SYMMETRIC ATTRACTORS FOR DIFFEOMORPHISMS AND FLOWS

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Abstract. Let $\Gamma \subset \mathbf{O}(n)$ be a finite group acting on $\mathbb{R}^n$. In this work we describe the possible symmetry groups that can occur for attractors of smooth (invertible) Γ-equivariant dynamical systems. In case $\mathbb{R}^n$ contains no reflection planes and $n \geq 3$, our results imply there are no restrictions on symmmetry groups. In case $n \geq 4$ (diffeomorphisms) and $n \geq 5$ (flows), we show that we may construct attractors which are Axiom A. We also give a complete description of what can happen in low dimensions.

Contents

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1. Introduction

In this work we are interested in describing the possible symmetry groups of attractors for smooth (invertible) equivariant dynamical systems.

Let $\Gamma \subset \mathbf{O}(n)$ be a finite group acting on $\mathbb{R}^n$. We start by reviewing some results on discrete Γ-equivariant dynamical systems on $\mathbb{R}^n$. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and Γ-equivariant. Given $x \in \mathbb{R}^n$, we let $\omega(x)$ denote the ω-limit set of x. The collection of ω-limit sets $\omega(x)$ represents the possible *asymptotic dynamics* of f.

If an element $\gamma \in \Gamma$ fixes the point $y \in \mathbb{R}^n$, we say that γ is a *symmetry of* y. (The set of all symmetries of y is usually called the *isotropy subgroup* of Γ at y and is denoted by Γ_y.) We say that the ω-limit set A has the *instantaneous symmetry* γ, if γ is a symmetry of each point of A.

Suppose that A is an ω-limit set of f. If A consists of a single point y, then y is a fixed point of f, and the group of instantaneous symmetries of A is precisely Γ_y. Next, suppose that A is a periodic orbit and $y \in A$. Then all points in A have isotropy group Γ_y and the group of instantaneous symmetries of A is equal to Γ_y. On the other hand, the group of symmetries preserving A is a cyclic extension of Γ_y and so may be strictly larger than the group of instantaneous symmetries of A.

Suppose that A is an ω-limit set for f which is not a periodic orbit. In this case the dynamics of $f|A$ are typically complicated and it is known that the symmetry group of A may be much larger than the group of instantaneous symmetries of A. (See Dellnitz, Golubitsky and Melbourne [10].) As the symmetry group of A reflects the average behavior of dynamics on A, we refer to this type of symmetry as symmetry on average.

In recent work of Melbourne, Dellnitz and Golubitsky [22], it has been shown that there are restrictions on symmetry on average. These restrictions depend on the specific representation of the group of symmetries and, in particular, on group elements which act as reflections.

Under the assumption that there are only trivial instantaneous symmetries, Ashwin and Melbourne [5] proved that, for continuous maps, the conditions obtained in [22, Section 4] are optimal. The ω-limit sets arising out of the constructions in [5] possess many desirable properties including those of topological transitivity and asymptotic stability. When there are nontrivial instantaneous symmetries, the restrictions in [22] are not optimal. However, the methods of [22, 5] are easily generalized to give necessary and sufficient conditions (see Melbourne [21]).

From the point of view of applications, it is important to obtain results on symmetry on average that apply to smooth *invertible* dynamical systems such as diffeomorphisms and flows. In addition, it is natural to ask under what conditions there exist Axiom A attractors with specified symmetry on average. We remark that in Field [11], constructions are given for structurally stable equivariant Smale diffeomorphisms. However, this work is not concerned with *attractors* supporting complex dynamics (but note [12]). Indeed, the hyperbolic sets constructed in [11] support dynamics conjugate to subshifts of finite type and so cannot be attractors.

It is our aim to extend the results of [22, 5] to smooth invertible dynamical systems and, where possible, construct Axiom A attractors. We shall concentrate primarily (as in [5]) on the case when the instantaneous symmetry is trivial, and treat the case when there is nontrivial instantaneous symmetry as a secondary issue. Throughout the remainder of the introduction, we shall assume that there are no nontrivial instantaneous symmetries. We work throughout with finite groups. The problems for continuous groups are somewhat different, see for example [21, 14, 4].

It is convenient to introduce some new terminology. If A is a nonempty subset of $\mathbb{R}^n$, we define the *symmetry group of A* by

$$\Sigma_A = \{\gamma \in \Gamma \mid \gamma A = A\}.$$

Note that Σ_A is a subgroup of Γ. Given the subgroup $\Sigma \subset \Gamma$, we say that A is Σ-*symmetric* if $\Sigma_A = \Sigma$ and there is at least one point in A with trivial isotropy (the last condition abstracts our assumption that there are no instantaneous symmetries).

Suppose that Σ is a subgroup of Γ. We say that Σ is *admissible for continuous maps* if there exists a continuous Γ-equivariant map $f : \mathbb{R}^n \to \mathbb{R}^n$ with a Σ-symmetric ω-limit set. The results of [22, 5] give necessary and sufficient conditions for admissibility of subgroups in terms of the representation of Γ (under the additional requirement that the ω-limit set be Liapunov stable). It is a consequence of the constructions in this paper that at least some of the admissibility results in [5] extend to smooth (C^∞) maps (see Remarks 4.2(2)).

On the other hand, it was shown in [21] that the restrictions given in [22] are not optimal for invertible dynamical systems. Our aim is to obtain necessary and sufficient conditions for admissibility for invertible equivariant dynamical systems. We start by describing our results for equivariant flows. We shall say that a subgroup $\Sigma \subset \Gamma$ is *admissible for flows* if there is a smooth Γ-equivariant flow on $\mathbb{R}^n$ with a Σ-symmetric ω-limit set A. Recall that an $(n-1)$-dimensional linear subspace V of $\mathbb{R}^n$ is called a *reflection hyperplane* of Γ if Γ contains a reflection with fixed point space V. We let L denote the union of all the reflection hyperplanes of Γ. The connected components of $\mathbb{R}^n \setminus L$ are permuted by elements of Γ. Obviously, an equivariant flow on $\mathbb{R}^n$ fixes the components of $\mathbb{R}^n \setminus L$. Hence, a Σ-symmetric ω-limit set A must lie in (the closure of) a single connected component C of $\mathbb{R}^n \setminus L$. In particular, C is fixed by the subgroup Σ. It follows that a necessary condition for Σ to be admissible for flows is that Σ fixes a connected component of $\mathbb{R}^n \setminus L$.

We prove that in high enough dimensions this condition is also sufficient and that admissibility can be realized by Axiom A attractors.

Definition 1.1. A subgroup $\Sigma \subset \Gamma$ is of *class I* if Σ fixes a connected component of $\mathbb{R}^n \setminus L$.

Theorem 1.2. *Suppose that Γ is a finite subgroup of $\mathbf{O}(n)$. If $n \geq 3$, a subgroup Σ is admissible for flows if and only if Σ is of class I. The ω-limit sets that realize admissibility can be taken to be asymptotically stable. Moreover, if $n \geq 5$ the ω-limit sets that realize admissibility can be taken to be Axiom A attractors.*

The problem of obtaining necessary and sufficient conditions for admissibility of diffeomorphisms is more subtle than the problem for flows. The difficulty occurs because ω-limits will generally not be contained in the closure of a single component of $\mathbb{R}^n \setminus L$. The new class of subgroups we consider is described in the following definition.

Definition 1.3. A subgroup $\Sigma \subset \Gamma$ is of *class II* if

(i) Σ is not of class I.

(ii) Σ contains an index two subgroup Δ of class I (fixing a connected component $C \subset \mathbb{R}^n \setminus L$ say).
(iii) There is a Γ-equivariant linear involution $B : \mathbb{R}^n \to \mathbb{R}^n$ such that $BC = \sigma C$ for all $\sigma \in \Sigma \setminus \Delta$.

Theorem 1.4. *Suppose that Γ is a finite subgroup of $\mathbf{O}(n)$, $n \geq 1$. A subgroup Σ is admissible for diffeomorphisms if and only if Σ is of class I or of class II. The ω-limit sets that realize admissibility can be taken to be asymptotically stable. Moreover, if $n \geq 4$ the ω-limit sets that realize admissibility can be taken to be Axiom A attractors.*

The heart of this paper concerns the proofs of the final statements of Theorems 1.2, 1.4. The idea is to extend the methods of Williams [23] to the equivariant context. We divide the paper into two parts. Roughly speaking, Part I deals with subgroups of class I and Part II deals with subgroups of class II. Thus in Part I we prove Theorem 1.2 and parts of Theorem 1.4. The proof of Theorem 1.4 is completed in Part II.

We conclude this introduction by noting that in many applications the representation of the group of symmetries Γ will contain no reflections. For example, in the infinite dimensional setting of a partial differential equation (PDE), it is highly unlikely that there will be any codimension one fixed-point spaces for the action. Hence for global dynamics in PDEs there are generally no restrictions on admissibility of subgroups and so any subgroup can be realized by an Axiom A attractor. Nevertheless, even within the context of PDEs, there is the likelihood that our results are relevant to the understanding of dynamics on center manifolds corresponding to low codimension bifurcations.

Part 1.

Let $\Gamma \subset \mathbf{O}(n)$ be a finite group. Our main aim in Part I is to prove Theorem 1.2. Thus we shall show that if $\Sigma \subset \Gamma$ is a subgroup of class I and $n \geq 5$, then there is a smooth Γ-equivariant flow on $\mathbb{R}^n$ with a Σ-symmetric Axiom A attractor. If $n < 5$, we prove Theorem 1.2 using a degenerate construction. We also obtain a classification of admissible subgroups for flows for all $n \geq 1$.

The Axiom A attractors we construct for flows are obtained by suspending Σ-symmetric solenoids for Γ-equivariant diffeomorphisms. Thus we start by proving the following special case of Theorem 1.4.

Theorem 1.5. *Suppose that Γ is a finite subgroup of $\mathbf{O}(n)$. If $n \geq 3$ and $\Sigma \subset \Gamma$ is of class I, then there exists a Γ-equivariant diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with a connected asymptotically stable Σ-symmetric ω-limit set. Moreover, if $n \geq 4$ the ω-limit set can be taken to be an Axiom A attractor.*

Part I is divided into five sections. Some basic results on equivariant extensions of smooth mappings are reviewed in Section 2. In Section 3, we define a smooth version of the Σ-graphs introduced in Ashwin and Melbourne [5]. Smooth Σ-graphs are the equivariant analogue of the branched 1-manifolds of [23]. By passing to the inverse limit we obtain Σ-solenoids.

In Section 4, we show that provided $\Sigma \subset \Gamma$ is a subgroup of class I with $n \geq 4$ it is possible to construct an Axiom A Γ-equivariant diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ having a Σ-solenoid as an attractor. When $n \geq 5$, we use a construction based on suspension to prove the corresponding result for flows. We briefly indicate how our main theorems extend to smooth Γ-manifolds.

