DYNAMICAL EQUIVALENCE OF NETWORK ARCHITECTURE FOR COUPLED DYNAMICAL SYSTEMS II: GENERAL CASE

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Abstract. We show that two networks of coupled dynamical systems are dynamically equivalent if and only if they are output equivalent. We also obtain necessary and sufficient conditions for two dynamically equivalent networks to be input equivalent. These results were previously described in the companion paper 'Dynamical equivalence of networks of coupled dynamical systems I: asymmetric inputs' but only proved there for the case of asymmetric inputs. In this paper, we allow for symmetric inputs. We also provide a number of examples to illustrate the main results in the case when there are both symmetric and asymmetric inputs.

1. Introduction

In this work we provide the proofs of two general results on equivalence of networks of coupled dynamical systems that were stated in the companion paper [1] (we refer to [1] for a general introduction and overview of our results and methodology as well as related references). In what follows we assume some familiarity with the notational conventions of [1] and, in particular, with the concepts of dynamical equivalence and input and output equivalence. (We give the formal definitions for symmetric inputs in sections 3 and 4.) In section 3, we show that networks $\mathcal{M}$ and $\mathcal{N}$ are dynamically equivalent if and only if they are output equivalent. In particular, if $\mathcal{M}$ and $\mathcal{N}$ both have n identical cells, then we have output equivalence of $\mathcal{M}$ and $\mathcal{N}$ if and only if we can order the cells of $\mathcal{M}$ and $\mathcal{N}$ so that the adjacency matrices of $\mathcal{M}$ and $\mathcal{N}$ span the same linear subspace of $M(n, n; \mathbb{Q})$. In section 4 we give necessary and sufficient conditions for the input equivalence of two dynamically equivalent networks architectures. We recall from [1] that dynamically equivalent networks with asymmetric

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inputs are always input equivalent. This is not the case when there are symmetric inputs and we provide, in sections 4 and 5, simple examples of dynamically equivalent networks $\mathcal{M}$, $\mathcal{N}$ that are not input equivalent (for the example in section 4, $\mathcal{M}$ is not input dominated by $\mathcal{N}$ and $\mathcal{N}$ is not input dominated by $\mathcal{M}$ — see [1, §3.1] and section 4 for the formal definition of input domination). As a corollary of our proofs, we obtain algorithms for moving from one network architecture to an input or output equivalent architecture so that each system in the second architecture is expressed in terms of cells from the first architecture and conversely. We illustrate these algorithms, as well as instances of the input and output equivalence theorems and the lemmas needed for their proof, by a number of examples. Unlike in [1], we give most examples in a form that emphasizes the relations of output or input domination and do not usually write down explicit dynamical equations.

Finally, we remark that although all the results are stated for the case of identical cell networks, the extension to networks with more than one class of cell is routine (see also [1]).

2. Preliminaries

2.1. **Adjacency and connection matrices.** Following [1], we define for $k \in \mathbb{N}$, $\mathbf{k} = \{1, \cdots, k\}$. It is also useful to define $\bar{\mathbf{k}} = \{0, \cdots, k\}$. Given $k, n \in \mathbb{N}$, let $M_k(n; \mathbb{Z}^+) \subset M(n; \mathbb{Z})$ denote the set of $n \times n$-matrices with entries in $\mathbb{Z}^+$ and valency k. That is, $M \in M_k(n; \mathbb{Z}^+)$ if the entries of M are positive integers and each column of M sums to k.

Let $\mathcal{M}$ be a coupled cell network consisting of n identical cells with r inputs and p input types. We suppose there are $r_i \geq 1$ (symmetric) inputs of type i, $i \in \mathbf{p}$. Necessarily, we have $\sum_{i \in \mathbf{p}} r_i = r$. Label the cells of $\mathcal{M}$ as $\mathbf{C}_1, \cdots, \mathbf{C}_n$. We recall the definition of the adjacency matrices matrices $M_0, \cdots, M_p \in M(n; \mathbb{Z}^+)$ of $\mathcal{M}$. We take M_0 to be the identity matrix. For $\ell \in \mathbf{p}$, we define $M_\ell = [m_{ij}^\ell] \in M_{r_\ell}(n; \mathbb{Z}^+)$ to be the matrix defined by $m_{ij}^\ell = k$ if there are exactly k inputs of type ℓ to C_j from the cell C_i. If there are no inputs of type ℓ from C_i, then $m_{ij}^\ell = 0$. We refer to M_ℓ as the *adjacency matrix of type* ℓ for $\mathcal{M}$. The jth column of M_ℓ identifies the source cells for all the inputs of type ℓ to the cell C_j.

We extend the definition of connection matrix given in [1] to allow for symmetric inputs. (We use connection matrix terminology in several of the proofs and technical definitions; it is not needed for the statements of the main results.) For $j \in \mathbf{n}$, we define $\mathfrak{m}_i^j \in \mathbf{n}^{r_i}$ by requiring that $\mathfrak{m}_i^j = (\mathfrak{m}_{i1}^j, \cdots, \mathfrak{m}_{ir_i}^j)$, where $\mathfrak{m}_{ik}^j \in \mathbf{n}$ and $\mathfrak{m}_{ik}^j$ identifies the source cell for the kth input of type i to $\mathbf{C}_j$, $k \in \mathbf{r}_i$. The vector $\mathfrak{m}^j = (\mathfrak{m}_1^j, \cdots, \mathfrak{m}_p^j) \in \prod_{i=1}^p \mathbf{n}^{r_i} \cong \mathbf{n}^r$ specifies all of the inputs to $\mathbf{C}_j$. The matrix $\mathfrak{m} = [\mathfrak{m}^1, \cdots, \mathfrak{m}^n] \in M(r, n; \mathbb{Z}^+)$ is a connection matrix for the network. If, in addition, we require that $1 \le \mathfrak{m}_{i1}^j \le \cdots \le \mathfrak{m}_{ir_i}^j \le n$, then $\mathfrak{m}_i^j$ is uniquely determined and we refer to the associated connection matrix as the *default* connection matrix of the network (or just *the* connection matrix of the network).

2.2. **Choose and pick cell.** In addition to the Add-Subtract and scaling cells defined in [1], we introduce one further construction that will be useful in simplifying the diagrams we use for some of the examples (this gadget is not used in any of the proofs).

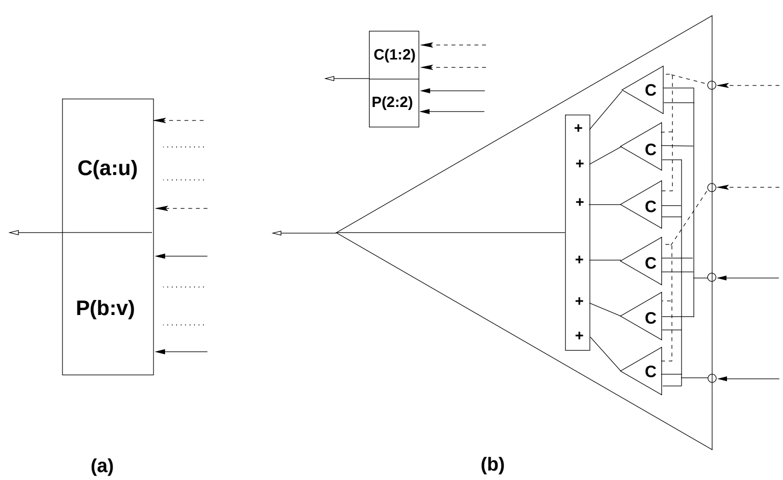

FIGURE 1. Choose and pick cells $\mathbf{C}(a, b : u, v)$, $\mathbf{C}(1, 2 : 2, 2)$

Let $\mathbf{C}$ be a cell which has r inputs all of the same type (we may allow other input types, but they will not effect the construction which only affects inputs of one type). Suppose that continuous dynamics are determined by the vector field $h : M \times M^r \to TM$, where $h(x_0; x_1, \cdots, x_r)$ is symmetric in the variables $x_1, \cdots, x_r$. Suppose that $a, b, u, v \in \mathbb{N}$ satisfy $u + v = r$ and $a \le u$, $b \le v$. The *choose and pick cell* $\mathbf{C}(a, b : u, v)$ is a new cell built by adding outputs from $\binom{v+b-1}{b}\binom{u}{a}$ class $\mathbf{C}$ cells. The cell $\mathbf{C}(a, b : u, v)$ will have two input types; u inputs of the first type, v inputs of the second type. More precisely, the cell $\mathbf{C}(a, b : u, v)$ has two components, denoted $C(a : u)$ and $P(b : v)$, corresponding to the two input types. If $x_1, \cdots, x_u$ are the inputs to the $C(a : u)$ component and

$y_1, \cdots, y_v$ are the inputs to the $P(b : v)$ component, then the output of $\mathbf{C}(a, b : u, v)$ is defined to be

$$\sum_{\substack{1 \le j_1 \le \cdots \le j_b \le v \\ 1 \le i_1 < \cdots < i_a \le u}} h(x_0; y_{j_1}, \cdots, y_{j_b}, x_{i_1}, \cdots, x_{i_a}).$$

The output of $\mathbf{C}(a, b : u, v)$ is symmetric in $x_1, \cdots, x_u$ and $y_1, \cdots, y_v$. We use the symbol for $\mathbf{C}(a, b : u, v)$ shown in figure 1(a). In figure 1(b), we show the connections for the choose and pick cell $\mathbf{C}(1, 2 : 2, 2)$.

3. Output Equivalence

Let $\mathcal{M}$ and $\mathcal{N}$ be coupled n identical cell networks. Denote the cells of $\mathcal{N}$ by $D_1, \cdots, D_n$ (this fixes an ordering of the cells). Suppose cells in $\mathcal{N}$ have s inputs and q input types with s_i inputs of type i, for $i \in \mathbf{q}$ ($s = \sum_{i=1}^q s_i$). Let $\mathbb{A}(\mathcal{N}) = \{N_0 = I,\ N_i \in M_{s_i}(n; \mathbb{Z}^+),\ i \in \mathbf{q}\}$ be the set of adjacency matrices and $\mathbf{A}(\mathcal{N})$ denote the subspace of $M(n; \mathbb{Q})$ spanned by $\mathbb{A}(\mathcal{N})$. Let $\mathfrak{n} = [\mathfrak{n}^1, \cdots, \mathfrak{n}^n]$ be a connection matrix for $\mathcal{N}$. In this section we always assume that $\mathfrak{n}$ is the default connection matrix (see §2.1) and so the vectors $\mathfrak{n}_i^j$ are uniquely determined by the condition $\mathfrak{n}_{i\ell}^j \le \mathfrak{n}_{i\ell'}^j$ if $\ell \le \ell'$. We adopt similar conventions for the network $\mathcal{M}$ but now suppose there are r inputs and p input types (see §2.1). Given an ordering of the cells of $\mathcal{M}$, we let $\mathbb{A}(\mathcal{M}) = \{M_0 = I,\ M_i \in M_{r_i}(n; \mathbb{Z}^+),\ i \in \mathbf{p}\}$ denote the set of adjacency matrices and $\mathbf{A}(\mathcal{M})$ denote the subspace of $M(n; \mathbb{Q})$ spanned by $\mathbb{A}(\mathcal{M})$. Denote the associated default connection matrix of $\mathcal{M}$ by $\mathfrak{m} = [\mathfrak{m}^1, \cdots, \mathfrak{m}^n]$.

Next we formalize the concepts of output dominance and output equivalence for networks with symmetric inputs.

Let $G_{\mathcal{N}} = \prod_{i=0}^q S_{s_i}$, where S_{s_i} denotes the symmetric group on s_i symbols and we have taken $s_0 = 1$ (so that $S_{s_0} = S_1$ is the trivial group consisting of the identity). We define $G_{\mathcal{M}} = \prod_{i=0}^p S_{r_i}$, where $r_0 = 1$.

We take the natural action of $G_{\mathcal{N}}$ on $\bar{\mathbf{s}}$ (we regard $\mathbf{s}$ as identified with $\{\mathbf{s}_1, \cdots, \mathbf{s}_q\}$ and $\bar{\mathbf{s}} = \{0\} \cup \mathbf{s}$ — see §2.1). Let $\mathbf{A}(r, s)$ denote the set of all maps $\gamma : \mathbf{r} \to \bar{\mathbf{s}}$. We have natural left and right actions of $G_{\mathcal{N}}$ and $G_{\mathcal{M}}$ on $\mathbf{A}(r, s)$ defined by

$$\begin{aligned} \gamma &\mapsto \sigma\gamma, \qquad \gamma \in \mathbf{A}(r, s),\ \sigma \in G_{\mathcal{N}}, \\ \gamma &\mapsto \gamma\beta, \qquad \gamma \in \mathbf{A}(r, s),\ \beta \in G_{\mathcal{M}}. \end{aligned}$$

A map $C : \mathbf{A}(r, s) \to \mathbb{Q}$ will be *$G_{\mathcal{N}}$-invariant* if $C(\gamma) = C(\sigma\gamma)$ for all $\sigma \in G_{\mathcal{N}}$.

