j=1}^{k} f_j(x,t)F_j(x),\ (f_j \in C^\infty(V\times\mathbb{R}^s)^G).$$

We define $\gamma = \gamma^s = I_{\mathcal{F}}\Pi^s : C^\infty_G(V\times\mathbb{R}^s, W) \to C^\infty(\mathbb{R}^s, \mathbb{R}^k)$ by

$$\gamma(f)(t) = (f_1(0,t), \ldots, f_k(0,t)),\ (t \in \mathbb{R}^s, f \in C^\infty_G(V\times\mathbb{R}^s, W))$$

When f is fixed we usually write γ_f rather than $\gamma^s(f)$.

Lemma 3.5. *The map $\gamma^s : C^\infty_G(V\times\mathbb{R}^s, W) \to C^\infty(\mathbb{R}^s, \mathbb{R}^k)$ is continuous with respect to the C^∞-topologies on $C^\infty(\mathbb{R}^s, \mathbb{R}^k)$ and $C^\infty_G(V\times\mathbb{R}^s, W)$.*

Proof. (following [1]) Let $\alpha : (C^\infty(V\times\mathbb{R}^s)^G)^k \to C^\infty_G(V\times\mathbb{R}^s, W)$ and $\beta : (C^\infty(V\times\mathbb{R}^s)^G)^k \to C^\infty(\mathbb{R}^s, \mathbb{R}^k)$ be defined by

$$\alpha(f_1,\ldots,f_k) = \sum_{j=1}^{k} f_jF_j, \text{ and } \beta(f_1,\ldots,f_k)(t) = (f_1(0,t),\ldots,f_k(0,t)),\ t \in \mathbb{R}^s.$$

Both α and β are continuous (with respect to the C^∞-topologies on function spaces). Since α is a continuous linear surjective map between Fréchet spaces, it follows by the Open Mapping Theorem that α is an open map. Since $\gamma\alpha = \beta$ it follows that for all open subsets V in $C^\infty(\mathbb{R}^s, \mathbb{R}^k)$, $\alpha(\beta^{-1}(V)) = \gamma^{-1}(V)$ is open and so γ is continuous. □

3.1. **The universal variety.**

Define the polynomial map $\boldsymbol{\vartheta} \in P_G(V\times\mathbb{R}^k, W)$ by

$$\boldsymbol{\vartheta}(x,t) = \sum_{j=1}^{k} t_jF_j(x),\ ((x,t) \in V\times\mathbb{R}^k).$$

Define $\Sigma = \boldsymbol{\vartheta}^{-1}(0) \subset V\times\mathbb{R}^k$ and note that Σ is a G-invariant algebraic subset of $V\times\mathbb{R}^k$. We sometimes refer to Σ as the *universal variety*, and $\boldsymbol{\vartheta}$ as the *universal polynomial* (for the pair (V,W)). We have

$$\Sigma \supset V = V\times\{0\} \subset V\times\mathbb{R}^k, \tag{3.2}$$

$$\Sigma \supset \mathbb{R}^k = \{0\}\times\mathbb{R}^k \subset V\times\mathbb{R}^k, \text{ if } W^G = \{0\}. \tag{3.3}$$

Every $f \in C^\infty_G(V\times\mathbb{R}^s, W)$ factorizes through $\boldsymbol{\vartheta}$. Specifically, if $f \in C^\infty_G(V\times\mathbb{R}^s, W)$, then we may write $f(x,s) = \sum_{j=1}^k f_j(x,s)F_j(x)$, where $f_j \in C^\infty(V\times\mathbb{R}^s)^G$. Define $\Gamma_f : V\times\mathbb{R}^s \to V\times\mathbb{R}^k$ by $\Gamma_f(x,s) = (x, f_1(x,s),\ldots,f_k(x,s))$. Then

$$\begin{aligned} f &= \boldsymbol{\vartheta}\circ\Gamma_f, \\ f^{-1}(0) &= \Gamma_f^{-1}(\Sigma). \end{aligned}$$

3.2. **The local definition of G-transversality.** Let $\mathcal{S}$ denote the canonical minimal stratification of Σ. Since Σ is algebraic, each stratum of $\mathcal{S}$ is a semialgebraic subset of $V \times \mathbb{R}^k$. Since the stratification is canonical and Σ is G-invariant, it follows that G permutes strata. In particular, group orbits of connected strata are G-manifolds. Our convention will be that if $S \in \mathcal{S}$ is a stratum then S is a G-manifold and S/G (rather than S) is connected. A smooth map is transverse to Σ if the map is transverse to each stratum of $\mathcal{S}$.