The remaining low-dimensional cases $n \leq 4$ ($n \leq 3$ for diffeomorphisms) are considered briefly in Section 5. Thus Theorem 1.2 holds trivially when $n = 1$, fails when $n = 2$, and can be proved when $n = 3$ and $n = 4$ (though by a rather unsatisfactory degenerate construction when Σ is not cyclic). In addition we complete the proof of Theorem 1.5.

Finally, in Section 6, we generalize our results for flows to the case where there are nontrivial instantaneous symmetries.

2. Background on equivariant extensions

We start by listing some well-known results about smooth Γ-equivariant extensions that we shall make frequent use of later. We continue to assume that $\Gamma \subset \mathbf{O}(n)$. If U is a set we write $\Gamma(U) = \bigcup_{\gamma \in \Gamma} \gamma U$.

If $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}^m$, we say that f is *smooth* if we can find an open neighborhood V of $\overline{U}$ and smooth map $\tilde{f} : V \to \mathbb{R}^m$ such that $\tilde{f}|U = f$. Of course, we may take $V = \mathbb{R}^n$.

Lemma 2.1. *Let A be a closed subset of $\mathbb{R}^n$. Suppose that there is a (closed) subgroup Σ of Γ such that for all $\gamma \in \Gamma$,*

(a) $A \cap \gamma A = A$, $(\gamma \in \Sigma)$.
(b) $A \cap \gamma A = \emptyset$, $(\gamma \notin \Sigma)$.

Then every smooth Σ-equivariant map $f : A \to \mathbb{R}^n$ extends uniquely to a smooth Γ-equivariant map $f : \Gamma(A) \to \mathbb{R}^n$.

Theorem 2.2. *Suppose that $f : U \to \mathbb{R}^n$ is a smooth Γ-equivariant map defined on an open Γ-invariant subset $U \subset \mathbb{R}^n$. Let $A \subset U$ be compact. Then there is a smooth Γ-equivariant map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\tilde{f}$ and f agree on a neighborhood of A. A similar result holds for smooth equivariant vector fields defined in U.*

Theorem 2.3. *Let M be a smooth Γ-manifold and let U be an open Γ-invariant subset of M. Suppose that $\phi : U \to M$ is a smooth Γ-equivariant embedding. Let $A \subset U$ be compact. Provided that ϕ is Γ-equivariantly isotopic to the inclusion of U in M, there is a smooth Γ-equivariant diffeomorphism $\tilde{\phi} : M \to M$ such that $\tilde{\phi}$ and ϕ agree on a neighborhood of A.*

Proof: See [17, Chapter 8, Theorem 1.4] and [7, Chapter VI, Theorem 3.1]. □

3. Smooth Σ-graphs and Σ-solenoids

In this section, we introduce the smooth Σ-graphs and their inverse limits which will form the basis for our construction of Σ-symmetric Axiom A attractors. Smooth Σ-graphs combine the features of Williams' branched 1-manifolds [23] and the (nonsmooth) Σ-graphs introduced in Ashwin and Melbourne [5].

In Subsection 3.1, we define a smooth Σ-graph to be a Σ-graph with a branched 1-manifold structure and obtain some elementary properties of smooth Σ-graphs. Then, in Subsection 3.2, we define Σ-solenoids. In particular, we show that there exists a Σ-solenoid for every finite group Σ.

3.1. **Smooth Σ-graphs.**

3.1.1. *Some graph theory.* We summarize the graph theory that we require (see [6] for further details). A *finite graph* G consists of a finite set of vertices and a finite set of edges that join pairs of vertices. In the sequel, all graphs will be assumed finite. A subset $J \subset G$ is a *subgraph* of G if J is a graph and the vertices and edges of J are vertices and edges of G. A *path* in G is a sequence of oriented edges where the initial vertex of each edge is the final vertex of the previous edge. A graph is *connected* if there is a path between any two vertices. If each pair of vertices in G is joined by an edge, the graph is *completely connected*. We say that a graph is a *completely connected oriented graph* if each pair of distinct vertices is joined by two edges with opposite orientations. The completely connected oriented graph on one vertex is defined to be the graph consisting of one vertex and one edge.

Each edge of a graph G can be made into a metric space isometric to the unit interval. Then the length of a path in G can be defined in the obvious way, and the distance between any two points in the same connected component is defined to be the length of the shortest path between the points. Let D be the maximum of the diameters of each component of G. If we define the distance between points in distinct connected components to be $D + 1$, then G has the structure of a compact metric space. Clearly, G is connected as a metric space if and only if it is connected as a graph. Viewing the graph G as a metric space, we shall regard each edge E as a (closed) subset of G (so $E \subset G$). We adopt the convention that when we remove an edge E from G, then the resulting set is, by definition, equal to the closure of $G \setminus E$. It follows that in order to disconnect a completely connected oriented graph it is necessary to remove at least eight edges. In particular, a completely connected oriented graph with three or fewer vertices cannot be disconnected by removing edges.

The *degree* of a vertex is the number of edges emanating from from the vertex. A connected graph is said to be *Eulerian* if it has at least one edge and each vertex has even degree. A completely connected oriented graph is an example of an Eulerian graph.

The sum of the degrees of the vertices of a graph is even (twice the number of edges). It follows that an Eulerian graph cannot be disconnected by removing a single edge. It is well-known that Eulerian graphs are characterized by the property that there exists an *Eulerian circuit*. That is, there is a continuous path tracing through each edge precisely once such that the initial vertex and the final vertex are equal. More generally, there exists an *Eulerian path* tracing through each edge of a graph precisely once if and only if the graph is connected and there are either two vertices or no vertices of odd degree. Any vertices of odd degree lie at the endpoints of the path.

3.1.2. *Branched* 1*-manifolds and smooth graphs.* We adapt the notion of *branched* 1*-manifold* defined in Williams [23]. For our purposes, it is sufficient to consider compact Hausdorff branched 1-manifolds.

We start by defining coordinate neighborhoods. Fix a C^∞ function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{aligned} \phi(x) &= 0,\ x \leq 0 \\ &> 0,\ x > 0 \end{aligned}$$

For integers $p \geq 1$, $q \geq 0$, $p \geq q$, we define the local branched 1-manifold $Y_{p,q} \subset \mathbb{R}^2$ by

$$Y_{p,q} = \{(x,y) \in \mathbb{R}^2 \mid y = i\phi(x) \text{ or } y = j\phi(-x),\ i = 0,\dots,p-1,\ j = 0,\dots,q-1\}.$$

A branched 1-manifold G will then consist of a Hausdorff topological space together with a differential atlas of coordinate neighborhoods each of which is diffeomorphic to some $Y_{p,q}$. In the usual way, we may define what it means for a continuous map between branched manifolds to be smooth. In particular, if G, G' are branched 1-manifolds which are smoothly embedded in $\mathbb{R}^n$, then $f : G \to G'$ will be smooth if and only if f extends smoothly to $\mathbb{R}^n$.

We say that a point $z \in G$ is a *point of type* (p, q) if there is a chart $\psi : U \to Y_{p,q}$ with $\psi(z) = 0$. We define the *defect* of z to be the positive integer $p - q$. Points of type (p, q) with $p \geq 2$ are called *branch points.* Let B denote the set of branch points. The *boundary* of G, ∂G, consists of all points of type $(p, 0)$. If every point is of type $(1, 1)$ and $(1, 0)$, that is $B = \emptyset$, then G is a 1-manifold (with boundary). Note that only points of type $(1, 1)$, $(1, 0)$ and $(2, 1)$ are allowed in [23]. Typically, we will be working with smooth graphs all of whose vertices have defect zero or one.

Suppose that G is a finite graph with vertex set $\mathcal{V}$. Then it is easy to give G the structure of a branched 1-manifold with branch set $B \subset \mathcal{V}$. Note, however, that we can generally give G many different structures of a branched manifold (the structure depends on how we arrange the edges at branch points).

Conversely, every branched 1-manifold determines (uniquely) a graph with vertex set equal to the set of points of type not equal to $(1, 1)$. In particular, the vertex set will contain the branch set B.

We define a *smooth graph* to be a finite graph with a designated branched 1-manifold structure.

Remark 3.1. If G is a smooth graph then every branch point of G is a vertex. Note, however, that vertices of G which are of types $(1,1)$ or $(1,0)$, relative to the designated branched 1-manifold structure, are not branch points.

3.1.3. *Smooth Σ-graphs.* The next definition is adapted from [5].

Definition 3.2. Let Σ be a finite group. A graph G is a Σ-*graph* if

(a) Σ acts isometrically on G, mapping vertices to vertices and edges to edges.
(b) Σ acts freely on the set of edges in G. In particular, if E is an edge, and $\sigma E = E$ for some $\sigma \in \Sigma$, then $\sigma = 1$.

Remark 3.3. Assumption (b) in the Definition 3.2 is equivalent to assuming the existence of a *fundamental* subgraph $J \subset G$ satisfying

(a) $G = \Sigma(J)$, and
(b) If $E, \sigma E \in J$ for some $\sigma \in \Sigma$, then $\sigma = 1$.

Example 3.4. An important example of a Σ-graph is provided by the *complete* Σ-*graph* $G(\Sigma)$ introduced in [5]. We recall that the graph $G(\Sigma)$ is the completely connected oriented graph with vertices $\sigma \in \Sigma$. The action of Σ on the vertices of $G(\Sigma)$ by left multiplication induces (and uniquely determines) an orientation-preserving isometric action of Σ on $G(\Sigma)$. Thus, if $E_{\tau,\tau'}$ denotes the oriented edge joining the vertex τ to the vertex τ', $\tau \neq \tau'$, and $\sigma \in \Sigma$, then $\sigma E_{\tau,\tau'} = E_{\sigma\tau,\sigma\tau'}$. If we define $J_\sigma = E_{1,\sigma}$, $\sigma \neq 1$, then a fundamental subgraph of $G(\Sigma)$ is given by $J = \bigcup_{\sigma\in\Sigma} J_\sigma$. Of course, Σ acts freely on the set of vertices of $G(\Sigma)$ and so freely on $G(\Sigma)$.

Definition 3.5. Let G be a smooth graph. We say that G is a *smooth* Σ-*graph* if

(1) G is a Σ-graph.
(2) Σ acts smoothly on G.

Remark 3.6. Suppose that G is a smooth Σ-graph and that v is a vertex of type (p,q) (relative to the designated branched 1-manifold structure). Then for every $\sigma \in \Sigma$, σv is a vertex of type (p,q).

Definition 3.7. A smooth graph is *balanced* if the graph is connected and every vertex has defect zero.

The following lemma will suffice for our needs.