Let M be a smooth manifold. We write points $\mathbf{X} \in M \times \prod_{i=1}^p M^{r_i}$ in the form $\mathbf{X} = (\mathbf{X}_0; \mathbf{X}_1, \cdots, \mathbf{X}_p)$, where $\mathbf{X}_i = (x_1^i, \cdots, x_{r_i}^i)$, $i \in \mathbf{p}$. We often write x_0 rather than $\mathbf{X}_0$ as the variable belongs to a single factor rather than a product of factors. We use similar notation for points in $M \times \prod_{i=1}^q M^{s_i}$. Given $j \in \mathbf{n}$, $i \in \mathbf{p}$, we let $\mathbf{X}_{\mathfrak{m}_i^j} \in M^{r_i}$ be the variables defined by the connection vector $\mathfrak{m}^j$. We similarly define $\mathbf{X}_{\mathfrak{n}_i^j} \in M^{s_i}$ for $i \in \mathbf{q}$. $G_{\mathcal{M}}$ acts on $\mathbf{X} \in M \times \prod_{i=1}^p M^{r_i}$ by

$$\begin{aligned}\beta\mathbf{X} &= (\mathbf{X}_0; \beta_1\mathbf{X}_1, \cdots, \beta_p\mathbf{X}_p) \\ &= (\mathbf{X}_0; x^1_{\beta_1(1)}, \cdots, x^1_{\beta_1(r_1)}, \cdots, x^p_{\beta_p(1)}, \cdots, x^p_{\beta_p(r_p)})\end{aligned}$$

for $\beta = (\beta_1, \cdots, \beta_p) \in G_{\mathcal{M}} = \prod_{i=0}^p S_{r_i}$. $G_{\mathcal{N}}$ similarly acts on $\mathbf{X} \in M \times \prod_{i=1}^q M^{s_i}$.

Let $f : M \times \prod_{i=1}^p M^{r_i} \to TM$ be a family of $G_{\mathcal{M}}$-invariant vector fields on the smooth manifold M. For $\gamma \in \mathbf{A}(r, s)$, define $f_\gamma : M \times \prod_{i=1}^q M^{s_i} \to TM$ by

$$f_\gamma(x_0; x_1, \cdots, x_s) = f(x_0; x_{\gamma(1)}, \cdots, x_{\gamma(r)}),$$

where $(x_0; x_1, \cdots, x_s) \in M \times \prod_{i=1}^q M^{s_i}$.

Definition 3.1. (Notation and assumptions as above.) Suppose that $f : M \times \prod_{i=1}^p M^{r_i} \to TM$ is $G_{\mathcal{M}}$-invariant, $g : M \times \prod_{i=1}^q M^{s_i} \to TM$, and $C : \mathbf{A}(r, s) \to \mathbb{Q}$ is $G_{\mathcal{N}}$-invariant. We say that f is $(C, \mathfrak{m}, \mathfrak{n})$-*output dominated by* g, written $f <^O_{(C,\mathfrak{m},\mathfrak{n})} g$, if

(1) $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$.
(2) For $j \in \mathbf{n}$ we have $g(x_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = f(x_j; \mathbf{X}_{\mathfrak{m}_1^j}, \cdots, \mathbf{X}_{\mathfrak{m}_p^j})$.

Remark 3.2. Since C is $G_{\mathcal{N}}$-invariant, $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$ is automatically $G_{\mathcal{N}}$-invariant, even if f is not $G_{\mathcal{M}}$-invariant. We use this remark below to obtain a useful simplification of the formula $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$.

The next three lemmas (lemmas 3.4,3.5,3.6) show that the number of terms in the relation between g and f can be reduced using the $G_{\mathcal{M}}$-invariant property of f. The reduced relation obtained will define a new $G_{\mathcal{N}}$-invariant map $\hat{C}$. Before we state and prove these lemmas, it may be helpful to illustrate the ideas by means of a simple example.

Example 3.3. Let the single input type networks $\mathcal{M}$ and $\mathcal{N}$ have non-identity adjacency matrices $M_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $N_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ respectively. As usual, $M_0 = N_0 = I$. We have $M_1 = I + N_1$. If $\mathcal{F} \in \mathcal{M}$ has model f and we define

$$g(x_0; x_1, x_2) = f(x_0; x_0, x_1, x_2), \tag{3.1}$$

then g models a system $\mathcal{G} \in \mathcal{N}$ with identical dynamics to $\mathcal{F}$. In this case, $G_{\mathcal{M}} = S_3$, $G_{\mathcal{N}} = \langle\sigma\rangle = S_2$, where $\sigma(x_1, x_2) = (x_2, x_1)$. Obviously, $g(x_0; \sigma(x_1, x_2)) = g(x_0; x_2, x_1) = f(x_0; x_0, x_2, x_1) = f(x_0; x_0, x_1, x_2)$ and so g is $G_{\mathcal{N}}$-invariant. Following definition 3.1, we may also define g by

$$g(x_0; x_1, x_2) = af(x_0; x_0, x_1, x_2) + bf(x_0; x_0, x_2, x_1),$$

where $a + b = 1, a, b \in \mathbb{R}$. Since f is $G_{\mathcal{M}}$-invariant, the expression for g is equal to that given by (3.1).

Lemma 3.4. *(Notation and assumptions as above.) If f is $G_{\mathcal{M}}$-invariant, then $f_\gamma = f_{\gamma\beta}$ for all $\beta \in G_{\mathcal{M}}$.*

Proof. The model f is $G_{\mathcal{M}}$-invariant and so we have $f(x_0; x_1, \cdots, x_r) = f(x_0; x_{\beta(1)}, \cdots, x_{\beta(r)})$ for all $\beta \in G_{\mathcal{M}}$. Hence, if $\beta \in G_{\mathcal{M}}$, $\gamma \in \mathbf{A}(r, s)$, we have

$$\begin{aligned} f_\gamma(x_0; x_1, \cdots, x_s) &= f(x_0; x_{\gamma(1)}, \cdots, x_{\gamma(r)}), \\ &= f(x_0; x_{\gamma\beta(1)}, \cdots, x_{\gamma\beta(r)}), \\ &= f_{\gamma\beta}(x_0; x_1, \cdots, x_s). \end{aligned}$$

Therefore $f_\gamma = f_{\gamma\beta}$. □

Let $\tilde{\mathbf{A}}(r, s) = \mathbf{A}(r, s)/G_{\mathcal{M}}$ denote the orbit space of $\mathbf{A}(r, s)$ under the right action by $G_{\mathcal{M}}$. Since the actions of $G_{\mathcal{N}}$ and $G_{\mathcal{M}}$ on $\mathbf{A}(r, s)$ commute, the $G_{\mathcal{N}}$-action on $\mathbf{A}(r, s)$ induces a (left) $G_{\mathcal{N}}$-action on $\tilde{\mathbf{A}}(r, s)$. Although a $G_{\mathcal{N}}$-invariant map $C : \mathbf{A}(r, s) \to \mathbb{Q}$ will not generally induce a map on $\tilde{\mathbf{A}}(r, s)$, we do have a trivial converse.

Lemma 3.5. *(Notation and assumptions as above.) If $\tilde{C} : \tilde{\mathbf{A}}(r, s) \to \mathbb{Q}$ is $G_{\mathcal{N}}$-invariant, then $\tilde{C}$ lifts to a $G_{\mathcal{N}} \times G_{\mathcal{M}}$-invariant map*

$$\hat{C} : \mathbf{A}(r, s) \to \mathbb{Q}.$$

We regard the orbit space $\mathbf{A}(r, s)/G_{\mathcal{M}}$ as the set of group orbits for the $G_{\mathcal{M}}$-action on $\mathbf{A}(r, s)$. It is convenient to fix a subset $R = \{\gamma \in \mathbf{A}(r, s)\}$ such that the $\{G_{\mathcal{M}}\gamma \mid \gamma \in R\}$ partitions $\mathbf{A}(r, s)$. That is, $\cup_{\gamma \in R} G_{\mathcal{M}}\gamma = \mathbf{A}(r, s)$ and $G_{\mathcal{M}}\gamma \cap G_{\mathcal{M}}\nu \neq \emptyset$ iff $\gamma = \nu$.

Lemma 3.6. *(Notation as above.) Suppose that f is $G_{\mathcal{M}}$-invariant and $C : \mathbf{A}(r, s) \to \mathbb{Q}$ is $G_{\mathcal{N}}$-invariant. Then there exists a $G_{\mathcal{N}} \times G_{\mathcal{M}}$-invariant map $\hat{C} : \mathbf{A}(r, s) \to \mathbb{Q}$ such that*

$$\sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma = \sum_{\gamma \in R} \hat{C}(\gamma) f_\gamma.$$

Proof. We have

$$\sum_{\gamma\in\mathbf{A}(r,s)} C(\gamma)f_\gamma = \sum_{\gamma\in R}\left(\sum_{\tau\in G_{\mathcal{M}}\gamma} C(\tau)f_\tau\right).$$

By lemma 3.4, $f_\tau = f_\nu$ for all $\tau,\nu\in G_{\mathcal{M}}\gamma$. Letting $[\gamma]\in\tilde{\mathbf{A}}(r,s)$ denote the coset defined by γ, we define $\tilde{C}([\gamma]) = \sum_{\tau\in G_{\mathcal{M}}\gamma} C(\tau)$, $\gamma\in R$. This defines a $G_{\mathcal{N}}$-invariant map $\tilde{C}:\tilde{\mathbf{A}}(r,s)\to\mathbb{Q}$. Let $\hat{C}:\mathbf{A}(r,s)\to\mathbb{Q}$ be the $G_{\mathcal{N}}\times G_{\mathcal{M}}$-invariant lift given by lemma 3.5. □

Remark 3.7. In the lemmas and examples in this paper, the lifted map $\hat{C}$ will be used to define the output relation between f and g.

Definition 3.8. Let $\mathcal{M}$ and $\mathcal{N}$ be coupled identical cell networks such that

(a) Both networks have n cells.
(b) Cells in $\mathcal{M}$ have p input types, cells in $\mathcal{N}$ have q input types.
(c) If we fix an ordering of the cells in $\mathcal{N}$, then the associated connection matrix is $\mathfrak{n} = [\mathfrak{n}^1,\cdots,\mathfrak{n}^n]$.

We write $\mathcal{M}\prec_O\mathcal{N}$ and say $\mathcal{M}$ *is output dominated by* $\mathcal{N}$, if there exist an ordering of the cells of $\mathcal{M}$, with associated connection matrix $\mathfrak{m}$, and a $G_{\mathcal{N}}$-invariant map $C:\mathbf{A}(r,s)\to\mathbb{Q}$, such that for every $\mathcal{F}\in\mathcal{M}$, there exists $\mathcal{G}\in\mathcal{N}$ for which $f_{\mathcal{F}} <^O_{(C,\mathfrak{m},\mathfrak{n})} g_{\mathcal{G}}$. (Recall $\mathcal{F}$ is modelled by $f_{\mathcal{F}}$, and $\mathcal{G}$ is modelled by $g_{\mathcal{G}}$.) If $\mathcal{M}\prec_O\mathcal{N}$ and $\mathcal{N}\prec_O\mathcal{M}$, we say $\mathcal{N}$ and $\mathcal{M}$ are *output equivalent* and write $\mathcal{M}\sim_O\mathcal{N}$.

Lemma 3.9. *The relation $\prec_O$ is transitive.*

Proof. Let $\mathcal{M},\mathcal{N},\mathcal{H}$ be coupled n identical cell networks with r,s,t inputs and p,q,u input types, respectively. Suppose $\mathcal{M}\prec_O\mathcal{H}$ and $\mathcal{H}\prec_O N$. We show that $\mathcal{M}\prec_O\mathcal{N}$. Fix an ordering of cells in $\mathcal{N}$ with associated connection matrix $\mathfrak{n}=[\mathfrak{n}^1,\cdots,\mathfrak{n}^n]$. Since $\mathcal{H}\prec_O N$, it follows by the definition of output domination that we have an associated ordering of the cells of $\mathcal{H}$, connection matrix $\mathfrak{h}=[\mathfrak{h}^1,\cdots,\mathfrak{h}^n]$ and $G_{\mathcal{N}}$-invariant map $C_1:\mathbf{A}(t,s)\to\mathbb{Q}$. If $\mathcal{K}\in\mathcal{H}$ is modelled by k, there exists $\mathcal{G}\in\mathcal{N}$ modelled by g such that

(1) $g=\sum_{\sigma\in\mathbf{A}(t,s)} C_1(\sigma)k_\sigma$.
(2) For $j\in\mathbf{n}$ we have $g(x_j;\mathbf{X}_{\mathfrak{n}^j_1},\cdots,\mathbf{X}_{\mathfrak{n}^j_q}) = k(x_j;\mathbf{X}_{\mathfrak{h}^j_1},\cdots,\mathbf{X}_{\mathfrak{h}^j_u})$.