Definition 3.6. Let $f \in C^\infty_G(V, W)$. The map f is G-transversal to $0 \in W$ at $0 \in V$ if $\Gamma_f : V \to V \times \mathbb{R}^k$ is transverse to Σ at $0 \in V$.

Remark 3.7. It follows from the openness property of transversality to a Whitney stratification that if f is G-transversal to $0 \in W$ at $0 \in V$, then $\Gamma_f : V \to V \times \mathbb{R}^k$ is transverse to Σ on some G-invariant neighbourhood U of $0 \in V$. In fact, the G-transversality of f to $0 \in W$ at $0 \in V$ implies the G-transversality of f to $0 \in W$ on a neighbourhood of $0 \in V$. However, we will not discuss this point further here.

We omit the verification that the definition is is independent of choice of minimal set of homogeneous generators for $P_G(V, W)$ (see [15, 10]). Granted this independence it still remains to show that the definition is independent of the coefficient functions f_j. We do this by proving that the transversality of Γ_f to Σ at $0 \in V$ is determined by the values of $f_1(0), \ldots, f_k(0)$ – which are uniquely determined by the choice of $\mathcal{F}$. The approach we outline has two advantages: it gives a more geometric and natural definition of G-transversality, and it gives immediate applications to equivariant bifurcation theory.

Henceforth we shall assume that $V^G = \{0\}$ and let $s \in \mathbb{N}$. We regard $\mathbb{R}^s$ as embedded in $V \times \mathbb{R}^s$ as $\{0\} \times \mathbb{R}^s$.

Theorem 3.8. *There exists a natural Whitney semialgebraic stratification $\mathcal{A}$ of $\mathbb{U}$ with the property that $f \in C^\infty_G(V \times \mathbb{R}^s, W)$ is G-transverse to $0 \in W$ on $K \subset \mathbb{R}^s \subset V \times \mathbb{R}^s$, if and only if $\Pi^s(f) : \mathbb{R}^s \to \mathbb{U}$ is transverse to $\mathcal{A}$ along K.*

Remarks 3.9. (1) We are restricting attention to G-transversality along sets of points in the domain $V \times \mathbb{R}^s$ with trivial isotropy. This allows us to avoid discussion of openness of G-transversality. Note, however, that we obtain transversality to Σ on an open neighbourhood of K in $V \times \mathbb{R}^s$.
(2) If we choose a minimal set of homogeneous generators $\mathcal{F}$ for $P_G(V, W)$, then $I_{\mathcal{F}}(\mathcal{C})$ is a Whitney stratification $\mathcal{C}_{\mathcal{F}}$ of $\mathbb{R}^k$ and $\Pi^s(f) : \mathbb{R}^s \to \mathbb{U}$ is transverse to $\mathcal{C}$ along K if and only if $\gamma_f : \mathbb{R}^s \to \mathbb{R}^k$ is transverse to $\mathcal{C}_{\mathcal{F}}$ along K. Consequently, if $s = 0$, the theorem implies that f is G-transverse to $0 \in W$ at 0 if and only if $\gamma_f(0)$ belongs to the top (k-) dimensional stratum of $\mathcal{C}_{\mathcal{F}}$. That is, G-transversality is determined by $(f_1(0), \ldots, f_k(0))$. Similar remarks hold for $s > 0$. In particular, if $s < k$, then a necessary condition for G-transversality along K is that $\gamma_f|K$ does not take values in the strata of $\mathcal{C}_{\mathcal{F}}$ which are of dimension less than $k - s$.

Proof. We sketch the construction of the stratification $\mathcal{A}_{\mathcal{F}}$. The proof that the stratification $I_{\mathcal{F}}^{-1}(\mathcal{A}_{\mathcal{F}})$ of $\mathbb{U}$ is independent of $\mathcal{F}$ is in [15].