Lemma 3.8. *Suppose that Σ acts transitively on the set of vertices of the Σ-graph G. Then G may be given the structure of a smooth balanced Σ-graph. In particular, the complete Σ-graph $G(\Sigma)$ has the structure of a smooth balanced Σ-graph.*

Proof. It suffices to define coordinate neighborhoods at each vertex of G. Denote the set of vertices of G by $\mathcal{V} = \{v_1, \ldots, v_p\}$. Since Σ acts transitively on $\mathcal{V}$, p divides $|\Sigma|$ and we can choose a fundamental subgraph J all of whose edges have the same initial vertex, say v_1. Suppose that J contains n edges, $E_1, \ldots, E_n$. It follows that each vertex of G has degree $2n$. We write the edge $E_i \in J$ in the form $E_i = E_{1,w_i}$, where $w_i \in \mathcal{V}$, $1 \leq i \leq n$. (We allow repetitions.) Every edge in G can be written as σE_i, for some $\sigma \in \Sigma$ and $i \in \{1, \ldots, n\}$, We demand that the edge E corresponds to the branch $(i-1)\phi$ in the coordinate neighborhood at σv_i and to the branch $(i-1)\psi$ in the coordinate neighborhood at σw_i. It is easy to verify that this assignment gives G the structure of a smooth Σ-graph and that each vertex is of type (n, n). □

Remark 3.9. In the sequel, we always regard $G(\Sigma)$ as having the smooth graph structure given by (the proof of) Lemma 3.8. Note that each vertex of $G(\Sigma)$ is a branch point of type $(m-1, m-1)$, where $|\Sigma| = m$.

3.1.4. *Twisted products.* Suppose that Σ is a subgroup of the finite group Γ and G is a Σ-graph. In the sequel, it is useful to associate a Γ-graph to G. In order to do this, we start by noting that $G \times \Gamma$ has the natural structure of a Γ-graph if we take the product of the trivial Γ-action on G with the action by composition on Γ. We also have a free action of Σ on $G \times \Gamma$ defined by

$$\sigma(\gamma, x) = (\gamma\sigma^{-1}, \sigma x), \quad (\sigma \in \Sigma, \ (\gamma, x) \in \Gamma \times G)$$

We define the *twisted product* $\Gamma \times_\Sigma G$ to be the orbit space $(\Gamma \times G)/\Sigma$. It is straightforward to check that $\Gamma \times_\Sigma G$ inherits the natural structure of a Γ-graph from that on $\Gamma \times G$.

We omit the routine verification of the following properties of the twisted product.

Lemma 3.10. *Let Γ be a finite group and Σ be a subgroup of Γ. Suppose that G is a Σ-graph.*

(1) *The mapping $i_G : G \rightarrow \Gamma \times_\Sigma G, x \mapsto (1, x)$, Σ-equivariantly embeds G as a subgraph of $\Gamma \times_\Sigma G$.*
(2) *If G is connected, then $\Gamma \times_\Sigma G$ has $|\Gamma|/|\Sigma|$ connected components.*
(3) *If J is a fundamental subgraph for G, then J is a fundamental subgraph for the Γ-graph $\Gamma \times_\Sigma G$.*
(4) *If G is a smooth Σ-graph, then $\Gamma \times_\Sigma G$ has the natural structure of a smooth Γ-graph and, with respect to these structures, i_G will be a smooth embedding.*

3.1.5. *Smooth embeddings.* We now consider the problem of finding a smooth Σ-equivariant embedding of a smooth Σ-graph in a Σ-manifold. Our methods are a straightforward extension of the continuous embedding results given in [5].

With a view to our subsequent applications, we shall suppose Σ is a subgroup of a finite group Γ and require that our embeddings extend to $\Gamma \times_\Sigma G$.

Proposition 3.11. *Suppose that Γ is a finite group and Σ is a subgroup of Γ. Let G be a smooth Σ-graph and that Σ acts freely on G. Suppose that M is a smooth Γ-manifold such that*

(a) *$dim(M) \geq 3$.*
(b) *There is a connected open non-empty Σ-invariant subset $M_\circ$ of M consisting of points of trivial Γ-isotropy.*

Under these assumptions on M, there exists a smooth Σ-equivariant embedding $\chi : G \to M_\circ$. Moreover, we may require that χ extends uniquely to a smooth Γ-equivariant embedding of the twisted product graph $\Gamma \times_\Sigma G$ in M.

Proof. Let J be a fundamental subgraph of G. Choose a set of vertices $\mathcal{V} \subset J$ such that every vertex of G lies on the Σ-orbit of exactly one $v \in \mathcal{V}$. For each $v \in \mathcal{V}$, choose a point $\chi(v) \in M_\circ$. Clearly we may do this so that the Γ-orbits of the points $\chi(v)$ are disjoint. Associated to each of the points $v \in \mathcal{V}$, we have a local coordinate system defined on a neighborhood N_v of v in G. In the obvious way, we may define a corresponding coordinate system at $\chi(v)$ and extend χ to an embedding of N_v into $M_\circ$. Further, we may require that the Γ-orbits of the sets $\chi(N_v)$, $v \in \mathcal{V}$ are mutually disjoint. It follows that χ extends to a smooth Σ-equivariant embedding of $\Sigma(\cup_{v\in\mathcal{V}} N_v)$ in $M_\circ$. Next we extend χ Σ-equivariantly to the edges. We do this by first defining χ on the edges in J and then extending Σ-equivariantly. Since we are assuming that $\dim(M) \geq 3$, it follows that we may always perturb χ so that χ is a Σ-equivariant embedding. Finally, again using the assumption that $\dim(M) \geq 3$, we may require that χ extends to the twisted product as a Γ-equivariant embedding. □

Remark 3.12. In general, Proposition 3.11 fails when $\dim(M) = 2$. For example, if $M = \mathbb{R}^2$, $\Gamma = \mathbb{Z}_4$, $\Sigma = \mathbb{Z}_2$ and G is a smooth Σ-graph, there is no smooth Γ-equivariant embedding of $\Gamma \times_\Sigma G$ in $\mathbb{R}^2$. (If $\Gamma = \Sigma = \mathbb{Z}_m$, then it is easy to find a smooth Σ-graph for which the conclusions of the proposition hold.)

3.1.6. *Smooth Eulerian paths.* Suppose that G is a connected smooth graph with at least one edge. If every vertex of G has even degree, there exists a continuous Eulerian circuit. It is natural to ask whether we can choose this path to have a smooth parametrization. More precisely, we define a *smooth Eulerian circuit* to be an Eulerian circuit with smooth parametrization $r : S^1 \to G$ with *non-vanishing* derivative. Similarly, if G contains precisely two vertices v_0, v_1 of odd degree, we may ask when there exists a smooth Eulerian path $r : [0,1] \to G$ joining v_0 to v_1.

Proposition 3.13. *Let G be a connected smooth graph containing at least one edge.*

(1) *There exists a smooth Eulerian circuit for G if and only if G is balanced, that is, every vertex has defect zero.*
(2) *If there are precisely two vertices v_0, v_1 in G of odd degree, then there is a smooth Eulerian path joining v_0 to v_1 if and only if the vertices v_0, v_1 have defect one and the remaining vertices have defect zero.*

Proof. It is easy to see that the conditions in (1), (2) are necessary: any smooth Eulerian path or circuit approaching a vertex 'from one side' must pass 'out the other side' in order to be smooth. Hence there must be equally many branches on each side of any vertex. The exception occurs in (2) where the vertices v_0 and v_1 each have one additional branch – corresponding to the initial and final edges in the Eulerian path.

It remains to prove that the conditions of (1) and (2) are sufficient. Our proof goes by induction on the number of edges of G. Suppose the graph G has vertex set $\mathcal{V}$. We denote the degree of the vertex v by $\deg(v)$. The result is obvious if G has one edge. So suppose we have proved sufficiency of conditions (1), (2) for all graphs G with less than n edges. It is obvious that the truth of (2) for graphs with $(n-1)$ edges implies the truth of (1) for graphs with n edges. Hence, it suffices to prove that the conditions of (2) are sufficient if G has n edges.

We may assume that there is at least one $v \in \mathcal{V}$ such that $\deg(v) > 2$. (If not, G is a smooth arc with initial and terminal points of type $(1,0)$.) It follows that there is a smooth arc through v consisting of two edges E_1, E_2, with common center point v. Replace the edges E_1, E_2 by a single edge E' which no longer passes through v. (Essentially, we just 'perturb' $E_1 \cup E_2$ off the vertex v and remove the vertex v from $E_1 \cup E_2$.) Denote the new graph by G' and note that G' has $(n-1)$ edges. Suppose first that G' is connected. We apply the inductive hypothesis to G' to obtain a smooth Eulerian path for G' which in turn determines a smooth Eulerian path for G (re-insert the vertex v). If G' is not connected, we may write $G' = G_1 \cup G_2$, where G_1 and G_2 are connected smooth graphs with fewer than n edges. One of the graphs, say G_1, must have all vertices with zero defect. The graph G_2 will then contain the vertices v_0, v_1 (which will have defect one) and the remaining vertices will have defect zero. We now apply our inductive hypothesis to construct a smooth Eulerian circuit $r : S^1 \to G_1$ and an Eulerian path in G_2 joining v_0 to v_1. Re-inserting the vertex v, we may combine the paths in the obvious way to obtain the required Eulerian path in G. □

3.2. **Σ-solenoids.** Suppose that G is a smooth graph without boundary and that $f : G \to G$ is a smooth map. We say that f satisfies condition (W) if:

- (W1) f is an expanding immersion and maps vertices to vertices.
- (W2) We may find a positive integer p, such that $f^p(E) = G$, for every edge $E \in G$.
- (W3) Every point of G has a neighborhood N such that $f(N)$ is an arc.

Remarks 3.14. (1) Conditions (W1 — W3) are modelled on Axioms 1, 2 and 3 of [23, §3] (see also Remark 3.16). Note, however, that our definition of smooth graph is, *a priori*, more general than Williams' definition of branched 1-manifold in that we allow branch points of type (p, q), $p > 2$. We may also designate points of type $(1, 1)$ as vertices.
(2) Since f maps vertices to vertices it follows that the f-image of an edge will always be a finite union of edges. If every vertex of G has degree at least three, then an expanding immersion $f : G \to G$ will automatically map vertices to vertices.

The next result follows from the Folklore Theorem of Adler and Flatto [1] – see [5, Appendix].

Proposition 3.15. *Let G be a smooth graph without boundary and $f : G \to G$ be a smooth map satisfying (W1), (W2).*

- (a) *All points of G are non-wandering under f. (That is, f satisfies Axiom 2 of* [23]*).*
- (b) *Periodic points of f are dense in G and f has sensitive dependence on initial conditions.*
- (c) *f is topologically mixing.*
- (d) *There is a unique Lebesgue-equivalent f-invariant ergodic probability measure on G.*

Remark 3.16. It can be shown that if f satisfies conditions (W1) and (W3) and all points of G are non-wandering under f then f satisfies

FIGURE 1. Maps of smooth $\mathbb{Z}_m$-graphs

condition (W2). Combined with Proposition 3.15, it follows that conditions (W1), (W2) and (W3) are essentially equivalent to Axioms 1, 2 and 3 of [23, §3].