Also, since $\mathcal{M}\prec_O\mathcal{H}$, we have an associated ordering of the cells of $\mathcal{M}$, connection matrix $\mathfrak{m}=[\mathfrak{m}^1,\cdots,\mathfrak{m}^n]$ and $G_{\mathcal{H}}$-invariant map $C_2:\mathbf{A}(r,t)\to\mathbb{Q}$. If $\mathcal{F}\in\mathcal{M}$ is modelled by f, there exists $\mathcal{K}\in\mathcal{H}$ modelled by k such that

(3) $k=\sum_{\gamma\in\mathbf{A}(r,t)} C_2(\gamma)f_\gamma$.

(4) For $j \in \mathbf{n}$ we have $k(x_j; \mathbf{X}_{\mathfrak{h}_1^j}, \cdots, \mathbf{X}_{\mathfrak{h}_u^j}) = f(x_j; \mathbf{X}_{\mathfrak{m}_1^j}, \cdots, \mathbf{X}_{\mathfrak{m}_p^j})$.

For $\sigma \in \mathbf{A}(t,s)$, set $x_i^\sigma = x_{\sigma(i)}$ for $i \in \mathbf{t}$. We have,

$$\begin{aligned}
g(x_0; x_1, \cdots, x_s) &= \sum_{\sigma \in \mathbf{A}(t,s)} C_1(\sigma) k_\sigma(x_0; x_1, \cdots, x_s) \\
&= \sum_{\sigma \in \mathbf{A}(t,s)} C_1(\sigma) k(x_0; x_{\sigma(1)}, \cdots, x_{\sigma(t)}) \\
&= \sum_{\sigma \in \mathbf{A}(t,s)} C_1(\sigma) k(x_0; x_1^\sigma, \cdots, x_t^\sigma) \\
&= \sum_{\sigma \in \mathbf{A}(t,s)} C_1(\sigma) \sum_{\gamma \in \mathbf{A}(r,t)} C_2(\gamma) f_\gamma(x_0; x_1^\sigma, \cdots, x_t^\sigma) \\
&= \sum_{\sigma \in \mathbf{A}(t,s)} C_1(\sigma) \sum_{\gamma \in \mathbf{A}(r,t)} C_2(\gamma) f(x_0; x_{\gamma(1)}^\sigma, \cdots, x_{\gamma(r)}^\sigma) \\
&= \sum_{\substack{\sigma \in \mathbf{A}(t,s) \\ \gamma \in \mathbf{A}(r,t)}} C_1(\sigma) C_2(\gamma) f(x_0; x_{\sigma\circ\gamma(1)}, \cdots, x_{\sigma\circ\gamma(r)}).
\end{aligned}$$

Let $\tilde{\mathbf{A}}(r,s) = \{\sigma \circ \gamma \in \mathbf{A}(r,s) \mid \gamma \in \mathbf{A}(r,t), \sigma \in \mathbf{A}(t,s)\} \subset \mathbf{A}(r,s)$. Define $C : \mathbf{A}(r,s) \to \mathbb{Q}$ by

$$C(\phi) = \begin{cases} C_1(\sigma)C_2(\gamma) & \text{if } \phi = \sigma \circ \gamma \in \tilde{\mathbf{A}}(r,s) \\ 0 & \text{if } \phi \in \mathbf{A}(r,s) \setminus \tilde{\mathbf{A}}(r,s) \end{cases}$$

Let $\beta \in G_{\mathcal{N}}$. We have $C(\beta(\sigma \circ \gamma)) = C((\beta\sigma) \circ \gamma) = C_1(\beta\sigma)C_2(\gamma) = C_1(\sigma)C_2(\gamma) = C(\sigma \circ \gamma)$. Therefore, C is $G_{\mathcal{N}}$-invariant and the relation between f and g given by

$$g(x_0; x_1, \cdots, x_s) = \sum_{\phi \in \tilde{\mathbf{A}}(r,s)} C(\phi) f_\phi(x_0; x_1, \cdots, x_s).$$

Hence $\mathcal{M} \prec_O \mathcal{N}$ (input matching conditions follow from (2,4)). $\square$

Example 3.10. Let $\mathcal{M}, \mathcal{K}, \mathcal{N}$ be single input type networks with non-identity adjacency matrices $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $K_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, $N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ respectively. Note that $M_1 = I + K_1/2$ and $K_1 = 2N_1$. We claim that $\mathcal{M} \prec_O \mathcal{K}$ and $\mathcal{K} \prec_O \mathcal{N}$. Indeed, if $f : M \times M^2 \to TM$ is the model for the system $\mathcal{F} \in \mathcal{M}$, then $h : M \times M^2 \to TM$ defined by

$$h(x; y, z) = \frac{1}{2}(f(x; y, x) + f(x; z, x))$$

models $\mathcal{H} \in \mathcal{K}$ with the same dynamics as $\mathcal{F} \in \mathcal{M}$. Similarly, if we define $g : M \times M \to TM$ by

$$g(x; y) = h(x; y, y)$$

then g models a system $\mathcal{G} \in \mathcal{N}$ with the same dynamics as $\mathcal{H}$. Observe that

$$g(x; y) = h(x; y, y) = \frac{1}{2}(f(x; y, x) + f(x; y, x)) = f(x; y, x)$$

It is easy to check that g models a system $\mathcal{G} \in \mathcal{N}$ with the same dynamics as $\mathcal{F} \in \mathcal{M}$ and so $\mathcal{M} \prec_O \mathcal{N}$.

Before we give the main result of this section, we state and prove a useful result about output domination (an analogous result holds for input domination — see lemma 4.3). We continue with our assumptions on $\mathcal{M}$ and $\mathcal{N}$ and assume that we have fixed an ordering of the cells in $\mathcal{N}$. Given an ordering of the cells in $\mathcal{M}$, denote the associated set of adjacency matrices by $M_0, M_1, \cdots, M_p$. For $j \in \mathbf{p}$, Let $\mathcal{M}_j$ denote the n-cell network with 1 input type and adjacency matrices $\{M_0, M_j\}$. Denote the connection matrix associated to $\{M_0, M_j\}$ by $\mathfrak{m}_j$.

Lemma 3.11. *(Notation and assumptions as above). The following conditions are equivalent.*

(1) $\mathcal{M} \prec_O \mathcal{N}$.
(2) *There exists an ordering of the cells in* $\mathcal{M}$ *such that* $\mathcal{M}_j \prec_O \mathcal{N}$, *for all* $j \in \mathbf{p}$.

Proof. Suppose first that $\mathcal{M} \prec_O \mathcal{N}$. By definition of output domination, we have an associated ordering of the cells of $\mathcal{M}$, connection matrix $\mathfrak{m}$ and $G_{\mathcal{N}}$-invariant map $C : \mathbf{A}(r, s) \to \mathbb{Q}$. If $\mathcal{F} \in \mathcal{M}$ has model f, there exists $\mathcal{G} \in \mathcal{N}$ with model g such that

(1) $g = \sum_{\gamma \in \mathbf{A}(r,s)} C(\gamma) f_\gamma$.
(2) For $j \in \mathbf{n}$ we have $g(x_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = f(x_j; \mathbf{X}_{\mathfrak{m}_1^j}, \cdots, \mathbf{X}_{\mathfrak{m}_p^j})$.

Now suppose that f depends only on the variables $(x_0, \mathbf{X}_j) \in M \times M^{s_j}$. Then the associated system can be identified with a system in $\mathcal{M}_j$. The input matching condition (2) implies trivially that we have the correct input matching for the connection matrix $\mathfrak{m}_j$ of $\mathcal{M}_j$. Hence $\mathcal{M}_j \prec_O \mathcal{N}$. Conversely, suppose that there exists an ordering of the cells in $\mathcal{M}$ such that $\mathcal{M}_j \prec_O \mathcal{N}$, for all $j \in \mathbf{p}$. For each $j \in \mathbf{p}$, there exists a $G_{\mathcal{N}}$-invariant map $C_j : \mathbf{A}(r_j, s) \to \mathbb{Q}$ such that if f^j is the model for $\mathcal{F}_j \in \mathcal{M}_j$, there exists $\mathcal{G}_j \in \mathcal{N}$ with model g^j such that

$$g^j = \sum_{\gamma \in \mathbf{A}(r_j,s)} C_j(\gamma) f^j_\gamma.$$

and the input matching conditions hold (with $\mathfrak{m}$ replaced by $\mathfrak{m}_j$). Now suppose $\mathcal{F} \in \mathcal{M}$ has model f. We define g by

$$g = \sum_{\gamma_1 \in \mathbf{A}(r_1,s)} \cdots \sum_{\gamma_p \in \mathbf{A}(r_p,s)} C_1(\gamma_1) \cdots C_p(\gamma_p) f_{\gamma_1 \cdots \gamma_p}. \tag{3.2}$$

where we define $f_{\gamma_1 \cdots \gamma_p}$ by making the natural identification between $\prod_{j=1}^{p} \mathbf{A}(r_j, s)$ and $\mathbf{A}(r, s)$ (that is, using the identification of $\mathbf{r}$ and $\{\mathbf{r}_1, \cdots, \mathbf{r}_p\}$). It is straightforward to verify that g does define a system $\mathcal{G} \in \mathcal{N}$ which satisfies the input matching conditions (2). $\square$

Theorem 3.12. *(Notation as above.)* $\mathcal{M} \sim_O \mathcal{N}$ *iff* $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ *iff* $\mathcal{M} \sim \mathcal{N}$.

In order to prove theorem 3.12 it suffices to show that

(A) $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N}) \Longrightarrow \mathcal{M} \prec_O \mathcal{N}$.
(B) $\mathcal{M} \prec_O \mathcal{N} \Longrightarrow \mathcal{M} \prec \mathcal{N}$.
(C) $\mathcal{M} \prec_O \mathcal{N} \Longrightarrow \mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$.
(D) $\mathcal{M} \prec \mathcal{N} \Longrightarrow \mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$.

Statement (B) is trivial. We prove (C,D) by reducing to the case of linear vector fields. Most of the work involves the proof of (A) and we start with the proof of (A) and conclude with the proofs of (C,D).

We break the proof of (A) into a number of lemmas. These lemmas also give an algorithm for computing an explicit output equivalence or domination. Throughout we assume that $\mathcal{M}$, $\mathcal{N}$ are identical cell networks and follow our established notational conventions. In particular, we assume given orderings of the cells of $\mathcal{M}$, $\mathcal{N}$ and associated adjacency and connection matrices and the inclusion $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$. The result extends to non-identical cell networks by applying the proof cell class by cell class (see [1] and note that the linear equivalence results in [3] apply to networks with multiple cell classes).

Lemma 3.13. *If* $p = q$, *and* $M_i = N_i$, $i \notin \{a, b\}$, $N_a = M_b$, $N_b = M_a$ *then* $\mathcal{M} \prec_O \mathcal{N}$.

Proof. If $a = b$, there is nothing to prove. Suppose without loss of generality that $a < b$. We have $r_i = s_i, i \in \mathbf{p} \setminus \{a, b\}$, $r_a = s_b, r_b = s_a$. Suppose that $\mathcal{F} \in \mathcal{M}$ has model $f : M \times \prod_{i=1}^{p} M^{r_i} \to TM$. Define $g : M \times \prod_{i=1}^{p} M^{s_i} \to TM$ by

$$\begin{aligned} g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_a, \cdots, \mathbf{X}_b, \cdots, \mathbf{X}_p) \\ = f(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_b, \cdots, \mathbf{X}_a, \cdots, \mathbf{X}_p). \end{aligned}$$

It is easy to check that g defines the required system $\mathcal{G} \in \mathcal{N}$. $\square$

Remark 3.14. As a consequence of lemma 3.13, we see that if the adjacency matrices of $\mathcal{M}$ are a permutation of those of $\mathcal{N}$, then $\mathcal{M} \sim_O \mathcal{N}$.