Denote the canonical stratification of Σ by $\mathcal{S}$. Extend $\mathcal{S}$ to a Whitney semialgebraic stratification $\mathcal{S}^\star$ of $V \times \mathbb{R}^k$ by adding the stratum $(V \times \mathbb{R}^k) \setminus \Sigma$. Define $\mathcal{A}_{\mathcal{F}} = \{S^G \mid S \in$

$\mathcal{S}^\star,\ S^G \neq \emptyset\}$. It is straightforward to verify that $\mathcal{A}_\mathcal{F}$ is a Whitney semialgebraic stratification of $\mathbb{R}^k$. Moreover, it is obvious that $\Gamma_f \pitchfork \Sigma$ along K if and only if $\gamma_f \pitchfork \mathcal{C}_\mathcal{F}$ along K. □

Remark 3.10. If $W^G = \{0\}$, then $\Sigma \supset \Sigma^G = \mathbb{R}^k$ and $\mathcal{C}_\mathcal{F} = \{S^G \mid S \in \mathcal{S},\ S^G \neq \emptyset\}$. If $(W,G) = (V,G)$, then $\mathcal{C}_\mathcal{F}$ is a *union* of $\mathcal{S}$-strata.

Examples 3.11. (1) Let $V = W = \mathbb{R}$ and take the nontrivial representation of $\mathbb{Z}_2$ on V. In this case $\Sigma = \{(x,t) \mid tx = 0\}$ and $\mathcal{C}_\mathcal{F} = \{\mathbb{R} \setminus \{0\}, \{0\}\}$. If $f \in C^\infty_{\mathbb{Z}_2}(V \times \mathbb{R}, W)$ then $f(x,t) = g(x,t)x$. and f is $\mathbb{Z}_2$-transversal to $0 \in W$ along $K \subset \mathbb{R}$ if and only if $g(0,\cdot) : \mathbb{R} \to \mathbb{R}$ is transverse to $0 \in \mathbb{R}$ along K.
(2) Let $\mathrm{O}(2)$ act on $V = \mathbb{C}^2$ as $e^{\imath\theta}(z_1,z_2) = (e^{2\imath\theta}z_2, e^{\imath\theta}z_2)$ and on $W = \mathbb{C}$ in the standard way. A minimal set of homogeneous generators $\mathcal{F}$ of $P_{\mathrm{O}(2)}(\mathbb{C}^2,\mathbb{C})$ is given by $F_1(z_1,z_2) = z_2$ and $F_2(z_1,z_2) = z_1\bar{z}_2$. The natural stratification $\mathcal{C}_\mathcal{F}$ of $\mathbb{R}^2$ is given by $\{\{(0,0)\}, \{t_1 = 0, t_2 \neq 0\}, \{t_1 \neq 0\}\}$. The top-dimensional stratum is *not* a stratum of the canonical stratification $\mathcal{S}$ of Σ. Consequently, even if $V^G = W^G = \{0\}$ it does not follow that the induced stratification of $\mathbb{R}^k$ is a union of strata of $\mathcal{S}$.

4. Applications to equivariant bifurcation theory

We give a simple application of equivariant transversality to generic equivariant bifurcation theory and conclude with some examples that illustrate some of the phenomena that can be expected for various classes of vector fields.

Suppose that (V,G) is a nontrivial absolutely irreducible representation of the compact Lie group G (see 2.3). Let $C^\infty_G(V \times \mathbb{R}, V)$ denote the space of smooth 1-parameter families of G-equivariant vector fields on V. Suppose $X \in C^\infty_G(V \times \mathbb{R}, V)$. Since (V,G) is a irreducible representation, $V^G = \{0\}$ and it follows by G-equivariance that

$$X_\lambda(0) = 0, \quad (\lambda \in \mathbb{R}).$$

We refer to $x = 0$ as the *trivial* solution of X. Since $DX_\lambda(0) \in L_G(V,V)$, it follows by absolute irreducibility that

$$DX_\lambda(0) = \sigma(\lambda) I_V,$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is smooth. Since bifurcations of the trivial solution occur at points where $DX_\lambda(0)$ is singular, it follows that the bifurcation set for the trivial solution is precisely the zero set of σ.