Example 3.17. Let $\mathbb{Z}_2$ act on $\mathbb{R}^2$ as multiplication by plus or minus the identity map. In Figure 1(a), we show a $\mathbb{Z}_2$-graph G which is $\mathbb{Z}_2$-equivariantly embedded in $\mathbb{R}^2$. The graph has two vertices and four oriented edges, which we have denoted by a_1, b_1, a_2 and b_2. We may define a smooth $\mathbb{Z}_2$-equivariant map $f : G \rightarrow G$ satisfying condition (W) by the rules:

$$\begin{array}{rcl} a_1 & \mapsto & a_1 b_2 a_2^{-1} \\ b_1 & \mapsto & b_1 a_2 b_2^{-1} \\ a_2 & \mapsto & a_2 b_1 a_1^{-1} \\ b_2 & \mapsto & b_2 a_1 b_1^{-1} \end{array}$$

Note that the map f satisfies (W3) – we have highlighted the image of arcs through both vertices.

In Figure 1(b), we show the corresponding picture for a $\mathbb{Z}_m$-graph. The defining rules for f are given by $a_i \mapsto a_i b_{i+1} a_{i+1}^{-1}$, $b_i \mapsto b_i a_{i+1} b_{i+1}^{-1}$, $1 \leq i \leq m \pmod m$.

Suppose that $f : G \to G$ satisfies condition (W). Define the *solenoid* $\mathcal{S}$ to be the inverse limit of the sequence

$$G \xleftarrow{f} G \xleftarrow{f} G \xleftarrow{f} \cdots$$

A typical point in $\mathcal{S}$ is a sequence $a = (a_0, a_1, a_2, \ldots)$, where $a_n \in G$ and $f(a_n) = a_{n-1}$. Define the *shift map* $h : \mathcal{S} \to \mathcal{S}$ by the formula

$$h(a) = (f(a_0), a_0, a_1, \ldots).$$

Then h is a homeomorphism with inverse

$$h^{-1}(a) = (a_1, a_2, a_3, \ldots).$$

Let Σ be a finite group, G be a smooth connected Σ-graph and $f : G \to G$ be Σ-equivariant and satisfy condition (W). For $\sigma \in \Sigma$, define $\sigma \cdot a = (\sigma a_0, \sigma a_1, \sigma a_2, \ldots)$. This defines an action of Σ on the solenoid $\mathcal{S}$. Clearly, the shift map h is equivariant with respect to this action. We call $\mathcal{S}$ a Σ-*solenoid*.

Lemma 3.18. *Suppose that Σ is a finite group with complete smooth Σ-graph $G(\Sigma)$. Then there exists a smooth Σ-equivariant map $f : G(\Sigma) \to G(\Sigma)$ satisfying condition (W).*

Proof: We give the details when $|\Sigma| \geq 3$. Our proof is similar to that of [5, Theorem 4.3], except that now we have to ensure that condition(W3) is satisfied.

Choose a fundamental subgraph J consisting of edges with common initial vertex 1 so that the edges do not all lie on the same side of 1. We designate two edges in $G(\Sigma) \setminus J$ that together form a smooth arc a_1 through 1. For definiteness, suppose that $a_1 = E_{1,\eta} \cup E_{\tau,1}$ for some $\eta, \tau \in \Sigma$. Since $|\Sigma| \geq 3$ we can arrange things so that $\tau \neq \eta^{-1}$. By equivariance, we have a 'designated arc' $a_\sigma = E_{\sigma,\sigma\eta} \cup E_{\sigma\tau,\sigma}$ through each vertex σ. Since $\tau \neq \eta^{-1}$, no pair of arcs has an edge in common.

Suppose that $E = E_{1,\sigma} \subset J$. There is a unique choice of edges $I \subset a_1$, $F \subset a_\sigma$ and orientations on I, F such that IEF is a smooth arc. We claim that, for each $E \subset J$ we may construct a smooth Eulerian path $\rho_E : E \to G(\Sigma) - E$ such that

(a) ρ fixes the vertices of E.
(b) The path begins in $I^{-1} \subset a_1$ and ends in $F^{-1} \subset a_\sigma$.
(c) ρ is an expanding immersion and the derivative of ρ near the end-points $1, \sigma$ of E is independent of σ.

Granted the claim, equivariant extension of the set of paths defined on J defines a smooth map $f : G(\Sigma) \to G(\Sigma)$ satisfying condition (W).

It remains to verify the claim. Suppose $E \subset J$ and I, F are as defined above. Set $D = IEF$. If D is not a loop, we can apply Proposition 3.13 to obtain a smooth Eulerian path P in $G(\Sigma) - D$. (Recall that it is necessary to remove at least eight edges to disconnect a completely connected oriented graph.) The path $I^{-1}PF^{-1}$ clearly satisfies (a,b). If D is a loop, choose an edge $H \not\subset D$ so that HI is a smooth arc.

Repeat the previous construction to obtain a path $I^{-1}H^{-1}PF^{-1}$ which again satisfies (a,b). Finally, reparametrizing P, we may assume that condition (c) holds. □

4. Symmetric Axiom A attractors

In this section we prove the statements in Theorem 1.2 and Theorem 1.5 concerning the realization of admissibility by Axiom A attractors. We suppose as usual that $\Gamma \subset \mathbf{O}(n)$ is a finite group acting on $\mathbb{R}^n$. Recall that $\Sigma \subset \Gamma$ is a subgroup of class I if Σ fixes a connected component of $\mathbb{R}^n \setminus L$. In Subsection 4.1, we construct Axiom A attractors for diffeomorphisms when $n \geq 4$. Then, in Subsection 4.2, we construct Axiom A attractors for flows, $n \geq 5$, using a method based on the suspension construction. The model for our attractors is the solenoid that arises as the inverse limit of a smooth Σ-graph. Our methods are an extension of those in Williams [23].

4.1. **Axiom A attractors for diffeomorphisms.**

Theorem 4.1. *Let Γ be a finite group acting on $\mathbb{R}^n$, $n \geq 4$. Suppose that Σ is a subgroup of Γ and that there is a simply connected Σ-invariant open subset $V \subset \mathbb{R}^n$ consisting of points of trivial isotropy. Let $h : \mathcal{S} \to \mathcal{S}$ be the shift map on a Σ-solenoid (corresponding to a smooth Σ-equivariant map $f : G \to G$ satisfying condition (W)). Then there exists a smooth Γ-equivariant diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with a Σ-symmetric attractor A such that*

(a) *A is a 1-dimensional Axiom A attractor for ϕ.*
(b) *$\phi : A \to A$ is Σ-equivariantly topologically conjugate to $h : \mathcal{S} \to \mathcal{S}$.*

Proof: Our proof is very similar to that of [23, Theorem C] and so we shall only sketch the main details. Basically, we have to take account of equivariance of maps and our more general definition of branch point.

Using Proposition 3.11, we may smoothly and Σ-equivariantly embed the Σ-graph G as a subset G of V. Since $n \geq 3$, we may assume the embedding extends Γ-equivariantly to $G \times_\Sigma \Gamma$. In the obvious way, we may regard f as inducing a smooth map $f : G \to G$.

Using standard arguments, we construct a closed Σ-invariant tubular neighborhood $U \subset V$ of G such that $U \cap \gamma U = \emptyset$, all $\gamma \in \Gamma \setminus \Sigma$. Note that near the branch points the picture differs from the usual one. However, just as in [23], it is easy to construct smooth disk bundles, with fibers normal to G along each branch, which merge in the vicinity of branch points and separate away from branch points. If the branch point is of type (p, p), $p > 1$, we may arrange to have p-disks touching

FIGURE 2. Tubular neighborhood near a branch point of type $(2,2)$.

simultaneously to form an (extended) solid 'figure of eight' and then simultaneously separating on the other side of the branch point. The tubular neighborhood near a branch point of type $(2,2)$ is shown in Figure 2. Extend f smoothly and Σ-equivariantly to $\tilde{f} : U \to U$ so that $\tilde{f}$ is fiber preserving. For this we use the fact that we may choose a neighborhood N of each vertex in G which is mapped by f to a smooth arc. In fact, we can define $\tilde{f}$ simply by collapsing each fiber to a point (the fiber through x is mapped onto the point $f(x)$).

Since $n \geq 3$, it follows using general position arguments that we may approximate $\tilde{f}$ by a smooth embedding $\phi : U \to U$. Indeed, we can do this so that ϕ is fiber preserving and Σ-equivariant. For the Σ-equivariance, we just have to observe that since Σ acts freely on V, there are no obstructions caused by equivariance when we perturb (equivalently, work on the image of a fundamental subgraph in the orbit space). We may also assume that ϕ uniformly contracts fibers of U and that ϕ is orientation preserving. Define $A = \cap_{i \geq 0} \phi^i(U)$. Just as in [23], it follows that A has the required properties (a,b).

It remains to extend ϕ to a diffeomorphism of $\mathbb{R}^n$. Take the Γ-equivariant extension of ϕ to $\Gamma(U) \subset \mathbb{R}^n$. Since we are assuming $n \geq 4$ and V simply connected, it follows that ϕ is Γ-equivariantly isotopic to the inclusion map of $\Gamma(U)$ in $\mathbb{R}^n$ and so, by Theorem 2.3, we may extend ϕ to a Γ-equivariant diffeomorphism of $\mathbb{R}^n$ that agrees with ϕ in a neighborhood of A. $\square$

Remarks 4.2. (1) Although our definition of smooth graph is more general than Williams' definition of one-dimensional branched manifold, both definitions in fact lead to the same class of solenoids. This can be seen in two ways. First of all, the diffeomorphism ϕ and attractor A we construct in Theorem 4.1 satisfy the hypotheses of [23, Theorem D]. Consequently, $\phi : A \rightarrow A$ is conjugate to the shift map of a solenoid obtained from a one-dimensional branched manifold. This may also be seen directly by splitting branch points of type (p, p) into $2p-2$ branch points of the type allowed by Williams.
(2) Using the technique of the first part of the proof of Theorem 4.1, it is easy to extend some of the results of [22, 5] to smooth maps. In particular, the (strong) admissibility results of [5] extend to the smooth context when the subgroup Σ does not contain any reflections.

Theorem 4.3. *Suppose that $\Sigma \subset \Gamma$ is a subgroup of class I and $n \geq 4$. Then there exists a Γ-equivariant diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with a connected Σ-symmetric Axiom A attractor.*

Proof: If there exists a simply connected Σ-symmetric set $V \subset \mathbb{R}^n$ of points of trivial isotropy then the theorem follows immediately from Lemma 3.18 and Theorem 4.1. In particular, if there are no fixed-point subspaces of codimension two then we are finished.

In general, there may exist 'codimension two obstructions' to the equivariant isotopy step in the proof of Theorem 4.1. We can circumvent such obstructions by making a further modification to the construction of $f : G \to G$ in Lemma 3.18. In place of $G(\Sigma)$ we take $G = G(\Sigma)'$ an 'augmented' complete complete smooth Σ-graph with the same vertices as before but twice as many edges. Indeed, to each edge $E \subset G(\Sigma)$ we assign a new edge E' that has the same vertices as E and such that EE' forms a smooth loop. Note that we can regard $G(\Sigma)$ as a subgraph of $G(\Sigma)'$.

We make the conventions $E'' = E$ and if $P = E_1 E_2 \cdots E_k$ is a path then $P' = E'_k \cdots E'_2 E'_1$. Then we define an equivalence relation on paths in $G(\Sigma)'$ by setting $IEE'F \sim IF$.