Lemma 3.15. *Let $p = 2$, and $M_1 = \sum_{i\in A} \alpha_i N_i$, $M_2 = \sum_{j\in B} \epsilon_j N_j$, where $A, B \subset \overline{\mathbf{q}}$, and $\alpha_i, \epsilon_j \in \mathbb{N}$, $i \in A, j \in B$. Then $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. Suppose that $A = \{a_1, \cdots, a_u\}, B = \{b_1, \cdots, b_w\} \subset \overline{\mathbf{q}}$. Suppose that $\mathcal{F} \in \mathcal{M}$ has model $f : M \times \prod_{i=1}^{2} M^{r_i} \to TM$. Define $g : M \times \prod_{i=1}^{q} M^{s_i} \to TM$ by

$$g(\mathbf{X}_0; \mathbf{X}_1, .., \mathbf{X}_k) = f(\mathbf{X}_0; \overline{\mathbf{X}_{a_1}^{\alpha_1}, \cdots, \mathbf{X}_{a_u}^{\alpha_u}}, \overline{\mathbf{X}_{b_1}^{\epsilon_1}, \cdots, \mathbf{X}_{b_w}^{\epsilon_w}}),$$

where $\mathbf{X}_i \in M^{s_i}$ (variables corresponding to inputs of type i, $i \in \mathbf{q}$) and $\mathbf{X}_i^\alpha$ denotes $\mathbf{X}_i$ repeated α times. It is straightforward to check that g defines the required system $\mathcal{G} \in \mathcal{N}$. □

Example 3.16. (Illustration of Lemma 3.15) Let $\mathcal{N}$ be the network with non-identity adjacency matrices $N_1 = \begin{pmatrix} 3 & 0 & 2 \\ 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$, $N_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\mathcal{Q}$ be the network with non-identity adjacency matrices $P = \begin{pmatrix} 4 & 0 & 2 \\ 2 & 4 & 2 \\ 0 & 2 & 2 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. It is straightforward to check $P = N_1 + N_2$, $Q = 2N_0$ and so $\mathbf{A}(\mathcal{Q}) \subseteq \mathbf{A}(\mathcal{N})$. Suppose that $\mathcal{F} \in \mathcal{Q}$ has model $h : M \times M^6 \times M^2 \to TM$. Following lemma 3.15, we define $g : M \times M^4 \times M^2 \to TM$ by

$$g(x_0; \overline{x_1, \cdots, x_4}, \overline{x_5, x_6}) = h(x_0; \overline{x_1, \cdots, x_4, x_5, x_6}, \overline{x_0, x_0})$$

It can be easily checked that g models the required system $\mathcal{G} \in \mathcal{N}$.

The next two lemmas handle the most difficult cases of output domination.

Lemma 3.17. *Let $p = 1$ and suppose $M_1 = N_1 - N_2$ then $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. Set $r_1 = r$, $s_2 = \tilde{s}$ so that $s_1 = r + \tilde{s}$. Suppose that $\mathcal{F} \in \mathcal{M}$ has model $f : M \times M^r \to TM$. Set $\mathbf{Z} = (\mathbf{X}_3, \cdots, \mathbf{X}_q) \in \prod_{i=3}^{q} M^{s_i}$ (the variables represented by $\mathbf{Z}$ play no role in what follows). Define $g : M \times \prod_{i=1}^{q} M^{s_i} \to TM$ by

$$\begin{aligned} g(x_0; \overline{x_1, \cdots, x_{r+\tilde{s}}}, \overline{y_1, \cdots, y_{\tilde{s}}}, \mathbf{Z}) \\ = \sum_{i=0}^{r} (-1)^i \sum_{\mathcal{C}_i} f(x_0; \overline{y_1^{a_1}, \cdots, y_{\tilde{s}}^{a_{\tilde{s}}}, x_{j_1}, \cdots, x_{j_{r-i}}}), \end{aligned} \tag{3.3}$$

where $\mathcal{C}_i$ is the set of all $(\tilde{s}+r-i)$-tuples $(a_1,\cdots,a_{\tilde{s}},j_1,\cdots,j_{r-i})$ satisfying $a_1+\cdots+a_{\tilde{s}}=i$, $1\le j_1<\cdots<j_{r-i}\le r+\tilde{s}$.

Let $x_{r+i}=y_i$, $i=1,\cdots,\tilde{s}$. It suffices to show that

$$g(x_0;\overline{x_1,\cdots,x_{r+\tilde{s}}},\overline{y_1,\cdots,y_{\tilde{s}}},\mathbf{Z})=f(x_0;x_1,\cdots,x_r).$$

Suppose $t\in\mathbf{r}$ and $b_1,\cdots,b_{\tilde{s}}\in\mathbb{Z}^+$ satisfy $\sum_{i=1}^{\tilde{s}}b_i=t$. We find the coefficient of $f(x_0;\overline{y_1^{b_1},\cdots,y_{\tilde{s}}^{b_{\tilde{s}}},x_{j_1},\cdots,x_{j_{r-t}}})$, where $j_v\in\mathbf{r}$, $v\in\mathbf{r}-\mathbf{t}$. Let $(b_1,\cdots,b_{\tilde{s}},j_1,\cdots,j_{r-t})\in\mathcal{C}_t$ and m denote the number of b_i that are greater than equal to 1. We find that $f(x_0;\overline{y_1^{b_1},\cdots,y_{\tilde{s}}^{b_{\tilde{s}}},x_{j_1},\cdots,x_{j_{r-t}}})$ appears in the sum for g when $t-m\le i\le t$ and has coefficient $(-1)^i\binom{m}{t-i}$. Hence, the coefficient of this term is $\sum_{i=t-m}^{t}(-1)^i\binom{m}{t-i}$. This is zero unless $m=0$ $(t=0)$, in which case the coefficient is 1 and we get $f(x_0;x_1,\cdots,x_r)$. Hence g defines the required system $\mathcal{G}\in\mathcal{N}$. □

Remark 3.18. Another way to write equation 3.3 is

$$\begin{aligned}(3.4)\quad & g(x_0;\overline{x_1,\cdots,x_{r+\tilde{s}}},\overline{y_1,\cdots,y_{\tilde{s}}},\mathbf{Z})\\ &=\sum_{i=0}^{r}(-1)^i\sum_{\substack{1\le j_1<\cdots<j_{r-i}\le r+\tilde{s}\\ 1\le k_1\le\cdots\le k_i\le\tilde{s}}}f(x_0;\overline{y_{k_1},\cdots,y_{k_i},x_{j_1},\cdots,x_{j_{r-i}}})\end{aligned}$$

Example 3.19. (Illustration of lemma 3.17) Let $\mathcal{Q}$ be the network of example 3.16 and $\mathcal{R}$ be the network with non-identity adjacency matrix $R_1=\begin{pmatrix}2&0&2\\2&2&2\\0&2&0\end{pmatrix}$. It is straightforward to check $R_1=P-Q$ and so $\mathbf{A}(\mathcal{R})\subseteq\mathbf{A}(\mathcal{Q})$. Hence, by lemma 3.17, we have $\mathcal{R}\prec_O\mathcal{Q}$. Suppose that $\mathcal{F}\in\mathcal{R}$ has model $e:M\times M^4\to TM$. We construct $\mathcal{G}\in\mathcal{Q}$ with model h such that e is output dominated by h. Noting remark 3.18, we define $h:M\times M^6\times M^2\to TM$ by

$$\begin{aligned}(3.5)\quad & h(x_0;\overline{x_1,\cdots,x_6},\overline{x_7,x_8})\\ &=\sum_{i=0}^{4}(-1)^i\sum_{\substack{1\le j_1<\cdots<j_{4-i}\le 6\\ 7\le k_1\le\cdots\le k_i\le 8}}e(x_0;\overline{x_{k_1},\cdots,x_{k_i},x_{j_1},\cdots,x_{j_{4-i}}})\end{aligned}$$

It can be easily checked that h models the required system $\mathcal{G}\in\mathcal{Q}$. We can define a new cell class $\mathbf{D}$, built from the cells of the system $\mathcal{F}$, which realizes the dynamics of $\mathcal{F}$ when these cells are coupled according to the network architecture $\mathcal{Q}$. See figure 2.

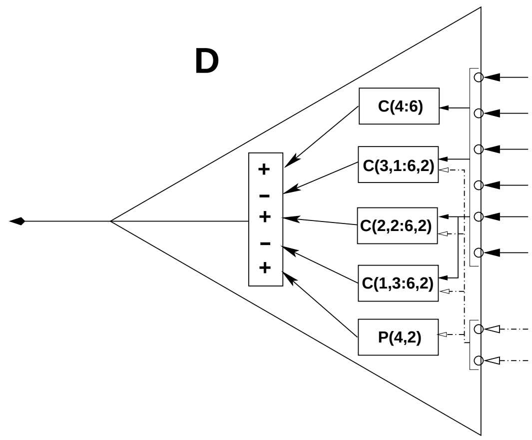

FIGURE 2. The cell **D**. The *choose and pick* cells are linear combinations of the vector field f modelling $\mathcal{F}$.

Lemma 3.20. *If $p = 1$ and $M_1 = \frac{1}{m}N_1$, then $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. Just as in the proof of lemma 3.17, the variables $\mathbf{X}_j \in M^{s_j}$ play no role if $j > 1$ and so it is no loss of generality to take $p = q = 1$. The computations do not use the internal variable which we also omit. Since $p = q = 1$ and there is no internal variable, all functions will be symmetric and we omit the overline signifying symmetry. Since the case when $m = 1$ is trivial we assume $m \geq 2$. Set $r_1 = r, s_1 = s$ and note that $s = mr$. Let $\mathcal{J}$ denote the set of all tuples $\boldsymbol{j} = (j_1, \cdots, j_u)$ of positive integers such that $j_1 \geq j_2 \geq \cdots \geq j_u \geq 1$ and $\sum_{i=1}^u j_i = r$. We define lexicographical ordering on $\mathcal{J}$:

$$\boldsymbol{j} = (j_1, \cdots, j_u) > \boldsymbol{j}' = (j'_1, \cdots, j'_{u'}),$$

if $\exists k \in \mathbf{u}$ such that

$$j_i = j'_i,\ i < k,\ \text{and}\ j_k > j'_k.$$

Note that $\boldsymbol{j} > \boldsymbol{j}'$ does not imply $u \lesseqgtr u'$. The unique maximal and minimal elements of $\mathcal{J}$ are (r) and $(1, 1, \cdots, 1)$ respectively.

Suppose $f : M \times M^r \to TM$ models $\mathcal{F} \in \mathcal{M}$. Define $g : M \times M^{rm} \to TM$ by

$$g(x_1, \cdots, x_{rm}) = \sum_{\boldsymbol{j} \in \mathcal{J}} c_{\boldsymbol{j}} \sum_{i_1, \cdots, i_u \in \mathbf{rm}} f(x_{i_1}^{j_1} \cdots, x_{i_u}^{j_u}), \tag{3.6}$$

where $c_{\boldsymbol{j}} \in \mathbb{Q}$ are constants to be determined. For fixed $\boldsymbol{j} \in \mathcal{J}$, define

$$g_{\boldsymbol{j}}(x_1, \cdots, x_{rm}) = \sum_{i_1, \cdots, i_u \in \mathbf{rm}} f(x_{i_1}^{j_1} \cdots, x_{i_u}^{j_u}).$$

Thus

$$g(x_1, \cdots, x_{rm}) = \sum_{\boldsymbol{j} \in \mathcal{J}} c_{\boldsymbol{j}} g_{\boldsymbol{j}}(x_1, \cdots, x_{rm}).$$

We remark that each $g_{\boldsymbol{j}}$ is symmetric in $(x_1, \cdots, x_{rm})$. Hence g is symmetric in $(x_1, \cdots, x_{rm})$.

Given $\boldsymbol{j} = (j_1, \cdots, j_u) \in \mathcal{J}$, define $\mathcal{J}(\boldsymbol{j}) \subset \mathcal{J}$ to consist of all $\boldsymbol{\ell} = (\ell_1, \cdots, \ell_{u'}) \geq \boldsymbol{j}$ such that each ℓ_t can be written as a sum $\sum_{i \in I_t} j_i$, $I_t \subset \mathbf{u}$.