It is natural to impose the generic condition that at bifurcation points λ_0, $\sigma'(\lambda_0) \neq 0$. It then follows that bifurcation points are isolated. Further, by a local smooth change of λ-coordinates, we may require that $\sigma(\lambda) = \lambda$. Since we shall only be interested in the local behaviour of the zero set of X near a generic bifurcation point, it is no loss of generality to restrict to the space $\mathcal{V}(V,G)$ of smooth equivariant families on V which are of the form

$$X_\lambda(x) = \lambda x + Q(x,\lambda),$$

where $Q(x,\lambda) = O(\|x\|^2)$ on compact subsets of $V \times \mathbb{R}$. For any $X \in \mathcal{V}(V,G)$ there is a non-degenerate change of stability of the trivial solution at $\lambda = 0$.

Let $\mathcal{G}(V,G) \subset \mathcal{V}(V,G)$ consist of those families which are G-transversal to $0 \in V$ at $(0,0) \in V \times \mathbb{R}$. If $X \in \mathcal{G}(V,G)$, the germ of $X^{-1}(0)$ is stable (topologically) under perturbation of X.

Suppose that $\mathcal{F} = \{F_1, \dots, F_k\}$ is a minimal set of homogeneous generators for the $P(V)^G$-module $P_G(V, V)$. Since (V, G) is absolutely irreducible we may take $F_1 = I_V$ and then $d_j \geq 2$, $j \geq 2$. Let $\Sigma \subset V \times \mathbb{R}^k$ denote the zero set of

$$\boldsymbol{\vartheta}(x, t) = \sum_{j=1}^{k} t_j F_j(x) = t_1 x + \sum_{j=2}^{k} t_j F_j(x).$$

Let $\Sigma = \cup_{\tau \in \mathcal{O}(V,G)} \Sigma_\tau$ denote the partition of Σ into points of the same isotropy type. It may be shown [12, 13] that each Σ_τ is a semialgebraic submanifold of $V \times \mathbb{R}^k$. To simplify our exposition, assume from now on that G is finite. It follows that $\dim(\Sigma_\tau) = k$, all $\tau \in \mathcal{O}(V, G)$. In particular, we have $\Sigma_{(G)} = \Sigma^G = \{0\} \times \mathbb{R}^k$. Observe that if $t_1 \neq 0$, then $(0, t_1) \notin \overline{\Sigma_\tau}$, all $\tau \neq (G)$. It follows that $\{t_1 \neq 0\} \subset \mathbb{R}^k$ is contained in a top (k) dimensional stratum of the minimal stratification of Σ.

Let $\mathcal{A} = \{A_0, \dots, A_k\}$ denote the natural stratification of $\mathbb{R}^k$ induced from the minimal stratification of Σ. We always have $A_k \supset \{t_1 \neq 0\}$.

If $X \in \mathcal{V}(V, G)$,

$$X(x, \lambda) = f_1(x, \lambda)x + \sum_{j=2}^{k} f_j(x, t) F_j(x),$$

where $f_j \in C^\infty(V \times \mathbb{R})^G$ and $f_1(0, \lambda) = \lambda$. Hence

$$\gamma_f(\lambda) = (\lambda, f_2(0, \lambda), \dots, f_k(0, \lambda)).$$

By definition, $\gamma_f \pitchfork \mathcal{A}$ at $0 \in \mathbb{R}$ if and only if $\gamma_f \pitchfork A_j$ at $0 \in \mathbb{R}$, $0 \leq j \leq k$. There are only two ways we can satisfy the condition $\gamma_f \pitchfork A_j$ at $0 \in \mathbb{R}$, $0 \leq j \leq k$.

(1) $\gamma_f(0) \in A_k$ (in particular, $A_k \supsetneq \mathbb{R}^{k-1}$).
(2) $\gamma_f(0) \in A_{k-1}$ (transversality to A_{k-1} is automatic since A_{k-1} is an open subset of $\{t_1 = 0\}$ and $\gamma_f \pitchfork \{t_1 = 0\}$).

If the first condition holds then no new branches of equilibria occur for X as λ passes through zero. As far as I am aware, there are no known examples where (1) holds.

In either case, the branching pattern for X – the germ of $X^{-1}(0)$ at the origin (see [16, 15]) – is entirely determined by $f_2(0, 0), \dots, f_k(0, 0)$. Consequently, if $X \in \mathcal{V}(V, G)$ and we write $X(x, \lambda) = \lambda x + Q(x, \lambda)$, then the dependence of Q on λ is irrelevant as far as the local homeomorphism type of $X^{-1}(0)$ is concerned. This remark still holds if we take account of stabilities along branches [12, 13, 15].