Next choose a fundamental subgraph J' as in the proof of Lemma 3.18 but with each edge replaced by the corresponding pair of edges. Construct 'designated arcs' in $G(\Sigma)$ as before. Suppose that $E \subset J$ is an edge with initial vertex 1 and final vertex σ. (Here E need not lie in $G(\Sigma)$.) We claim there is a smooth Eulerian path P in $G(\Sigma)' - E'$ that has same vertices as E, that begins and ends in the designated arcs a_1 and a_σ and such that $P \sim E$. We can then construct a smooth Σ-equivariant map $f : G(\Sigma)' \to G(\Sigma)'$ satisfying condition (W). Moreover, we can embed $G(\Sigma)'$ in $\mathbb{R}^n$ in such a way that each loop EE' is contractible to a constant in the set of points of trivial isotropy. It then follows from our construction that the embedding $\phi : \Gamma(U) \to \Gamma(U)$ is Γ-equivariantly isotopic to the inclusion map of $\Gamma(U)$ in $\mathbb{R}^n$.

It remains to verify the claim. Suppose first that E does not lie in a_1 or in a_σ. Let I be the edge in a_1 with initial vertex 1 and on the same side of 1 as E. Similarly F is the edge in a_σ with final vertex σ and on the same side of σ as E. By construction $II'EF'F$ is an arc. Let ξ denote the set of edges I, I', F, F', E, E'. Then we may choose a smooth Eulerian path Q in $G(\Sigma) - \xi$ such that EQ is an arc. Then $P = II'EQQ'F'F$ is the required path.

Finally, if say E lies in a_1, then we can dispense with the edges I, I'. So ξ consists of the edges F, F', E, E' and $P = EQQ'F'F$. □

Remark 4.4. It is important to note that the equivariant embedding $\phi : U \to U$ constructed in the proof of Theorem 4.3 is isotopic to the identity *within* U (as opposed to $\mathbb{R}^n$). We make use of this remark in the next subsection.

If Σ is cyclic, we can reduce the dimension restriction in Theorem 4.3.

Lemma 4.5. *Suppose that* $\Gamma \subset O(3)$ *and* $\Sigma \subset \Gamma$ *is a subgroup of class I. Suppose further that* Σ *is cyclic. Then there exists a* Γ*-equivariant diffeomorphism* $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ *with a connected* Σ*-symmetric Axiom A attractor.*

Proof: In this case we use the graph G defined in Example 3.17 rather than the augmented graph defined in the proof of Theorem 4.3. We let $f : G \to G$ be the smooth map on G defined in Example 3.17. We embed G in a component of $\mathbb{R}^3 \backslash L$ just as above. The proof of Theorem 4.3 now extends straightforwardly, granted the observation that we can perturb $\tilde{f} : U \to U$ to an Σ-equivariant embedding so that no links are generated between images of edges. We omit the straightforward details. □

4.2. **Symmetric Axiom A attractors for flows.** Suppose that $\Sigma \subset \Gamma$ is a subgroup of class I and $n \geq 5$. Under these conditions, we shall show that there exists a smooth Γ-equivariant flow on $\mathbb{R}^n$ which has an Axiom A attractor with symmetry group Σ. Our construction is based on the standard technique of suspending a diffeomorphism. Let C be a connected component of $\mathbb{R}^n \setminus L$ fixed by Σ.

4.2.1. *Suspending a graph.* Let S^1 denote the unit circle. Let $\mathcal{Z}$ denote the product of the complete smooth Σ-graph $G(\Sigma)$ with S^1. Taking the trivial action of Σ on S^1, we see that Σ acts on $\mathcal{Z}$. We may extend the smooth structure on $G(\Sigma)$ to $\mathcal{Z}$ in the obvious way. In particular, if I is an open arc in S^1, then we view $G(\Sigma) \times I \subset \mathcal{Z}$ as a 'ribboned' graph

FIGURE 3. Neighborhood of $\{v\} \times S^1$ in $\mathcal{Z}$.

with ribbons touching along a common arc at each vertex of $G(\Sigma)$. If $v \in G(\Sigma)$ is a vertex, we refer to $\{v\} \times S^1$ as a *vertex loop*.

4.2.2. *Embedding a suspended graph.*

Lemma 4.6. *Provided $n \geq 5$, there exists a smooth Σ-equivariant embedding χ of $\mathcal{Z}$ in C. Moreover, we may require that χ extends Γ-equivariantly to a smooth Γ-equivariant embedding of $\Gamma(\mathcal{Z})$ in $\mathbb{R}^n$.*

Proof: Choose a vertex $v \in G(\Sigma)$ and a connected neighborhood V of v in $G(\Sigma)$ such that if $\sigma \in \Sigma$ then $\sigma(V) \cap V \neq \emptyset$ if and only if $\sigma = e$. We refer the reader to Figure 3.

Choose an embedding $g : V \times S^1 \to C$. (We can always construct g provided that $n \geq 3$.) The embedding extends uniquely to a Σ-equivariant embedding $g : \Sigma(V \times S^1) \to C$.

Let J be a fundamental subgraph for $G(\Sigma)$. If $E \subset J$ is an edge, we have the corresponding tube $E \times S^1 \subset \mathcal{Z}$. The map g is already defined on $E \cap \Sigma(V \times S^1)$. Since $n \geq 4$, we may extend g to a smooth embedding of $E \times S^1$ in C. (Note that the S^1-fibers in E inherit an orientation from the orientation of S^1. If $n = 3$, it will not always be possible to extend g as an *embedding* as we will not be able to match the orientations induced from the S^1-fibers.) Repeat the construction for all edges in the fundamental subgraph J. Again g extends uniquely to a smooth Σ-equivariant immersion $\chi : \mathcal{Z} \to C$. Since $n \geq 5$ and Σ act freely on C, we can always equivariantly perturb χ so that if E, F are distinct edges in $G(\Sigma)$ then the embedded surfaces $\chi(E \times S^1)$, $\chi(F \times S^1)$ can only intersect along vertex loops. In particular, we may require that χ is a Σ-equivariant embedding. Finally, a further perturbation allows us to require that if $\gamma \in \Gamma$ then $\gamma\chi(\mathcal{Z}) \cap \chi(\mathcal{Z}) \neq \emptyset$ if and only if $\gamma \in \Sigma$. It follows that χ extends to a Γ-equivariant embedding of $\Gamma(\mathcal{Z})$ in $\mathbb{R}^n$. $\square$

4.2.3. *A tubular neighborhood of the embedded suspension.* Let $\chi : \mathcal{Z} \to C$ be the embedding given by Lemma 4.6 and set $Z = \chi(\mathcal{Z})$. The natural projection from $\mathcal{Z} = G(\Sigma) \times S^1$ onto S^1 induces a smooth Σ-equivariant map $\pi : Z \to S^1$. Observe that each fiber $\pi^{-1}(\theta)$ is (smoothly) diffeomorphic to $G(\Sigma)$, $\theta \in S^1$. Let $p : N \to Z$ be the normal bundle of Z and let U be a corresponding open and Σ-invariant tubular neighborhood of Z. For each $\theta \in S^1$, let $U(\theta) \subset U$ be the union of p-fibers over $\pi^{-1}(\theta)$. We may think of each $U(\theta)$ as an $(n-1)$-dimensional Σ-manifold containing the embedded graph $\chi(G(\Sigma) \times \{\theta\})$.

4.2.4. *Refining the graph.* We continue with our previous assumptions. Suppose now that we embed $G(\Sigma)$ in the graph $G(\Sigma)'$ and consider the corresponding embedding of $\mathcal{Z}' = G(\Sigma)' \times S^1$ in $\mathcal{Z}$. It is straightforward to extend the embedding χ to a smooth Σ-equivariant embedding $\chi' : \mathcal{Z}' \to C$. Moreover, we may do this so that for each $\theta \in S^1$, $\chi'(G(\Sigma)' \times \{\theta\}) \subset U(\theta)$.

4.2.5. *Constructing the flow.* Since $n \geq 5$ and so $n - 1 \geq 4$, it follows from Remark 4.4 that we can construct a Σ-equivariant embedding $\phi : U(0) \to U(0)$ which is Σ-equivariantly smoothly isotopic to the identity map within $U(0)$ and is such that ϕ has an Axiom A Σ-symmetric solenoid $A(0) \subset U(0)$. We may spread the isotopy round $U = \cup_{0 \leq \theta \leq 2\pi} U(\theta)$ and so construct a smooth Σ-equivariant flow X on U with Axiom A attractor conjugate to the suspension of $\phi : A(0) \to A(0)$. Now extend X Γ-equivariantly to $\mathbb{R}^n$. □

4.3. **Extension to Γ-manifolds.** It follows easily from our methods and Proposition 3.11 that our main theorems extend to Γ-manifolds.

Suppose that M is a smooth connected Γ-manifold. We let $M_\circ$ denote the open subset of M consisting of points of trivial isotropy. Provided that $M_\circ \neq \emptyset$, it is well-known (see [7]) that $M_\circ$ is open and dense in M. If Γ contains no involutions with codimension one fixed point spaces, then $M_\circ$ will be connected. (The converse may or may not hold.)

Theorem 4.7. *Let Γ be a finite group acting on the smooth connected Γ-manifold M. Suppose that $M_\circ \neq \emptyset$ and dim$(M) \geq 4$. Suppose that Σ is a subgroup of Γ which fixes a connected component of $M_\circ$. Then there exists a Γ-equivariant diffeomorphism $\phi : M \to M$ with a connected Σ-symmetric Axiom A attractor. The analogous result holds for flows when dim$(M) \geq 5$.*

5. Symmetric attractors in low dimensions

In Section 4 we showed that the condition that a subgroup $\Sigma \subset \Gamma$ is of class I is sufficient as well as necessary for flows provided $n \geq 5$ (and sufficient for diffeomorphisms provided $n \geq 4$). A solution for the low-dimensional cases $n \leq 4$ ($n \leq 3$) is given in this section. Unless Σ is cyclic our construction is highly degenerate. Indeed in some cases, such as when $\Gamma = \Sigma = \mathbb{I}$ is the 60 element icosahedral group acting on $\mathbb{R}^3$, it seems unlikely that a reasonable construction is possible.

Proposition 5.1. *Suppose that $n \geq 3$ and that $\Sigma \subset \Gamma$ is cyclic and of class I. Then Σ is admissible for flows and admissibility can be realized by a periodic sink. If $\Sigma = \mathbf{1}$ we require only $n \geq 1$ and the sink can be taken to be an equilibrium.*

Proof. First suppose that $\Sigma = \mathbf{1}$. Choose a point $y \in \mathbb{R}^n$ with trivial isotropy and a neighborhood U of y. Let $X : U \to \mathbb{R}^n$ be a smooth vector field with a sink at y. Shrink U if necessary so that $\gamma U \cap U = \emptyset$ for nontrivial $\gamma \in \Gamma$. It follows from Lemma 2.1 and Theorem 2.2 that X extends Γ-equivariantly to a smooth vector field on $\mathbb{R}^n$.