Suppose we are given $y_1, \cdots, y_r$ and $\boldsymbol{j} \in \mathcal{J}$. Suppose $x_1, \cdots, x_p = y_1, \cdots, x_{(r-1)m+1}, \cdots, x_{rm} = y_r$. Then there exist strictly positive integers $A_{\boldsymbol{\ell}}^{\boldsymbol{j}}$ such that

$$g_{\boldsymbol{j}}(y_1^m, \cdots, y_r^m) = \sum_{\boldsymbol{\ell} \in \mathcal{J}(\boldsymbol{j})} A_{\boldsymbol{\ell}}^{\boldsymbol{j}} f_{\boldsymbol{\ell}}(y_1, \cdots, y_r),$$

where

$$f_{\boldsymbol{\ell}}(y_1, \cdots, y_r) = \sum f(y_{i_1}^{\ell_1}, \cdots, y_{i_{u'}}^{\ell_{u'}}),$$

and the sum is taken all distinct u'-tuples $(i_1, \cdots, i_{u'})$ of integers in $\mathbf{r}$. Each $f_{\boldsymbol{\ell}}$ is symmetric in $y_1, \cdots, y_r$. We have

$$g(y_1^m, \cdots, y_r^m) = \sum_{\boldsymbol{j} \in \mathcal{J}} c_{\boldsymbol{j}} \left(\sum_{\boldsymbol{\ell} \in \mathcal{J}(\boldsymbol{j})} A_{\boldsymbol{\ell}}^{\boldsymbol{j}} f_{\boldsymbol{\ell}}(y_1, \cdots, y_r) \right).$$

We choose the coefficients $c_{\boldsymbol{j}}$ so that $g(y_1^m, \cdots, y_r^m) = f(y_1, \cdots, y_r)$. The term $f(y_1, \cdots, y_r)$ only occurs once in the sum we have for g (when $\boldsymbol{j}$ is the minimal element $(1, 1, 1, \cdots, 1)$ of $\mathcal{J}$). Hence $c_{(1,\cdots,1)}$ is uniquely determined. Our choice of order on $\mathcal{J}$ orders the the rows of the matrix of the linear system and our construction implies that the matrix is in upper triangular form. Hence we can solve for the coefficients $c_{\boldsymbol{j}}$. □

Example 3.21. (Illustration of lemma 3.20) Let $\mathcal{R}$ be the network of example 3.19 and $\mathcal{M}$ be the network with non-identity adjacency matrix $M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. We have $M_1 = \frac{R}{2}$. Suppose that $\mathcal{F} \in \mathcal{M}$ has model $f : M \times M^2 \to TM$. Following lemma 3.20, define e :

$M \times M^4 \to TM$ by

$$e(x_0; x_1, \cdots, x_4) = \sum_{\boldsymbol{j} \in \mathcal{J}} c_{\boldsymbol{j}} \sum_{i_1, \cdots, i_u \in \mathbf{4}} f(x_0; x_{i_1}^{j_1}, \cdots, x_{i_u}^{j_u})$$

where, $\mathcal{J} = \{(1,1),(2)\}$ (lemma 3.20) and we have omitted the overlines denoting symmetric inputs. Setting $a = c_{(1,1)}, b = c_{(2)}$, we have

$$e(x_0; x_1, \cdots, x_4) = a \sum_{i_1, i_2 \in \mathbf{4}} f(x_0; x_{i_1}, x_{i_2}) + b \sum_{i_1 \in \mathbf{4}} f(x_0; x_{i_1}, x_{i_1}). \tag{3.7}$$

After substituting $x_1 = x_2 = u$, $x_3 = x_4 = v$, we get the following terms: $f(x_0; u, u)$, $f(x_0; u, v)$, $f(x_0; v, v)$. The coefficient of $f(x_0; u, u)$ and $f(x_0; v, v)$ is $4a + 2b$ and the coefficient of $f(x_0; u, v)$ is $8a$. Since we require $e(x_0; u, u, v, v) = f(x_0; u, v)$, we obtain $a = \frac{1}{8}, b = -\frac{1}{4}$. It is straightforward to check that e models the required system $\mathcal{G} \in \mathcal{R}$. We can define a new cell class $\mathbf{D}$, built from the class $\mathbf{C}$ cells of the system $\mathcal{F}$, which realizes the dynamics of $\mathcal{F}$ when these cells are coupled according to the network architecture $\mathcal{R}$. See figure 3.

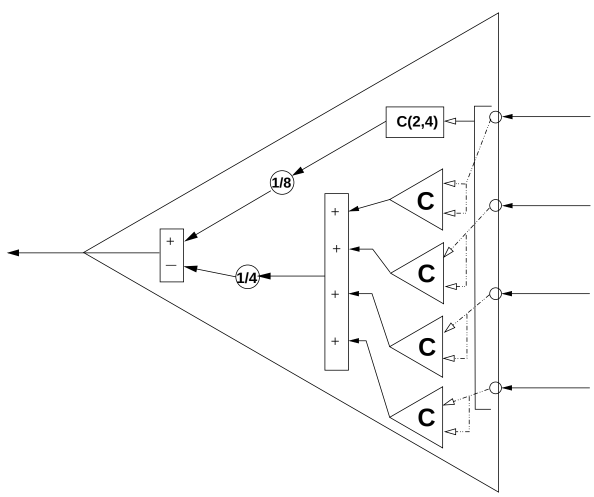

FIGURE 3. The cell $\mathbf{D}$, built from class $\mathbf{C}$ cells

Lemma 3.22. *If $p = 1$, then $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$ implies $\mathcal{M} \prec_O \mathcal{N}$.*

Proof. Since $M_1 \in \mathbf{A}(\mathcal{N})$, we may write $M_1 = \sum_{i \in A} \lambda_i N_i - \sum_{i \in B} \lambda_i N_i$, where A, B are disjoint subsets of $\overline{\mathbf{q}}$ and for $i \in A \cup B$, $\lambda_i = \frac{a_i}{b_i}$, where $a_i, b_i \in \mathbb{N}$, and $(a_i, b_i) = 1$.

Let $\lambda = \mathrm{lcm}\{b_i \mid i \in A \cup B\}$ and define $\alpha_i = \lambda\lambda_i \in \mathbb{Z}^+, i \in A \cup B$. If we define $P = \sum_{i\in A} \alpha_i N_i$, $Q = \sum_{i \in B} \alpha_i N_i$, then

$$M_1 = \frac{1}{\lambda}(P - Q)$$

Let $\mathcal{N}_1$ be the network with adjacency matrices $\{I, P, Q\}$, and $\mathcal{M}_1$ be the network with adjacency matrices $\{I, R = P - Q\}$. Note that

(1) If $Q = 0$, $\mathcal{M}_1 = \mathcal{N}_1$.
(2) If $\lambda = 1$, $\mathcal{M}_1 = \mathcal{M}$.
(3) If $Q = 0$ and $\lambda = 1$, $\mathcal{N}_1 = \mathcal{M}_1 = \mathcal{M}$.

We claim that

$$\mathcal{M} \prec_O \mathcal{M}_1 \prec_O \mathcal{N}_1 \prec_O \mathcal{N}.$$

Assuming the claim, the transitivity of $\prec_O$ (Lemma 3.9) gives $\mathcal{M} \prec_O \mathcal{N}$. The claim follows since lemma 3.15 implies $\mathcal{N}_1 \prec_O \mathcal{N}$, lemma 3.17 implies $\mathcal{M}_1 \prec_O \mathcal{N}_1$ and lemma 3.20 implies $\mathcal{M} \prec_O \mathcal{M}_1$. □

Example 3.23. (Illustration of lemma 3.22.) Let $\mathcal{N}$ be the network defined in example 3.16 and $\mathcal{M}$ be the network of example 3.21. We have $M_1 = \frac{N_1}{2} + \frac{N_2}{2} - N_0$. Following the notation of the proof of lemma 3.22, we have $\lambda = 2$, $P = N_1 + N_2$, $Q = 2N_0$. Note that P, Q are the non-identity adjacency matrices of the second network $\mathcal{Q}$ of example 3.16. We have $\mathcal{Q} \prec_O \mathcal{N}$ (example 3.16); $\mathcal{R} \prec_O \mathcal{Q}$ (example 3.19), and $\mathcal{M} \prec_O \mathcal{R}$ (example 3.21). Since $\prec_O$ is transitive, $\mathcal{M} \prec_O \mathcal{N}$. By using the output relations between g and h from example 3.16, h and e from example 3.19, and e and f from example 3.21, it can be shown (after some computation) that the output relation between g and f is given by

$$g(x_0; \overline{x_1, \cdots, x_4}, \overline{x_5, x_6}) = \frac{1}{4} \sum_{1 \le j_1 < j_2 \le 6} f(x_0; \overline{x_{j_1}, x_{j_2}}) + f(x_0; \overline{x_0, x_0}) - \frac{1}{8} \sum_{1 \le j_1 \le 6} f(x_0; \overline{x_{j_1}, x_{j_1}}) - \frac{1}{2} \sum_{1 \le j_1 \le 6} f(x_0; \overline{x_0, x_{j_1}})$$

Lemma 3.24. *If* $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$*, then* $\mathcal{M} \prec_O \mathcal{N}$ *(statement (A) is true).*

Proof. By lemma 3.11, it suffices to show that $\mathcal{M}_j \prec_O \mathcal{N}$ for all $j \in \mathbf{p}$. By lemma 3.13, we may assume $j = 1$. The result follows from lemma 3.22. □

Lemma 3.25. *If* $\mathcal{M} \prec_O \mathcal{N}$ *then* $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$ *(statement (C) is true).*

Proof. Suppose $\mathcal{M} \prec_O \mathcal{N}$. The method we use is based on the linear equivalence ideas described in [3]. Specifically, we prove that $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$ by restricting to the case where phase spaces equal $\mathbb{R}$ and vector fields are linear. (Notice that output domination preserves linearity of vector fields.)

Let $\mathcal{F} \in \mathcal{M}$ have (linear) model $f : \mathbb{R} \times \prod_{i=1}^p \mathbb{R}^{r_i} \to \mathbb{R}$. Then there exists a system $\mathcal{G} \in \mathcal{N}$ with linear model $g : \mathbb{R} \times \prod_{i=1}^q \mathbb{R}^{s_i} \to \mathbb{R}$ such that for each $j \in \mathbf{n}$ we have

$$g(x_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = f(x_j; \mathbf{X}_{\mathfrak{m}_1^j}, \cdots, \mathbf{X}_{\mathfrak{m}_p^j}), \tag{3.8}$$

where $\mathbf{X}_{\mathfrak{n}_i^j} = (x_{\mathfrak{n}_{i1}^j}, \cdots, x_{\mathfrak{n}_{is_i}^j})$, $i \in \mathbf{q}$, and $\mathbf{X}_{\mathfrak{n}_i^j} = (x_{\mathfrak{m}_{i1}^j}, \cdots, x_{\mathfrak{m}_{ir_i}^j})$, $i \in \mathbf{p}$. Let $k \in \mathbf{p}$ and take

$$f(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_p) = \sum_{i=1}^{r_k} x_{ki}$$

where $\mathbf{X}_v = (x_{v1}, \cdots, x_{vr_v})$, $v \in \mathbf{p}$. The corresponding g given by output domination is linear and so, noting the symmetry of inputs, we may write

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = c_{k0}x_0 + \sum_{i=1}^{q} c_{ki} \sum_{\ell=1}^{s_i} x_{i\ell},$$

where $\mathbf{X}_i = (x_{i1}, \cdots, x_{is_i})$, $i \in \mathbf{q}$, and the $c_{\alpha\beta}$ are constants. From (3.8) we get

$$c_{k0}x_j + \sum_{i=1}^{q} c_{ki} \sum_{\ell=1}^{s_i} x_{\mathfrak{n}_{i\ell}^j} = \sum_{i=1}^{r_k} x_{\mathfrak{m}_{ki}^j}, \ j \in \mathbf{n}.$$

Putting these equations in matrix form, we obtain

$$\sum_{i=0}^{q} c_{ki} N_i = M_k.$$

Hence for each $k \in \mathbf{q}$, we have shown that $M_k \in \mathbf{A}(\mathcal{N})$ and so $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$. □

Lemma 3.26. *If $\mathcal{M} \prec \mathcal{N}$ then $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$ (statement (D) is true).*

Proof. (Sketch) Working within the class of C^1-vector fields with phase space $\mathbb{R}$, it follows that if $\mathcal{F}$ has linear model f, then there exists $\mathcal{G} \in \mathcal{N}$ with C^1-model g such that $\mathcal{G}$ has identical dynamics to $\mathcal{F}$. The statement remains true if we replace g by the derivative of g at $0 \in \mathbb{R} \times \mathbb{R}^q$ and then the method of proof of lemma 3.25 applies (essentially we reduce to linear equivalence, cf [3]). With a little more work, we can remove the assumption that g is C^1 — identical dynamics to a linear system implies the flow is linear and from this one can show that we can always choose g to be linear. □

Proof of theorem 3.12. Lemmas 3.24, 3.25, 3.26 give statements A,C,D and, as noted previously, statement B is trivial. Interchange $\mathcal{M}$ and $\mathcal{N}$ to obtain the reverse relations. □

4. Input Equivalence

We start by giving the definition of input equivalence applicable to networks with symmetric inputs. This a straightforward extension of the definition given in [1, §3] for networks with asymmetric inputs. Aside from assuming that models are defined on vector spaces V rather than manifolds M, we closely follow the notational conventions established in sections 2 and 3. In particular, $\mathcal{M}$ and $\mathcal{N}$ will be coupled n identical cell networks. We fix an ordering of the cells of $\mathcal{N}$. Suppose cells in $\mathcal{N}$ have s inputs and q input types. Let $\mathbb{A}(\mathcal{N}) = \{N_0 = I,\ N_i \in M_{s_i}(n;\mathbb{Z}^+),\ i \in \mathbf{q}\}$ be the set of adjacency matrices and $\mathbf{A}(\mathcal{N})$ denote the subspace of $M(n;\mathbb{Q})$ spanned by $\mathbb{A}(\mathcal{N})$. Let $\mathfrak{n} = [\mathfrak{n}^1, \cdots, \mathfrak{n}^n]$ be the default connection matrix for $\mathcal{N}$.

We suppose cells in $\mathcal{M}$ have r inputs and p input types. Given an ordering of the cells of $\mathcal{M}$, we let $\mathbb{A}(\mathcal{M}) = \{M_0 = I,\ M_i \in M_{r_i}(n;\mathbb{Z}^+),\ i \in \mathbf{p}\}$ denote the set of adjacency matrices and $\mathbf{A}(\mathcal{M})$ denote the subspace of $M(n;\mathbb{Q})$ spanned by $\mathbb{A}(\mathcal{M})$. Let $\mathfrak{m} = [\mathfrak{m}^1, \cdots, \mathfrak{m}^n]$ be the default connection matrix for $\mathcal{M}$.