Example 4.1 ([16]). For $n \geq 2$, let $H_n \subset \mathrm{O}(n)$ denote the group of signed $n \times n$ permutation matrices. We have $H_n = \Delta_n \rtimes S_n$, where Δ_n is the group of diagonal matrices, entries ± 1 and S_n is the symmetric group on n-symbols. The group H_n is the symmetry group of the n-dimensional hypercube and is a finite reflection group. A *basis* for the $P(\mathbb{R}^n)^{H_n}$-module of equivariants is given by

$$F_j(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^{2j+1}, \quad 1 \leq j \leq n.$$

It is shown in [16] that the natural stratification of $\mathbb{R}^n$ has filtration

$$\mathbb{R}^n \supset \mathbb{R}^{n-1} \supset \dots \supset \mathbb{R} \supset \{0\}.$$

If we write $X(x,\lambda) = (\lambda + g(x,\lambda))x + bF_2(x) + O(\|x\|^5)$, then X is generic if and only if $b \neq 0$. Results for other finite reflection groups may be found in [16, 17].

Example 4.2. The minimal number $k = k(V,G)$ of generators for the $P(V)^G$-module $P_G(V,V)$ will usually be (much) larger than the dimension of V. As a result computation of the natural stratification $\mathcal{A}$ of $\mathbb{R}^k$ may be very difficult. Nevertheless, it is often possible to gain a lot of information about the codimension one strata of $\mathcal{A}$. For each $\tau \in \mathcal{O}(V,G)$, $\tau \neq (G)$, let $A^\star_\tau = \mathbb{R}^k \cap \overline{\Sigma_\tau}$. We have

(1) $A^\star_\tau \subset \{t_1 = 0\}$.
(2) If $\tau = (H)$ and V^H is odd-dimensional, then $A^\star_\tau = \{t_1 = 0\}$. (In case $\dim(V^H) = 1$, this amounts to the equivariant branching lemma of Cicogna and Vanderbauwhede. See also [13, Example 4.3.10]).

In [17], results are given that identify a large class of subgroups G of the hyperoctahedral group H_n for which one can compute all $\tau \in \mathcal{O}(\mathbb{R}^n, G)$ such that $\dim(A^\star_\tau) = k-1$ (the 'symmetry breaking isotropy types'). For example, if $G = \Delta_3 \rtimes S_3 \subset H_3$ (an example studied by Guckenheimer and Holmes [20],[17, §15.5]), then

$$\mathcal{O}(\mathbb{R}^3, G) = \{(G), (S_3), (\mathbb{Z}_2^2 \subset \Delta_3), (\mathbb{Z}_2 \subset \Delta_3), (e)\}.$$

The isotropy types $(S_3), (\mathbb{Z}_2^2)$ are both maximal (that is, maximal isotropy subgroups of G) and have one-dimensional fixed point spaces. It follows from (2) that $A^\star_{(S_3)} = A^\star_{(\mathbb{Z}_2)} = \mathbb{R}^{k-1}$. Thus far, we have not needed any quantitative information on the equivariants. To proceed further, it suffices to note that there are no quadratic equivariants (since $-I \in G$, all equivariants are odd) and that there are two cubic equivariants in a minimal set of homogeneous generators. These may be taken to be $F_2(x,y,z) = (y^2, z^2, x^2)$ and $F_3(x,y,z) = (z^2, x^2, y^2)$. With these choices, one can easily show that

$$A^\star_{(\mathbb{Z}_2)} = \{t_1 = 0,\ t_2 t_3 \geq 0\}.$$

Clearly $A^\star_{(\mathbb{Z}_2)}$ is of codimension one in $\mathbb{R}^k$ and is not equal to $\{t_1 = 0\}$. This provides the simplest example for which $A^\star_\tau$ is a proper semialgebraic, non algebraic, subset of $\mathbb{R}^k$. We refer to [17, 15] for more details and generalizations.

Much of what we have described above for equivariant vector fields can be extended to other classes of vector fields. We conclude with an example of a reversible equivariant vector field with a *forced kernel*.