Next suppose that $n \geq 3$ and Σ is a cyclic subgroup of class I fixing a connected component C say of $\mathbb{R}^n \setminus L$. Choose a smoothly embedded Σ-symmetric circle $A \subset C$ consisting of points with trivial isotropy. Since $n \geq 3$, we may perturb A so that $\gamma A \cap A = \emptyset$ for $\gamma \in \Gamma \setminus \Sigma$. Then we define a smooth Σ-equivariant flow on a neighborhood U of A for which A is a periodic sink. Now extend as before to a Γ-equivariant vector field on $\mathbb{R}^n$. □

Corollary 5.2. *If $\Gamma \subset \mathbf{O}(n)$ is a finite group generated by reflections, then a subgroup $\Sigma \subset \Gamma$ is admissible for flows if and only if $\Sigma = \mathbf{1}$.*

Proof. Since Γ is generated by reflections, Γ acts freely on the components of $\mathbb{R}^n \setminus L$. Hence, the only candidate for an admissible subgroup is $\Sigma = \mathbf{1}$. In this case admissibility follows from Proposition 5.1. □

Corollary 5.3. (a) *Suppose $n = 1$ and $\Gamma = \mathbf{1}$ or $\Gamma = \mathbb{Z}_2$. Then $\Sigma \subset \Gamma$ is admissible for flows if and only if $\Sigma = \mathbf{1}$.*
(b) *Suppose $n = 2$ and $\Gamma = \mathbb{D}_m$ or $\Gamma = \mathbb{Z}_m$, $m \geq 1$. Then $\Sigma \subset \mathbb{D}_m$ is admissible for flows if and only if $\Sigma = \mathbf{1}$. If $\Sigma \subset \mathbb{Z}_m$, then Σ is admissible for flows if and only if $\Sigma = \mathbb{Z}_m$ or $\Sigma = \mathbf{1}$.*

Proof. Except for $\Gamma = \mathbb{Z}_m$ in part (b), the groups Γ are generated by reflections and Corollary 5.2 applies. If $\Gamma = \mathbb{Z}_m$, there are no reflections and every subgroup of Σ is of class I. However, there is a topological obstruction to admissibility of subgroups $\Sigma = \mathbb{Z}_k$ for $1 < k < m$, see the argument in [5, Theorem 7.2(a)]. Finally, $\Sigma = \mathbf{1}$ can be realized by a sink and $\Sigma = \mathbb{Z}_m$ can be realized by a periodic sink. For example, the vector field

$$X(z) = (1 - |z|^2)z + \imath z,$$

on $\mathbb{C} = \mathbb{R}^2$ is $\mathbf{SO}(2)$-equivariant (hence $\mathbb{Z}_m$-equivariant) and has the unit circle as a periodic sink. □

It still remains to consider admissibility of class I subgroups $\Sigma \subset \Gamma$ when the following conditions are satisfied:

- $n = 3$ or $n = 4$, and
- Σ is not cyclic.

In the remainder of this section, we show that such subgroups are in fact admissible for flows and diffeomorphisms, thereby completing the proof of Theorems 1.2, 1.5. At the same time, our construction is highly degenerate and the resulting ω-limit set A is certainly not structurally stable.

We only sketch the construction of a vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ with the required properties. The ω-limit set A is taken to be an embedded complete Σ-graph. Choose a smooth path P in a neighborhood of A that 'spirals' into A (tracking a smooth Eulerian circuit in A for example). We can choose a neighborhood V of P satisfying $A \cap V = \emptyset$ and a vector field X on V such that

(i) P is a trajectory of X,
(ii) $X|_V$ has a nonnegative component in the direction of the path P,
(iii) X has support lying in V,
(iv) As V approaches A, the vector field X and all its derivatives approach zero.

The vector field X extends uniquely and Γ-equivariantly to $\Gamma(V)$. We extend X to a Γ-equivariant vector field on $\mathbb{R}^n$ by setting $X(x) = 0$ for $x \in \mathbb{R}^n \setminus \Gamma(V)$. It follows from property (iv) that X is smooth. By construction, the Σ-symmetric set A is invariant and consists of equilibria. Property (i) implies that A is an ω-limit set for X. It follows from properties (ii) and (iii) that A is Liapunov stable.

With a little care, we can modify the construction so as to make A asymptotically stable. Let $Z \subset \mathbb{R}^n$ be an embedded Eulerian ribbon, isomorphic to $G(\Sigma) \times (-1, 1)$. Then we can choose a neighborhood U of Z in $\mathbb{R}^n$ with fibers that are smooth discs away from branch points. At branch points, fibers are finite unions of smooth discs meeting at the branch point.

Given this local product structure, it is sufficient to construct a smooth vector field X on Z for which A is asymptotically stable. We may assume that the spiral path P and neighborhood V lie in Z and hence that there is a flow-invariant open spiral tube $W \subset Z$ such that every point in the spiral has ω-limit set A.

Now consider vector fields on Z of the form $X+Y$ where Y contracts along fibers of the ribbon Z towards A. If $z \in Z \setminus A$, then the trajectory of z will repeatedly cross the spiral W. We choose Y so small so that every time a trajectory of X enters W, the trajectory traverses the entire ribbon at least once before exiting W. It follows that the ω-limit set of every point of $Z \setminus A$ is equal to A.

Remark 5.4. The constructions in this section for flows yield symmetric connected attractors for diffeomorphisms on passing to the time-one map. In particular, Theorem 1.5 is proved. We require the degenerate construction only when $n = 3$ and Σ is not cyclic. (If $n = 3$ and Σ is cyclic, it follows from Lemma 4.5 that the attractor can be taken to be an Axiom A solenoid. The only other case to consider is when $n = 2$ and $\Gamma = \Sigma = \mathbb{Z}_m$, $m \geq 2$ in Corollary 5.3. Here we can take the attractor to be a normally hyperbolic irrational rotation on a circle.)

6. Attractors in fixed-point spaces

So far we have concentrated attention on attractors that have trivial instantaneous symmetry. In this section we generalize our results to attractors with nontrivial instantaneous symmetry. We largely follow the argument of [21] for continuous maps.

Suppose as usual that $\Gamma \subset \mathbf{O}(n)$ is a finite group acting on $\mathbb{R}^n$. Recall that if A is a subset of $\mathbb{R}^n$ we define the symmetry group Σ_A to be the subgroup of Γ leaving A invariant. We define the group of *instantaneous* symmetries T_A to be the subgroup of Γ which fixes A pointwise:

$$T_A = \{\gamma \in \Gamma \mid \gamma x = x \text{ for all } x \in A\}.$$

Observe that $A \subset \mathrm{Fix}(T_A)$. It is easily shown [21] that T_A is an isotropy subgroup of Γ and that $T_A \subset \Sigma_A \subset N(T_A)$. Hence we may restrict to pairs of subgroups (Σ, T) of Γ satisfying these properties. We generalize the notion of a Σ-symmetric set by saying that a subset $A \subset \mathbb{R}^n$ is (Σ, T)-*symmetric* if $\Sigma_A = \Sigma$, $T_A = T$ and there is a point $y \in A$ with $\Gamma_y = T$.

Definition 6.1. Let $\Gamma \subset \mathbf{O}(n)$ be a finite group acting on $\mathbb{R}^n$ and let (Σ, T) be a pair of subgroups of Γ. We say that (Σ, T) is *admissible for flows* if there is a Γ-equivariant flow on $\mathbb{R}^n$ with a (Σ, T)-symmetric Liapunov stable ω-limit set A.

We impose the additional criterion of Liapunov stability in Definition 6.1 so that we can make use of the following result (but note Remark 6.5).

Proposition 6.2. *Suppose that A is a Liapunov stable ω-limit set and that $\gamma \in \Gamma$. Either $\gamma A = A$ or $\gamma A \cap A = \emptyset$.*

Proof. This was originally proved under slightly different hypotheses by Chossat and Golubitsky [8, Proposition 1.1] and was reformulated as stated here in [22, Proposition 4.8]. □

Just as in [21], we obtain a necessary condition for admissibility of a pair of subgroups (Σ, T) by restricting to $\mathrm{Fix}(T)$. A Γ-equivariant flow on $\mathbb{R}^n$ restricts to a $N(T)/T$-equivariant flow on $\mathrm{Fix}(T)$. At the same time, a (Σ, T)-symmetric attractor for the flow on $\mathbb{R}^n$ is a Σ/T-symmetric attractor for the flow restricted to $\mathrm{Fix}(T)$. As an immediate consequence we have

Proposition 6.3. *Let (Σ, T) be a pair of subgroups of Γ and set $\Gamma' = N(T)/T$ and $\Sigma' = \Sigma/T$. If (Σ, T) is admissible for flows then Σ' is admissible for flows as a subgroup of Γ'.*

The conditions in Proposition 6.3 are not optimal (even if $\mathrm{Fix}(T)$ is of high dimension). This is due to the presence of *hidden symmetries.* That is, elements $\gamma \in \Gamma \setminus N(T)$ with the property that $\gamma\mathrm{Fix}(T) \cap \mathrm{Fix}(T) \neq \emptyset$, see [15, 16]. In the remainder of this section, we obtain necessary and sufficient conditions for a pair (Σ, T) to be admissible for flows. As in the case of continuous maps [21], it suffices to take account of 'hidden reflections'. To this end, let K_T denote the set of elements $\tau \in \Gamma$ such that $\mathrm{Fix}(\tau)$ intersects $\mathrm{Fix}(T)$ in a codimension one subspace. Define

$$L_T = \bigcup_{\tau \in K_T} \mathrm{Fix}(\tau).$$

Note that the connected components of $\mathrm{Fix}(T) \setminus L_T$ are permuted by elements of $N(T)$ and also by Γ-equivariant homeomorphisms.

Theorem 6.4. *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a finite group, T is an isotropy subgroup of Γ with $\dim \mathrm{Fix}(T) \geq 3$, and Σ is a subgroup satisfying $T \subset \Sigma \subset N(T)$. Then (Σ, T) is admissible for flows if and only if Σ fixes a connected component of $\mathrm{Fix}(T) \setminus L_T$. Admissibility can be realized by an Axiom A attractor if $\dim \mathrm{Fix}(T) \geq 5$.*

Proof. First, suppose that (Σ, T) is admissible for flows. That is, there exists a Γ-equivariant flow on $\mathbb{R}^n$ with a connected attractor $A \subset \mathrm{Fix}(T)$ and $\Sigma_A = \Sigma$. It follows that $A \cap \mathrm{Fix}(\tau) = \emptyset$ for any

$\tau \in N(T)$ that acts as a reflection on $\mathrm{Fix}(T)$. Moreover, by Proposition 6.2, $A \cap \mathrm{Fix}(\tau) = \emptyset$ for all $\tau \in \Gamma \setminus \Sigma$. Hence $A \cap L_T = \emptyset$ and so A lies in a connected component $C \subset \mathrm{Fix}(T) \setminus L_T$. It follows that Σ fixes the component C.