Let $L = (L_1, \cdots, L_p) \in \prod_{i=1}^p M(r_i, 1+\sum_{j=1}^q s_j;\mathbb{Q})$ and define the linear map $\mathbf{L} : V \times \prod_{i=1}^q V^{s_i} \to \prod_{i=1}^p V^{r_i}$ in the obvious (V-independent) way. Recall that $\mathbf{L}$ is $G_{\mathcal{M},\mathcal{N}}$-equivariant if there exists a homomorphism $h : G_{\mathcal{N}} \to G_{\mathcal{M}}$ such that

$$\mathbf{L}(\gamma(\mathbf{X})) = h(\gamma)\mathbf{L}(\mathbf{X}),\ \text{for all } \gamma \in G_{\mathcal{N}},\ \mathbf{X} \in V \times \prod_{i=1}^q V^{s_i}.$$

If $f : V \times \prod_{i=1}^p V^{r_i} \to V$ is $G_{\mathcal{M}}$-invariant, define $g : V \times \prod_{i=1}^q V^{s_i} \to V$ by

$$g(\mathbf{X}_0;\mathbf{X}_1, \cdots, \mathbf{X}_q) = f(\mathbf{X}_0;\mathbf{L}(\mathbf{X}_0;\mathbf{X}_1, \cdots, \mathbf{X}_q)). \tag{4.9}$$

Since $\mathbf{L}$ is $G_{\mathcal{M},\mathcal{N}}$-equivariant, g is $G_{\mathcal{N}}$-invariant. We write $f <^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$, if

(1) (4.9) is satisfied.
(2) For $j \in \mathbf{n}$, we have $g(x_j;\mathbf{X}_{\mathfrak{n}^j_1}, \cdots, \mathbf{X}_{\mathfrak{n}^j_q}) = f(x_j;\mathbf{X}_{\mathfrak{m}^j_1}, \cdots, \mathbf{X}_{\mathfrak{m}^j_p})$.

Definition 4.1. (Notation and assumptions as above.) The coupled cell network $\mathcal{M}$ is *input dominated* by $\mathcal{N}$, denoted $\mathcal{M} \prec_I \mathcal{N}$, if there exist a linear map $\mathbf{L}$, an ordering of the cells of $\mathcal{M}$, with associated connection matrix $\mathfrak{m}$, such that for every $\mathcal{F} \in \mathcal{M}(\mathbb{L})$, there exists $\mathcal{G} \in \mathcal{N}(\mathbb{L})$ for which $f_{\mathcal{F}} <^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g_{\mathcal{G}}$. If $\mathcal{N} \prec_I \mathcal{M}$ and $\mathcal{M} \prec_I \mathcal{N}$, we say $\mathcal{M}$ and $\mathcal{N}$ are *input equivalent* and write $\mathcal{M} \sim_I \mathcal{N}$.

Remarks 4.2. (1) As we shall see later (remark 4.13), the map $\mathbf{L}$ may not preserve default connection matrices. However, since inputs are symmetric, it is no loss of generality to require the default connection matrix of $\mathcal{M}$ in definition 4.1. When we come to prove our main theorem, we allow for general connection matrices.
(2) We write $\mathcal{M} \prec_{I,\mathbb{Z}} \mathcal{N}$ if $\mathcal{M} \prec_I \mathcal{N}$ and we can require the entries of $\mathbf{L}$ to lie in $\mathbb{Z}$). We similarly define $\mathcal{M} \sim_{I,\mathbb{Z}} \mathcal{N}$.

Lemma 4.3. *(Notation and assumptions as above). The following conditions are equivalent.*

1. $\mathcal{M} \prec_I \mathcal{N}$.
2. *There exists an ordering of the cells in $\mathcal{M}$ such that $\mathcal{M}_j \prec_I \mathcal{N}$, for all $j \in \mathbf{p}$.*

Proof. The proof follows by observing that

$$f_{\mathcal{F}} <^{\imath}_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g_{\mathcal{G}} \iff f_{\mathcal{F}_j} <^{\imath}_{(\mathbf{L_j},\mathfrak{m}^{\mathfrak{j}},\mathfrak{n})} g_{\mathcal{G}}$$

for all $j \in \mathbf{p}$ where $\mathbf{L} = [\mathbf{L}_1, \cdots, \mathbf{L}_p]$, $\mathbf{L}_j : V \times \prod_{i=1}^q V^{s_i} \to V^{r_j}$, $\mathcal{F}_j \in \mathcal{M}_j$, and $\mathfrak{m}^{\mathfrak{j}}$ is the connection matrix induced on $\mathcal{M}_j$ by $\mathfrak{m}$. ☐

As a consequence of lemma 4.3, it will be no loss of generality in what follows to assume that $\mathcal{M}$ has just one input type; that is, $p = 1$. We simplify notation by setting $r_1 = r$. With these conventions, we have $G_{\mathcal{M}} = S_r \approx S_1 \times S_r$.

Suppose that the linear map $\mathbf{L} : V \times \prod_{i=1}^q V^{s_i} \to V^r$ is defined by the matrix $L \in M\left(r, 1 + \sum_{i=1}^q s_i, \mathbb{Q}\right)$. The map $\mathbf{L}$ is $G_{\mathcal{M},\mathcal{N}}$-equivariant if there exists a homomorphism $h : G_{\mathcal{N}} \to G_{\mathcal{M}} = S_r$ such that

$$\mathbf{L}(\gamma(\mathbf{X})) = h(\gamma)\mathbf{L}(\mathbf{X}),$$

for all $\gamma \in G_{\mathcal{N}}$, $\mathbf{X} \in V \times \prod_{i=1}^q V^{s_i}$.

Given a $G_{\mathcal{M}}$-invariant map $f : V \times V^r \to V$ and $G_{\mathcal{M},\mathcal{N}}$-equivariant linear map $\mathbf{L}$ as above, define the $G_{\mathcal{N}}$-invariant map $g : V \times \prod_{i=1}^q V^{s_i} \to V$ by

$$g(\mathbf{X}_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = f(\mathbf{X}_0; \mathbf{L}(\mathbf{X}_0; \mathbf{X}_1, \cdots, \mathbf{X}_q)).$$

Let $L = [L_1, \cdots, L_r]$, where $L_i \in \mathbb{Q} \times \prod_{i=1}^q \mathbb{Q}^{s_i}$ denotes the i^{th} row of L, $i \in \mathbf{r}$. Since $\mathbf{L}(\gamma(\mathbf{X})) = h(\gamma)\mathbf{L}(\mathbf{X})$ for all $\gamma \in G_{\mathcal{N}}$, $\mathbf{X} \in V \times \prod_{i=1}^q V^{s_i}$, we have $[L_1, \cdots, L_r](\gamma \mathbf{X}) = h(\gamma)[L_1, \cdots, L_r](\mathbf{X})$. That is,

$$[\gamma L_1, \cdots, \gamma L_r](\mathbf{X}) = h(\gamma)[L_1, \cdots, L_r](\mathbf{X}),$$

where, γL_i is defined using the natural permutation action of $G_{\mathcal{N}}$ on $\mathbb{Q} \times \prod_{j=1}^{q} \mathbb{Q}^{s_j}$, $i \in \mathbf{r}$. This is true for all $\mathbf{X}$, hence

$$[\gamma L_1, \cdots, \gamma L_r] = h(\gamma)[L_1, \cdots, L_r],$$

for all $\gamma \in G_{\mathcal{N}}$.

Definition 4.4. Suppose a finite group G acts on a non-empty set X. For $x \in X$, let $Gx = \{gx \mid g \in G\}$ denote the *G-orbit of x.*

Remark 4.5. We have $|Gx| = |G|/|G_x|$ where $G_x = \{g \in G \mid gx = x\}$ denotes the *isotropy subgroup* of G at x.

Theorem 4.6. *There exists $u(\leq r) \in \mathbb{N}$, $t_1, \cdots, t_u \in \mathbb{N}$, with $\sum_{i=1}^{u} t_i = r$ such that $\{L_1, \cdots, L_r\} = \bigcup_{i=1}^{u} G_{\mathcal{N}} L^i$ where $|G_{\mathcal{N}} L^i| = t_i$. (We allow $L^i = L^j$ for $i \neq j$, $i, j \in \mathbf{u}$.)*

Proof. $[\gamma L_1, \cdots, \gamma L_r] = h(\gamma)[L_1, \cdots, L_r]$ for all $\gamma \in G_{\mathcal{N}}$. Therefore, for each $i \in \mathbf{r}$, $\gamma L_i \in \{L_1, \cdots, L_r\}$ for all $\gamma \in G_{\mathcal{N}}$. Hence, the $G_{\mathcal{N}}$-orbit of L_i is contained in $\{L_1, \cdots, L_r\}$. Suppose $L_k = L_j$ for some $k \neq j$, then $\gamma L_k = \gamma L_j$ for all $\gamma \in G_{\mathcal{N}}$. So if an element is repeated m times then its full orbit is repeated m times. Therefore, there exist $u(\leq r) \in \mathbb{N}$, L_{j_i}, for $j_1, \cdots, j_u \in \mathbf{r}$ with $|G_{\mathcal{N}} L_{j_i}| = t_i$ and $\sum_{i=1}^{u} t_i = r$ such that $\{L_1, \cdots, L_r\} = \bigcup_{j=1}^{u} G_{\mathcal{N}} L_{j_i}$. Define $L^i = L_{j_i}$, $i \in \mathbf{u}$. □

Remark 4.7. For each $i \in \mathbf{u}$, there are t_i choices for L_{j_i}.

From now on, we write the matrix of $\mathbf{L}$ in the form

$$\left[G_{\mathcal{N}} L^1, \cdots, G_{\mathcal{N}} L^u\right].$$

That is, we group rows according to the group orbits of $G_{\mathcal{N}}$. Note that this ordering is imposing a condition on the order of inputs of $\mathcal{M}$.

Example 4.8. Suppose $p, q = 1, s = s_1 = 2, r = 3$, $L = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$. Then we can take $L^1 = (0, 1, 2)$, $L^2 = (0, 1, 1)$. We have $t_1 = 2$ and $t_2 = 1$. If we write L in the form $[G_{\mathcal{N}} L^1, G_{\mathcal{N}} L^2]$, then $L = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

4.1. **Splittings and connection matrices.** We recall from [1] that dynamically equivalent networks with asymmetric inputs are always input equivalent. This is not always the case for networks with symmetric inputs as we show in the next example.

Example 4.9. Let $\mathcal{M}$ be the network with non-identity adjacency matrix $M_1 = \begin{pmatrix} 2 & 4 \\ 2 & 0 \end{pmatrix}$ and $\mathcal{N}$ be the network with non-identity adjacency matrix $N_1 = \begin{pmatrix} 3 & 6 \\ 3 & 0 \end{pmatrix}$. We have $M_1 = \frac{2}{3}N_1$ and so $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$. Suppose

$$g(x_0; x_1, \cdots, x_6) = f(x_0; \mathbf{L}(x_0; x_1, \cdots, x_6)),$$

where $\mathbf{L} : V \times V^6 \to V^4$ is a $\mathcal{G}_{\mathcal{M},\mathcal{N}}$-equivariant linear map. The only possible form of L is $\begin{pmatrix} a_1 & b_1 & b_1 & b_1 & b_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 & b_2 & b_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 & b_3 & b_3 & b_3 & b_3 \\ a_4 & b_4 & b_4 & b_4 & b_4 & b_4 & b_4 \end{pmatrix}$. It is easy to check that there does not exist any $a_i, b_i \in \mathbb{Q}$ for which f is input dominated by g. This shows $\mathcal{M} \not\prec_I \mathcal{N}$. Similarly we can show that $\mathcal{N} \not\prec_I \mathcal{M}$ (in this case, L has two possible forms). This provides an example of network architectures $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \sim_O \mathcal{N}$ ($\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$) but $\mathcal{M} \not\prec_I \mathcal{N}$ and $\mathcal{N} \not\prec_I \mathcal{M}$.

The previous example shows that $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ is not sufficient for $\mathcal{M} \sim_I \mathcal{N}$. Note that $\mathcal{M} \sim_I \mathcal{N} \Rightarrow \mathcal{M} \sim \mathcal{N} \Rightarrow \mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$. Thus $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{N})$ is a necessary condition for $\mathcal{M} \sim_I \mathcal{N}$. In theorem 4.11, we give sufficient conditions for input equivalence to hold. The sufficiency conditions come from the structure of the $\mathcal{G}_{\mathcal{M},\mathcal{N}}$-equivariant linear map $\mathbf{L}$. If $\mathcal{M} \prec_I \mathcal{N}$ and we fix a connection matrix for $\mathcal{N}$, then $\mathbf{L}$ determines a connection matrix for $\mathcal{M}$ which may not be the default connection matrix (whatever the choice of $\mathbf{L}$). In order to analyze the relationship between connection matrices of $\mathcal{M}$ and $\mathcal{N}$, we introduce the idea of splitting a valency k adjacency matrix into a sum of k valency one matrices. We find that there is a one-one correspondence between splittings and connection matrices.