Example 4.3 ([15], see also [8])**.** Let W denote the index 2 finite reflection subgroup $\Delta'_4 \rtimes S_4$ of H_4. As basis for the $\mathbb{R}$-algebra $P(\mathbb{R}^4)^W$ we take (see [16])

$$p_1(x) = \frac{1}{2}\|x\|^2,\ p_2(x) = \frac{1}{4}\sum_{i=1}^4 x_i^4,\ p_3(x) = \frac{1}{6}\sum_{i=1}^6 x_i^6,\ p_4(x) = x_1x_2x_3x_4.$$

Corresponding generators for the equivariants are given by $\phi_i = \mathrm{grad}(p_i)$. It follows from smooth invariant theory that every smooth W-equivariant vector field $X : \mathbb{R}^4 \to \mathbb{R}^4$ may be written (uniquely) in the form

$$X = \sum_{j=1}^4 f_i(p_1, \ldots, p_4)F_i, \tag{4.4}$$

where $f_i \in C^\infty(\mathbb{R}^4)$. Define two, non-isomorphic, representations of $G = H_4$ on $\mathbb{R}^4$. The first representation of G will be the standard representation $\rho : G = H_4 \to \mathrm{O}(4)$ defined previously. For the second representation, let $\sigma : G \to \mathrm{O}(1) = \mathbb{Z}_2$ be the representation defined by mapping W to $+1$ and $G \setminus W$ to -1. We then define $\rho_\sigma : G \to \mathrm{O}(4)$ by $\rho_\sigma(g) = \sigma(g)\rho(g)$. It is easy to verify that these two representations of G are absolutely irreducible and non-isomorphic. We write the first representation as $(\mathbb{R}^4, G)$, the second as $(\mathbb{R}^4_\sigma, G)$. Obviously every $P \in P_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$ may be written in the form (4.4). While $F_4 \in P_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$, the polynomials $F_1, F_2, F_3 \in P_W(\mathbb{R}^4, \mathbb{R}^4)$ do not lie in $P_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$. In order that $f_i(p_1, \dots, p_4)F_i \in P_{H_4}(\mathbb{R}^4, \mathbb{R}^4_\sigma)$, $i \neq 4$, it is necessary and sufficient that $f_i(p_1, \dots, p_4)(gx) = -f_i(p_1, \dots, p_4)(x)$, for all g such that $\sigma(g) = -1$. Similarly, $f_4(p_1, \dots, p_4)F_4 \in P_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$ only if $f_4(p_1, \dots, p_4)(gx) = f_4(p_1, \dots, p_4)(x)$, for all g such that $\sigma(g) = -1$. It follows straightforwardly that if we define

$$\bar{F}_i = p_4 F_i, \ i = 1, 2, 3, \ \bar{F}_4 = F_4,$$

then $\bar{F}_1, \dots, \bar{F}_4$ generate the $P(\mathbb{R}^4)^G$-module $P_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$. In particular, by smooth invariant theory, every $X \in C^\infty_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$ may be written (uniquely) in the form

$$X = \sum_{j=1}^{4} f_i(p_1, p_2, p_3, p_4^2)\bar{F}_i, \tag{4.5}$$

where $f_i \in C^\infty(\mathbb{R}^4)$. Here we have used the fact that p_1, p_2, p_3, p_4^2 generate $P(\mathbb{R}^4)^{H_4}$. Elements of $C^\infty_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$ are *reversible* equivariant vector fields (see [8]).

Let $F(x, t) = \sum_{j=1}^4 t_j \bar{F}_j(x)$. Clearly, $F(x_1, x_2, x_3, x_4, t) = 0$ if any two of x_1, x_2, x_3, x_4 are zero. Hence

$$A^\star_\tau = \mathbb{R}^4, \ \text{ if } \tau = (G_{1,0,0,0)}), (G_{1,1,0,0)}), (G_{1,2,0,0}).$$

It is not hard to compute the remaining $A^\star_\tau$. For example, we have

$$A^\star_{(G_{(1,1,1,0)})} = A^\star_{(G_{(1,1,1,1)})} = \{t_1 = 0\},$$

$$A^\star_{(G_{(1,1,2,2)})} = \{t_1, t_2 = 0\}, \ A^\star_{(G_{(1,2,3,4)})} = \{(0, 0, 0, 0)\}.$$

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