Next suppose that Σ fixes a connected component of $\mathrm{Fix}(T) \setminus L_T$. Let $\Gamma' = N(T)/T$ and $\Sigma' = \Sigma/T$. By Proposition 6.3 and the results in Section 5 (and Section 4) we can construct a Σ'-symmetric (Axiom A) attractor $A \subset C$ for a Γ'-equivariant flow on $\mathrm{Fix}(T)$. Moreover, there is no difficulty extending Γ-equivariantly to the whole of $\mathbb{R}^n$ provided that

$$\gamma A \cap A = \emptyset, \text{ for all } \gamma \in \Gamma \setminus N(T). \tag{6.1}$$

Since $A \subset C$, condition (6.1) is satisfied for $\gamma \in K_T$. Moreover, $\mathrm{Fix}(\gamma) \cap \mathrm{Fix}(T)$ has codimension greater than one in $\mathrm{Fix}(T)$, $\gamma \notin K_T \cup T$, and we may assume that all points in A have isotropy precisely T. Condition (6.1) holds unless A contains points $x, \gamma x$ with $\gamma \notin N(T) \cup K_T$ and $\gamma x = \sigma x$ for some $\sigma \in \Sigma$. But then $\gamma^{-1}\sigma \in T$ and $\gamma \in N(T)$. □

Remarks 6.5. (1) The necessary and sufficient condition for a pair (Σ, T) to be admissible for flows is also a sufficient condition for (Σ, T) to be admissible for diffeomorphisms. Moreover, admissibility can be realized by a connected Axiom A attractor provided $\dim \mathrm{Fix}(T) \geq 4$.
(2) Results of [2, 3] indicate that weaker notions of stability than Liapunov or asymptotic stability are appropriate for attractors in proper invariant subspaces. As in [21] it is possible to generalize Theorem 6.4 to include such notions of stability.

Part 2.

We continue to assume that $\Gamma \in \mathbf{O}(n)$ is a finite group acting on $\mathbb{R}^n$. In Part I, we classified the admissible symmetries of ω-limit sets for smooth Γ-equivariant flows on $\mathbb{R}^n$. We also verified part of Theorem 1.4 by finding sufficient conditions for admissibility for Γ-equivariant diffeomorphisms. In Part II, we complete the proof of Theorem 1.4 as well as the classification of admissible symmetries of ω-limit sets for Γ-equivariant diffeomorphisms. The proof of Theorem 1.4 depends on a number of results about finite reflection groups as well as extensions of the techniques used in Part 1.

We also consider the question of strong admissibility, where we require that the ω-limit set is connected and Liapunov stable. Provided $n \geq 3$, the ω-limit sets in Theorem 1.4 can be taken to be connected for class I subgroups and to have two connected components for class

II subgroups. For certain class II subgroups, we can reduce the number of connected components to one.

Proposition 6.6. *Suppose that* Γ *is a finite subgroup of* $\mathbf{O}(n)$*,* $n \geq 3$*, and that* Σ *is a class II subgroup. Then* Σ *is strongly admissible for diffeomorphisms if and only if* Σ *contains a reflection that lies in the center of* Γ*. The* ω*-limit sets that realize strong admissibility can be taken to be asymptotically stable but cannot be taken to be Axiom A attractors.*

We review background material on finite reflection groups in Section 7 and give a reasonably computable characterization of subgroups of class I and class II. In Section 8, we prove that admissible subgroups are of class I and II and classify the admissible subgroups of finite reflection groups.

The next two sections are concerned with the construction of Axiom A attractors when $n \geq 4$. In Section 9, we prove a result of independent interest where a result of [22] on the symmetry of connected components of an attractor is shown to be optimal. This result is used in Section 10 where we construct symmetric Axiom A attractors for subgroups of class II. In particular, this completes the proof of Theorem 1.4 when $n \geq 4$.

In Section 11 we consider the low-dimensional cases where $n \leq 3$ and so conclude the proof of Theorem 1.4. Finally, in Section 12, we obtain the characterization of strong admissibility described in Proposition 6.6.

7. Background from finite reflection groups

Suppose that Γ is a finite group. Let R denote the normal subgroup of Γ generated by the set of reflections in Γ. The group R is (by definition) a *finite reflection group* and the questions we consider depend crucially on the action of Γ on the fundamental domains of R. In Subsection 7.1, we recall and develop the ideas we need from the representation theory of finite reflection groups. In Subsection 7.2 we consider the full group Γ and prove results about the action of Γ-equivariant homeomorphisms on the fundamental domains of R. Finally, in Subsection 7.3, we give a computable characterization of subgroups of class I and class II.

7.1. **Finite reflection groups.** In this subsection, we recall the standard results about finite reflection groups that we need. In addition, we obtain a useful characterization of central reflections.

Proposition 7.1. *Suppose that R is generated by reflections and that L is the union of the reflection hyperplanes corresponding to reflections in R. Then each connected component of $\mathbb{R}^n \setminus L$ is convex and a fundamental domain for the action of R. In particular, R acts transitively and fixed-point freely on the set of connected components of $\mathbb{R}^n \setminus L$.*

Proof. This is a well-known fact about representations of finite reflection groups (see, for example, [18, Chapter 1]). □

Proposition 7.2. *Suppose that R is a finite reflection group acting on $\mathbb{R}^n$. We may write $\mathbb{R}^n = V \oplus W$, $V = V_1 \oplus \cdots \oplus V_p$ and $R = R_1 \times \cdots \times R_p$, where V and W are R-invariant subspaces and, for each $j \in \{1, \dots, p\}$, we have*

(a) *(V_j, R_j) is a nontrivial irreducible finite reflection group.*
(b) *R_j acts trivially on $(\oplus_{i \neq j} V_i) \oplus W$*

Furthermore, if C is a fundamental domain for $(\mathbb{R}^n, R)$, then $C = C_1 \times \cdots \times C_p \times W$ where C_j is a fundamental domain for (V_j, R_j), $j = 1, \dots, p$.

Proof. Statements (a,b) are standard results in the classification theory of finite reflection groups (see, for example, [18, Chapter 2, §2]). The final assertion is an immediate consequence of the fact that R acts on V as the product $R_1 \times \cdots \times R_p$. □

Remarks 7.3. (1) The group R acts trivially on the subspace W in Proposition 7.2.
(2) The linear maps that commute with the action of R leave each of the subspaces $V_1, \dots, V_p, W$ invariant and are scalar multiples of the identity on each V_j.

Let $Z(R)$ denote the *center* of R. As an immediate consequence of Proposition 7.2 and Remarks 7.3, we have

Corollary 7.4. *Suppose $\tau \in R$ is a reflection. There exists a unique $j \in \{1, \dots, p\}$ and reflection $\tau_j \in R_j$ such that τ is induced from τ_j. Moreover, if $\tau \in Z(R)$ then $\tau_j \in Z(R_j)$ and $(V_j, R_j) \cong (\mathbb{R}, \mathbb{Z}_2)$.*

Definition 7.5. Two fundamental domains C and C' are *adjacent* if the closures $\overline{C}$ and $\overline{C'}$ have $(n-1)$-dimensional intersection.

Lemma 7.6. *Let C be a fundamental domain for R and suppose that $\tau \in R$ is a reflection. Then $\tau \in Z(R)$ if and only if*

(i) *C and τC are adjacent, and*
(ii) *there is an R-equivariant involution B such that $\tau C = BC$.*

Proof. First we reduce to the case where R is a finite reflection group acting irreducibly on $\mathbb{R}^n$. Let $\mathbb{R}^n = V \oplus W$, $V = V_1 \oplus \cdots \oplus V_p$ be as in Proposition 7.2. It follows from Corollary 7.4 that we may identify τ with a reflection $\tau_j \in R_j$. Moreover, $\tau \in Z(R)$ if and only if $\tau_j \in Z(R_j)$. Observe that the fundamental domains C and τC are adjacent if and only if C_j and $\tau_j C_j$ are adjacent. An R-equivariant involution B satisfying $\tau C = BC$ restricts to an R_j-equivariant involution satisfying $\tau_j C_j = BC_j$. Conversely, an R_j-equivariant involution satisfying $\tau_j C_j = BC_j$ extends to an involution on $\mathbb{R}^n$ satisfying condition (ii) by setting B to be the identity on $(\oplus_{i\neq j} V_i) \oplus W$. Hence, without loss of generality, we may suppose that $R = R_j$ and $\mathbb{R}^n = V_j$.

Now let R be a finite reflection group acting irreducibly on $\mathbb{R}^n$. The only equivariant involutions are $\pm I$. If $\tau \in Z(R)$ then $n = 1$ so that conditions (i) and (ii) are trivially valid. Conversely, suppose that the reflection τ satisfies conditions (i) and (ii). Since $B = -I$ it follows that C and $-C$ are adjacent. This is possible only when $n = 1$ in which case R is abelian. □

If $J \subset R$ is a maximal isotropy subgroup of R then $\mathrm{Fix}(J)$ is one-dimensional. We shall call $\mathrm{Fix}(J)$ an *axis of symmetry* for R.

Proposition 7.7. *Suppose that R is a finite reflection group acting irreducibly on $\mathbb{R}^n$ and that C is a fundamental domain for R. Then $\overline{C} \setminus \{0\}$ intersects precisely n axes of symmetry, say $A_1 \ldots, A_n$. Moreover, if $\overline{C'} \setminus \{0\}$ intersects $A_1 \ldots, A_n$ then $C' = \pm C$.*

Proof. This result follows from standard facts about simple root systems. Again we refer to [18] for more details. □

We shall require one further basic fact about finite reflection groups.

Proposition 7.8. *Suppose that J is an isotropy subgroup of R. Then J is generated by reflections.*

7.2. **Groups that contain reflections.** Throughout this subsection, Γ denotes a finite group acting on $\mathbb{R}^n$ and R is the normal subgroup generated by reflections. It follows from the definition of R and Proposition 7.1 that Γ acts transitively on the set of fundamental domains for R.

The decomposition in Proposition 7.2 holds for the subgroup R but with the additional property that the subspaces V and W are Γ-invariant.

Proposition 7.9. *Let C denote a fundamental domain for R and let Γ_C be the subgroup of Γ that fixes the component C. Then*

(a) *Γ_C is an isotropy subgroup of Γ.*

(b) Γ *is the semi-direct product of* R *and* Γ_C.

Proof. Choose $x \in C$. Then $y = \sum_{\gamma\in\Gamma_C} \gamma x$ lies in C since C is convex. By construction, the isotropy subgroup Σ_y contains Γ_C. Conversely, if $\sigma \in \Sigma_y$, then σ fixes y and hence C so that $\sigma \in \Gamma_C$. Statement (a) follows.