Definition 4.10. Let $P \in M_k(n; \mathbb{Z}^+)$. A *splitting* $(P_1, \cdots, P_k)$ of P is an ordered decomposition of P into a sum $P = P_1 + \cdots + P_k$, where each $P_j \in M_1(n; \mathbb{Z}^+)$.

Suppose that the network $\mathcal{M}$ has one input type and connection matrix $\mathfrak{m}$, where $\mathfrak{m}$ is not necessarily the default connection matrix. Denote the adjacency matrices of $\mathcal{M}$ by $M_0 = I$ and M_1. The connection matrix $\mathfrak{m}$ naturally determines a unique splitting $M^1 + \cdots + M^r$

of M_1. Indeed, if we let $M^k = [m_{ij}^k]$, $k \in \mathbf{r}$, then we define $m_{ij}^k = 1$ if input k of cell j comes from cell i, else $m_{ij}^k = 0$. That is, $m_{ij}^k = 1$ iff $\mathfrak{m}_{1k}^j = i$. Conversely, every splitting of M_1 uniquely determines a connection matrix $\mathfrak{m}$ for $\mathcal{M}$. All of this applies equally well if $\mathcal{M}$ has multiple input types.

Let $\mathfrak{n}$ be a connection matrix for $\mathcal{N}$ (not necessarily the default). For $k \in \mathbf{q}$, let $\mathbf{N}_k = (N_{k1}, \cdots, N_{ks_k})$ denote the splitting of N_k naturally determined by $\mathfrak{n}$. Set $\mathbf{N} = \{\mathbf{N}_1, \cdots, \mathbf{N}_q\}$. We refer to $\mathbf{N}$ as the splitting determined by $\mathfrak{n}$.

Let $\mathbf{a} = (a_0; a_1, \cdots, a_q) \in \mathbb{Q} \times \prod_{j=1}^q \mathbb{Q}^{s_j}$. We write $\mathbf{a} = (a_{ji})_{j \in \overline{\mathbf{q}}, i \in \mathbf{s_j}}$, where $a_j = (a_{j1}, \cdots, a_{js_j}) \in \mathbb{Q}^{s_j}$, $j \in \overline{\mathbf{q}}$. If $\mathbf{N} = \{\mathbf{N}_1, \cdots, \mathbf{N}_q\}$ is the set of splittings of the adjacency matrices $\{N_1, \cdots, N_q\}$ determined by $\mathfrak{n}$, then we define

$$\mathbf{a} \star \mathbf{N} = a_0 N_0 + \sum_{j=1}^{q} \sum_{i=1}^{s_j} a_{ji} N_{ji} \in M(n, n; \mathbb{Q}).$$

Theorem 4.11. *(Notation and assumptions as above; in particular $p = 1$.) The following statements are equivalent*

(1) $\mathcal{M} \prec_I \mathcal{N}$.
(2) *Suppose that $\mathfrak{n}$ is a connection matrix for $\mathcal{N}$ with associated splitting $\mathbf{N}$. There exist $u \in \mathbb{N}$, $L^i \in \mathbb{Q} \times \prod_{v=1}^q \mathbb{Q}^{s_v}$, $i \in \mathbf{u}$, such that $\{\mathbf{b} \star \mathbf{N} \mid b \in G_{\mathcal{N}} L^i, i \in \mathbf{u}\}$ is a splitting of M_1.*
(3) *There exist $u \in \mathbb{N}$, $L^i \in \mathbb{Q} \times \prod_{v=1}^q \mathbb{Q}^{s_v}$, $i \in \mathbf{u}$ such that for every connection matrix $\mathfrak{n}$ of $\mathcal{N}$ with associated splitting $\mathbf{N}$, $\{\mathbf{b} \star \mathbf{N} \mid b \in G_{\mathcal{N}} L^i, i \in \mathbf{u}\}$ is a splitting of M_1.*

Before giving the proof of theorem 4.11, we make two remarks, the first of which shows how theorem 4.11 simplifies in the case of asymmetric inputs.

Remarks 4.12. (1) If all the inputs of the networks $\mathcal{M}$ and $\mathcal{N}$ are asymmetric then $q = s$, $s_i = 1$, $\mathbf{N} = \{N_1 = N_{11}, \cdots, \mathcal{N}_q = N_{q1}\}$ and M_1 is a splitting of itself. Thus (3) of theorem 4.11 implies that there exist $u \in \mathbb{N}$, $L^i = (a_{i0}; a_{i1}, \cdots, a_{iq}) \in \mathbb{Q} \times \mathbb{Q}^q$, $i \in \mathbf{u}$ such that for every connection matrix $\mathfrak{n}$ of $\mathcal{N}$, $\{\sum_{j=0}^q a_{ij} N_j, i \in \mathbf{u}\}$ is a splitting of M_1. Since $M_1 \in M_1(n, \mathbb{Z})$, we must have $u = 1$. Therefore, the condition simplifies to $M_1 = \sum_{j=0}^q a_{ij} N_j$. Hence $M_1 \in \mathbf{A}(\mathcal{N})$; the condition obtained for networks with asymmetric inputs in [1].
(2) Condition (3) of the theorem shows that for computations, we can always take $\mathfrak{n}$ to be the default connection matrix.

Proof of Theorem 4.11 (1) $\Rightarrow$ (2). Suppose $\mathcal{M} \prec_I \mathcal{N}$. Then there is a linear transformation $\mathbf{L}$ with matrix $L = [G_\mathcal{N} L^1, \cdots, G_\mathcal{N} L^u]$. Let $\mathfrak{n}$ be a connection matrix for $\mathcal{N}$ and denote the corresponding splittings of $N_1, \cdots, N_q$ by $\mathbf{N}$. For each $j \in \mathbf{n}$, we have

$$\mathbf{L}(\mathbf{X}_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = \mathbf{X}_{\mathfrak{m}_1^j}$$

where $\mathfrak{m}$ is a connection matrix for the network $\mathcal{M}$. Thus $\{\mathbf{b} \star \mathbf{N} \mid b \in G_\mathcal{N} L^i, i \in \mathbf{u}\}$ is a splitting of M_1.
(2) $\Rightarrow$ (3). Suppose statement (2) holds for the connection matrix $\mathfrak{n}$ and let $\widehat{\mathfrak{n}}$ be any other connection matrix for $\mathcal{N}$. Then for each $j \in \mathbf{n}$, $\widehat{\mathfrak{n}^j} = \gamma^j \mathfrak{n}^j$ for some $\gamma^j \in G_\mathcal{N}$ ($\gamma^j \mathfrak{n}^j$ is the natural action of $G_\mathcal{N}$ on $\{j\} \times \prod_{i=1}^q \mathbf{n}^{s_i}$). For $j \in \mathbf{n}$, let $\mathbf{N}^j$ denote the set of j^{th} columns of all matrices in $\mathbf{N}$. Since $\{b \star \mathbf{N} \mid b \in G_\mathcal{N} L^i, i \in \mathbf{u}\}$ is a splitting of M_1, $\{[\gamma^1(b) \star \mathbf{N}^1, \cdots, \gamma^n(b) \star \mathbf{N}^n] \mid b \in G_\mathcal{N} L^i, i \in \mathbf{u}\} = \{[b \star \gamma^1(\mathbf{N}^1), \cdots, b \star \gamma^n(\mathbf{N}^n)] \mid b \in G_\mathcal{N} L^i, i \in \mathbf{u}\}$ is a splitting of M_1. Hence (2) holds for $\widehat{\mathfrak{n}}$.
(3) $\Rightarrow$ (1). Take $L = [G_\mathcal{N} L^1, \cdots, G_\mathcal{N} L^u]$. Fix a connection matrix $\mathfrak{n} = [\mathfrak{n}^1, \cdots, \mathfrak{n}^n]$ for $\mathcal{N}$ and denote the associated family of splittings of $N_1, \cdots, N_q$ by $\mathbf{N}$ as above. Since $\{\mathbf{b} \star \mathbf{N} \mid b \in G_\mathcal{N} L^i, i \in \mathbf{u}\}$ is a splitting of M_1, we have a connection matrix $\mathfrak{m} = [\mathfrak{m}^1, \cdots, \mathfrak{m}^n]$ for $\mathcal{M}$, where $\mathfrak{m}^j = (\mathfrak{m}_1^j) \in \mathbf{n}^r$ satisfies

$$\mathbf{L}(\mathbf{X}_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) = \mathbf{X}_{\mathfrak{m}_1^j},\ j \in \mathbf{n}.$$

Hence for all $j \in \mathbf{n}$,

$$\begin{aligned} g(\mathbf{X}_j; \mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j}) &= f(\mathbf{X}_j; \mathbf{L}(\mathbf{X}_{\mathfrak{n}_1^j}, \cdots, \mathbf{X}_{\mathfrak{n}_q^j})) \\ &= f(\mathbf{X}_j; \mathbf{X}_{\mathfrak{m}_1^j}) \end{aligned}$$

This implies $\mathcal{M} \prec_I \mathcal{N}$. $\square$

Remark 4.13. If we have $\mathcal{M} \prec_I \mathcal{N}$ and we take the default connection matrix for $\mathcal{N}$, then the connection matrix on $\mathcal{M}$ given by theorem 4.11(2) will generally *not* equal the default connection matrix of $\mathcal{M}$. For example, suppose that $\mathcal{N}$ is the network with non-identity adjacency matrix $N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{M}$ is the network with non-identity adjacency matrix $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. We have $M_1 = N_0 + N_1$ and may easily check directly that $\mathcal{M} \prec_I \mathcal{N}$. If $\mathcal{F} \in \mathcal{M}$ has model $f : V \times V^2 \to V$, then we define g modelling $\mathcal{G} \in \mathcal{N}$ either by $g(x; y) = f(x; \overline{x, y})$ or by $g(x; y) = f(x; \overline{y, x})$. Here the only choices of L are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Neither of these choices gives the default connection matrix for $\mathcal{M}$.

Corollary 4.14. *(Notation and assumptions as above.) Suppose that $M_1 \in M_1(n;\mathbb{Z}^+)$, then $\mathcal{M} \prec_I \mathcal{N}$ iff $M_1 \in \mathbf{A}(\mathcal{N})$.*

Proof. ($\Rightarrow$): Since $\mathcal{M} \prec_I \mathcal{N}$, there is a linear transformation $\mathbf{L}$ with matrix $L = [\mathbf{a}] \in M(1, \sum_{j=0}^q s_j, \mathbb{Q})$, where $\mathbf{a} = [a_0; a_1, \cdots, a_q] \in \mathbb{Q} \times \prod_{j=1}^q \mathbb{Q}^{s_j}$, such that $f \prec^\imath_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$. Since L has only one row, $\mathcal{G}_{\mathcal{N}}\mathbf{a} = \{\mathbf{a}\}$. Therefore, for $j \in \mathbf{q}$, we may write $a_j = \lambda_j \mathbf{1} \in \mathbb{Q}^{s_j}$ where $\lambda_j \in \mathbb{Q}$. If we take $u = 1$, and $L^1 = \mathbf{a}$, then $M_1 = \sum_{j=0}^q \lambda_j N_j$.
($\Leftarrow$): Let $M_1 = \sum_{j=0}^q \lambda_j N_j$. Take $\mathbf{L}$ to be the linear transformation with matrix $L = [\mathbf{a}] \in M(1, \sum_{j=0}^q s_j, \mathbb{Q})$, $\mathbf{a} = [a_0; a_1, \cdots, a_q] \in \mathbb{Q} \times \prod_{j=1}^q \mathbb{Q}^{s_j}$, where $a_j = \lambda_j \mathbf{1} \in \mathbb{Q}^{s_j}$, $j \in \mathbf{q}$. Hence we have

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = \lambda_0 f(x_0; \sum_{j=0}^q \lambda_j \sum_{i=1}^{s_j} x_{ji}),$$

where $s_0 = 1$, $x_{01} = x_0$ and $\mathbf{X}_i = (x_{i1}, \cdots, x_{is_i})$ denotes variables corresponding to the inputs of type i, $i \in \mathbf{q}$. It is straightforward to check that $f \prec^\imath_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$. □

Corollary 4.15. *(Notation and assumptions as above.) If M_1 has a splitting $(Q_1, \cdots, Q_r)$ such that $\{Q_1, \cdots, Q_r\} \subseteq \mathbf{A}(\mathcal{N})$, then $\mathcal{M} \prec_I \mathcal{N}$.*

Proof. For each $i \in \mathbf{r}$, let $Q_i = \sum_{j=0}^q \lambda_{ij} N_j$. Define

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = \lambda_0 f(x_0; \sum_{j=0}^q \lambda_{1j} \sum_{i=1}^{s_j} x_{ji}, \cdots, \sum_{j=0}^q \lambda_{rj} \sum_{i=1}^{s_j} x_{ji}),$$

where $s_0 = 1$, $x_{01} = x_0$ and $\mathbf{X}_i = (x_{i1}, \cdots, x_{is_i})$ denotes variables corresponding to the inputs of type i, $i \in \mathbf{q}$. It is straightforward to check that $f \prec^\imath_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$. □

Corollary 4.16. *(Notation and assumptions as above.) Suppose that $M_1 \in \mathbf{A}(\mathcal{N}, \mathbb{Z}^+)$, that is, $M_1 = \sum_{j=0}^q \alpha_j N_j$, $\alpha_j \in \mathbb{Z}^+$, $j \in \overline{\mathbf{q}}$. Then $\mathcal{M} \prec_I \mathcal{N}$.*

Proof. Define

$$g(x_0; \mathbf{X}_1, \cdots, \mathbf{X}_q) = \lambda_0 f(x_0; x_0^{\alpha_0}; \mathbf{X}_1^{\alpha_1}, \cdots, \mathbf{X}_q^{\alpha_q}),$$

where $\mathbf{X}_i = (x_{i1}, \cdots, x_{is_i})$ denotes variables corresponding to the inputs of type i, $i \in \mathbf{q}$. It is straightforward to check that $f \prec^\imath_{(\mathbf{L},\mathfrak{m},\mathfrak{n})} g$. □

Corollary 4.17. *(Notation and assumptions as above.) If we can write $M_1 = A + S$ where $A \in \mathbf{A}(\mathcal{N}, \mathbb{Z}^+)$, and there exists a splitting $(S_1, \cdots, S_t)$ of S such that $S_i \in \mathbf{A}(\mathcal{N})$, $i \in \mathbf{t}$, then $\mathcal{M} \prec_I \mathcal{N}$.*

Proof. Define the S component of M_1 using corollary 4.15 and the A component using corollary 4.16. $\square$

Theorem 4.18. *(Notation and assumptions as above except that we allow $p \geq 1$.) The following statements are equivalent*

(1) $\mathcal{M} \prec_I \mathcal{N}$.
(2) *Suppose that $\mathfrak{n}$ is a connection matrix for $\mathcal{N}$. For $j \in \mathbf{p}$, there exist $u_j \in \mathbb{N}$, $L^i_j \in \mathbb{Q} \times \prod_{v=1}^q \mathbb{Q}^{s_v}$, $i \in \mathbf{u_j}$, such that $\{\mathbf{b} \star \mathbf{N} \mid b \in G_{\mathcal{N}} L^i_j, i \in \mathbf{u_j}\}$ is a splitting of M_j.*

Proof. The result is immediate from theorem 4.11 and lemma 4.3. $\square$

Corollary 4.19. *Let $\mathcal{M}$ and $\mathcal{N}$ be coupled n identical cell networks. Assume cells in $\mathcal{M}$ have r inputs, cells in $\mathcal{N}$ have s inputs. Suppose that $\mathcal{M}$ has adjacency matrices $M_0 = I, M_1, \cdots, M_p$ and $\mathcal{N}$ has adjacency matrices $N_0 = I, N_1, \cdots, N_q$. We assume that for each $i \in \mathbf{p}$ either $r_i = 1$ or $s_j > r_i > 1$, for all $j \in \mathbf{q}$. Under these conditions the following statements are equivalent*

(1) $\mathcal{M} \prec_I \mathcal{N}$.
(2) *For all $i \in \mathbf{p}$, there exists a splitting $(P_{i,1}, \cdots, P_{i,r_i})$ of M_i such that $P_{i,j} \in \mathbf{A}(\mathcal{N})$, for all $j \in \mathbf{r}_i$.*

Proof. (Sketch.) (2) $\Rightarrow$ (1) is trivial. In order to prove (1) $\Rightarrow$ (2), we may assume $p = 1$. Set $r = r_1$. For every $\mathbf{a} \in \mathbb{Q} \times \prod_{j=1}^q \mathbb{Q}^{s_j}$, $G_{\mathcal{N}}\mathbf{a}$ has one element or at least $\min_{j \in \mathbf{q}} s_j$ elements. Since $r < s_j$ for all $j \in \mathbf{q}$, we have $r < \min_{j \in \mathbf{q}} s_j$. Therefore L must be of the form $[L^1, \cdots, L^r]$ where $L^i \in \mathbb{Q} \times \prod_{j=1}^q \mathbb{Q}\mathbf{1}^{s_j}$. $\square$

Remark 4.20. If the network $\mathcal{M}$ has asymmetric inputs and $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$, hypothesis (2) of corollary 4.19 is automatically satisfied (and so we recover the result for networks with asymmetric inputs — see lemma [1, §3.13]). However, if $\mathcal{M}$ has symmetric inputs and $\mathbf{A}(\mathcal{M}) \subseteq \mathbf{A}(\mathcal{N})$, then it need not be the case that (2) is satisfied (see example 4.9, note that the only splitting of M_1 is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) and so $\mathcal{M}$ may not be input dominated by $\mathcal{N}$, even if we assume linear phase spaces or scaling signalling. We give some examples in the next section.

5. Examples

We conclude with two examples of network architectures that are both input and output equivalent as well as an example of self-output equivalence.

Example 5.1. If $p = q = 1$, $N_1 = bS$, and $M_1 = aS$ for $S \in M_1(n; \mathbb{Z}^+)$, $a, b \in \mathbb{N}$, then $\mathcal{M} \sim_O \mathcal{N}$ and $\mathcal{M} \sim_I \mathcal{N}$. Here $r = a$, $s = b$.
(a) Suppose $\mathcal{F} \in \mathcal{M}$ has model $f : M \times M^a \to TM$. Define $g : M \times M^b \to TM$ by

$$\begin{aligned} g_O(x_0; \overline{x_1, \cdots, x_b}) &= \frac{1}{b}\left[f(x_0; x_1^a) + \cdots + f(x_0; x_b^a)\right] \\ g_I(x_0; \overline{x_1, \cdots, x_b}) &= f(x_0; (\frac{1}{b}\sum_{i=1}^{b} x_i)^a), \end{aligned}$$

where x^a signifies that x repeated a-times. It is easy to verify that g_O and g_I give the required output and input dominations of f. Hence, $\mathcal{M} \prec_O \mathcal{N}$ and $\mathcal{M} \prec_I \mathcal{N}$. The reverse order is obtained by interchanging a and b. Note that the input relations are same as were defined in corollary 4.15.

Example 5.2. Let $\mathcal{M}$ be the network with non-identity adjacency matrix $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathcal{N}$ be the network with non-identity adjacency matrix $N_1 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. Note that $N_1 = 2M_1$ and so $\mathbf{A}(\mathcal{M}) = \mathbf{A}(\mathcal{M})$. We show that $\mathcal{M} \sim_O \mathcal{N}$ and $\mathcal{M} \sim_I \mathcal{N}$. (a) Suppose that $\mathcal{G} \in \mathcal{N}$ has model g. Let the system $\mathcal{F} \in \mathcal{M}$ have model

$$f(x_0; \overline{x_1, x_2}) = g(x_0; \overline{x_1, x_2, x_1, x_2})$$

Then f output and input dominates g and so $\mathcal{N} \prec_I \mathcal{M}$ and $\mathcal{N} \prec_O \mathcal{M}$.
(b) Suppose that $\mathcal{F} \in \mathcal{M}$ has model f. Define g by

$$\begin{aligned} g_O(x_0; \overline{x_1, \cdots, x_4}) &= \frac{1}{4}\sum_{1 \le i < j \le 4} f(x_0; \overline{x_i, x_j}) - \frac{1}{8}\sum_{1 \le i \le 4} f(x_0; \overline{x_i, x_i}) \\ g_I(x_0; \overline{x_1, x_2, x_3, x_4}) &= f(x_0; x_0, \frac{x_1 + x_2 + x_3 + x_4}{2} - x_0) \end{aligned}$$

Then g_O and g_I give the required output and input dominations of f and so $\mathcal{M} \prec_O \mathcal{N}$ and $\mathcal{M} \prec_I \mathcal{N}$.

Example 5.3. For every network $\mathcal{M}$ we have $\mathcal{M} \sim_O \mathcal{M}$. However, there may be many ways of achieving this self output equivalence. For example, consider the two cell network $\mathcal{M}$ with asymmetric inputs shown in figure 4.

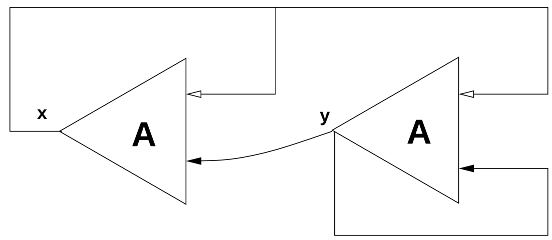

FIGURE 4. A two cell network $\mathcal{M}$ with asymmetric inputs

Suppose $\mathcal{F} \in \mathcal{M}$ has model f. It can be shown that the two-parameter family defined for $c, d \in \mathbb{R}$ by

$$\begin{aligned} f_{c,d}(x_0; x_1, x_2) = {} & cf(x_0; x_0, x_0) + df(x_0; x_0, x_1) \\ & - (c+d)f(x_0; x_0, x_2) - (c+d)f(x_0; x_1, x_0) \\ & + (1+c+d)f(x_0; x_1, x_2) + df(x_0; x_2, x_0) \\ & - df(x_0; x_2, x_1), \end{aligned}$$

gives all output equivalences $\mathcal{M} \sim_O \mathcal{M}$. For example, if we take $c = 0$, $d = -1/2$, then

$$\begin{aligned} g(x_0; x_1, x_2) &= f_{0,-1/2}(x_0; x_1, x_2) \\ &= \frac{1}{2}(-f(x_0; x_0, x_1) + f(x_0; x_0, x_2) + f(x_0; x_1, x_0) \\ &\qquad + f(x_0; x_1, x_2) - f(x_0; x_2, x_0) + f(x_0; x_2, x_1)). \end{aligned}$$

In terms of ordinary differential equations, if the model for a cell is $f(x_0; x_1, x_2) = x_0 x_1 x_2^2 + x_0$ and we define

$$\begin{aligned} g(x_0; x_1, x_2) &= f_{0,-1/2}(x_0; x_1, x_2) \\ &= \frac{1}{2}(-x_0^2 x_1^2 + x_0^2 x_2^2 + x_0^3 x_1 + x_0 x_1 x_2^2 - x_0^3 x_2 + x_0 x_2 x_1^2 + 2x_0). \end{aligned}$$

then $x' = f(x; x, y)$, $y' = f(y; x, y)$ and $x' = g(x; x, y), y' = g(x; x, y)$ have identical dynamics, even though the models f and g are quite different. Note, however, that if f is a linear vector field or is of the form $f(x; y, z) = au(x; y) + bv(x; z)$ then $f = g$. In particular, it seems we cannot usefully develop this idea using the concept of linear self-equivalence [3].

Continuing with our choice of $c = 0$, $d = -1/2$, if we define the new cell class $\mathbf{A}^\star$ as in figure 5. Although the new cell is different from the original cell $\mathbf{A}$, when it is incorporated in the network $\mathcal{M}$, it will give the same dynamics.

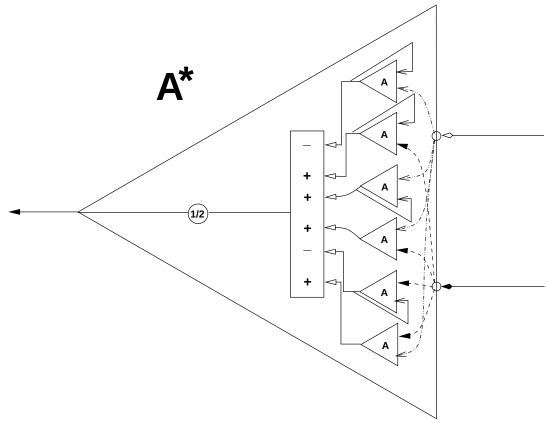

FIGURE 5. The cell $\mathbf{A}^\star$

This construction leads naturally to a number of observations and questions and we conclude by briefly discussing some of these issues. First, to what extent can this process be reversed? That is, given a network of 'complex' cells, when is it equivalent to the same network but built of simpler cells? Secondly, is there a way of choosing the specific output equivalence so as to protect against failure of individual units comprising the new cells? For example, if we build the network $\mathcal{M}$ from the cells $\mathbf{A}^\star$, what is the effect on network dynamics of the failure of a single $\mathbf{A}$-cell in $\mathbf{A}^\star$? Is there an optimal way of choosing the output equivalence so as to minimize the effect of failure of individual units? Are there potential applications to numerical analysis (for example, in the solution of partial differential equations)? There are also questions related to the effects of *inflation* [2] on $\mathbf{A}$-cells in $\mathbf{A}^\star$. Another potentially interesting question is to extend the notion of input equivalence to allow for nonlinear combinations of inputs. This would seem to be of particular interest for scalar scalar signalling networks and self-loops [1].

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