Since C is a fundamental domain for R, each element of Γ can be written uniquely as a product of elements of Γ_C and R. Since R is normal, it follows that Γ is the semi-direct product of R and Γ_C. □

Proposition 7.10. *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a* Γ*-equivariant homeomorphism. Let* C *be a fundamental domain for the action of* R *on* $\mathbb{R}^n$ *and let* $C = C_1 \times \cdots \times C_p \times W$ *be the decomposition of* C *given in Proposition 7.2. Then*

(a) $f(C) = e_1C_1 \times \cdots \times e_pC_p \times W$*, where* $e_i \in \{-1, +1\}$*,* $i \in \{1, \ldots, p\}$.
(b) *The subspaces* $V_+ = \oplus_{e_i=1} V_i$ *and* $V_- = \oplus_{e_i=-1} V_i$ *are* Γ*-invariant.*

Proof. Write $f(C) = C'$, $C' = C'_1 \times \cdots \times C'_p \times W$. To prove (a) we must show for example that $C'_1 = \pm C_1$. Let $\dim V_1 = m$ and let $A_1, \ldots, A_m$ be the axes of symmetry for R_1 that intersect $\overline{C_1} \setminus \{0\}$ as in Proposition 7.7. Since f is equivariant, $f(A_j) = A_j$ for each j. It follows that $\overline{C'_1}$ intersects each of the axes A_j so that $C'_1 = \pm C_1$ as required.

Next we prove (b). Suppose that $\gamma \in \Gamma$ and that A is an axis of symmetry for some R_j. Then γA is an axis of symmetry for $\gamma R_j \gamma^{-1}$. By equivariance f preserves or reverses orientations on A and γA together. Hence A and γA both lie either in V_+ or V_-. In particular, the axes of symmetry in V_+ and in V_- are preserved by the action of Γ. The result now follows from the fact that the axes of symmetry for R_j span V_j. □

Lemma 7.11. *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a* Γ*-equivariant homeomorphism and that* C *is a fundamental domain for* R*. Then there is an equivariant involution* $B \in \mathcal{B}$ *such that* $f(C) = B(C)$.

Proof. We have shown that $\mathbb{R}^n = V_+ \oplus V_i \oplus W$ where V_+, V_- and W are Γ-invariant subspaces. Hence we can define a Γ-equivariant involution by $B|_{V_+} = I_{V_+}$, $B|_{V_-} = -I_{V_-}$, $B|_W = I_W$. Then $BC = f(C)$ as required. □

Remark 7.12. If we assume that f is a diffeomorphism, we may give an elementary proof of Lemma 7.11. Indeed, df_0 is an invertible linear map commuting with Γ and satisfying $df_0(C) = f(C)$. Replace df_0 by the linear map B which is the identity on W and equal to df_0 on

V. Note that B is Γ-equivariant and we still have $BC = f(C)$. By Remark 7.3(2), $B|_V$ is diagonal and we may scale so that B has entries consisting of plus and minus one. Hence $B^2 = I$ as required.

7.3. **Computation of subgroups of class I & II.** The definitions in Section 1 depend on explicit details about the action of Γ that may be difficult to verify. In this subsection, we derive alternative characterizations of class I and II subgroups that can be easily applied in practice. We follow the viewpoint of [5] that the isotropy subgroups and reflections in the representation of Γ are relatively computable and seek characterizations of the class I and II subgroups in terms of this information and group theory alone.

The characterization of subgroups of class I is straightforward thanks to Proposition 7.9. Recall that Γ_C is the subgroup of Γ that fixes a connected component $C \subset \mathbb{R}^n \setminus L$.

Proposition 7.13. *The subgroup Γ_C is a maximal reflection-free isotropy subgroup of Γ and contains all subgroups of class I up to conjugacy.*

Recall that the *centralizer* of Γ (in $\mathbf{O}(n)$) is defined to be

$$C(\Gamma) = \{\rho \in \mathbf{O}(n),\ \rho\gamma = \gamma\rho \text{ for all } \gamma \in \Gamma\}.$$

We denote by $\mathcal{B}$ the set of all elements in $C(\Gamma)$ of order two. Thus $\mathcal{B}$ consists of nontrivial Γ-equivariant involutions. If $B \in \mathcal{B}$, $\mathbb{Z}_2(B)$ denotes the two element group generated by B. (In general, we shall use this notation $\mathbb{Z}_2(B)$ even if $B \notin C(\Gamma)$.)

Suppose that Σ is a class II subgroup of Γ. We recall that Σ contains an index two subgroup Δ (fixing a connected component C of $\mathbb{R}^n \setminus L$) and that there exists $B \in \mathcal{B}$ such that $B(C) = \sigma C$, all $\sigma \in \Sigma \setminus \Delta$. The simplest case is when $B \in \Sigma$ since we can easily list the subgroups of Γ of the form $\Sigma = \Delta \oplus \mathbb{Z}_2(B)$ where Δ is of class I and $B \in \mathcal{B}$. We now consider the remaining cases $B \in \Gamma \setminus \Sigma$ and $B \notin \Gamma$.

First of all, suppose that Σ and Δ are subgroups of Γ and Δ is of index two in Σ. Given $B \in \mathcal{B}$, $B \notin \Sigma$, we may define a new group $\Sigma_B \subset \mathbf{O}(n)$ by

$$\Sigma_B = \Delta \cup B(\Sigma \setminus \Delta)$$

It is obvious that Δ is of index two in Σ_B and that $(\Sigma_B)_B = \Sigma$. Moreover, $\Sigma_B \subset \Gamma$ if and only if $B \in \Gamma$.

Proposition 7.14. *Let Σ and Δ be subgroups of Γ with Δ of index two in Σ and let $B \in \mathcal{B}$ be an equivariant involution with $B \in \Gamma \setminus \Sigma$. Suppose further that Σ is not of class I but that Δ is of class I. Then Σ is of class II if and only if Σ_B is of class I.*

Proof. Observe that Σ_B fixes a connected component C of $\mathbb{R}^n \setminus L$ if and only if Δ fixes C and $B(\Sigma \setminus \Delta)$ fixes C. □

If the involution $B \notin \Gamma$, then Σ_B is a subgroup of the group $\Gamma \oplus \mathbb{Z}_2(B)$. Note that elements of Γ, $C(\Gamma)$, and hence $\Gamma \oplus \mathbb{Z}_2(B)$, permute the connected components of $\mathbb{R}^n \setminus L$.

Proposition 7.15. *Assume the same hypotheses as in Proposition 7.14 except that $B \notin \Gamma$. Then Σ is of class II if and only if there is an isotropy subgroup J in $\Gamma \oplus \mathbb{Z}_2(B)$ such that $\Sigma_B \subset J$ and $J \cap \Gamma$ is reflection-free.*

Proof. As before, Σ is of class II if and only if Σ_B fixes a connected component C of $\mathbb{R}^n \setminus L$. Suppose that Σ_B fixes C. Let $J = (\Gamma \oplus \mathbb{Z}_2(B))_C$ and R' denote the normal subgroup of $\Gamma \oplus \mathbb{Z}_2(B)$ generated by reflections. If $R = R'$, then $\Gamma \oplus \mathbb{Z}_2(B)$ is the semi-direct product of R with J and it follows from Proposition 7.9(a) that J is an isotropy group for the action of $\Gamma \oplus \mathbb{Z}_2(B)$. If $R' \neq R$, then C is not a fundamental domain for the action of R' and J will be the isotropy of any point in C which has Γ-isotropy Γ_C and lies on a a reflection hyperplane of $R' \setminus R$. In either case, it is obvious that $J \cap \Gamma$ is reflection-free. The converse is equally straightforward (see also [5, Theorem 3.2]). □

8. Admissible subgroups are of class I or II

In Subsection 8.1, we prove that the conditions for admissibility described in Theorem 1.4 are necessary. In Subsection 8.2, we classify the admissible subgroups of finite reflection groups.

8.1. Admissible subgroups for homeomorphisms.

In this subsection, we show that admissible subgroups for homeomorphisms are of class I or II. We begin by showing that an ω-limit set for an equivariant continuous one-to-one map intersects at most two connected components of $\mathbb{R}^n \setminus L$.

Proposition 8.1. *Let $\Gamma \subset \mathbf{O}(n)$ be a finite group. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous one-to-one Γ-equivariant map. Then f^2 fixes each connected component of $\mathbb{R}^n \setminus L$.*

Proof. Let $\tau \in \Gamma$ be a reflection. Since f is one-to-one and Γ-equivariant, the subspace $\mathrm{Fix}(\tau)$ is backward as well as forwards invariant under f, and hence f permutes the two connected component of $\mathbb{R}^n \setminus \mathrm{Fix}(\tau)$. That is, f either fixes the connected components or interchanges them. Hence the components are fixed by f^2.

Now suppose that C is a connected component of $\mathbb{R}^n \setminus L$. We may write $L = \mathrm{Fix}(\tau_1) \cup \cdots \cup \mathrm{Fix}(\tau_k)$ and hence $C = C_1 \cap \cdots \cap C_k$, where

C_j is a connected component of $\mathbb{R}^n \setminus \mathrm{Fix}(\tau_j)$. Then $f^2(C) = f^2(C_1) \cap \cdots \cap f^2(C_k) = C_1 \cap \cdots \cap C_k = C$ as required. □

Corollary 8.2. *Suppose that Γ and f are as in the proposition, and that A is an ω-limit set for f. Then A intersects at most two connected components of $\mathbb{R}^n \setminus L$. Moreover, if A is Σ-symmetric, Σ a subgroup of Γ, then Σ contains at most one reflection in which case A intersects two connected components.*

Let Σ be a subgroup of Γ. If $A \subset \mathbb{R}^n$ is a Σ-symmetric ω-limit set, then γA is a $\gamma\Sigma\gamma^{-1}$-symmetric ω-limit set. It follows that we need only consider representatives of conjugacy classes of subgroups and orbits of ω-limit sets. Since Γ acts transitively on the connected components of $\mathbb{R}^n \setminus L$, we may assume without loss of generality that $A \cap C \neq \emptyset$ for some fixed component C. Recall that Γ_C is the subgroup of Γ that fixes C.

Theorem 8.3. *Suppose that $\Gamma \subset \mathbf{O}(n)$, $n \geq 1$. Let Σ be a subgroup of Γ. Suppose that A is a Σ-symmetric ω-limit set for the Γ-equivariant homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $A \cap C \neq \emptyset$, where C is a connected component of $\mathbb{R}^n \setminus L$. Either $\Sigma \subset \Gamma_C$ or*

(a) *There is a unique involution $r \in R \cap C(\Gamma_C)$ such that $\Sigma \subset \Gamma_C \oplus \mathbb{Z}_2(r)$.*
(b) *$\Delta = \Sigma \cap \Gamma_C$ is of index two in Σ.*
(c) *There exists an equivariant involution $B \in \mathcal{B}$ such that $BC = \sigma C$ for all $\sigma \in \Sigma \setminus \Delta$.*

In particular, Σ is of class I or class II.

Proof. (a,b) Let $\sigma \in \Sigma \setminus \Gamma_C$. Since A is an ω-limit set, it follows that $f(C) = \sigma C$ and by equivariance of f that Γ_C fixes the component σC as well as the component C. By Proposition 7.9(b) there is a unique element $r \in R$ such that $\sigma C = rC$. Moreover $r^{-1}\sigma$ fixes C and so lies in Γ_C. It follows that Σ lies in the group generated by Γ_C and r and that $\Delta = \Sigma \cap \Gamma_C$ is of index two in Σ.

Next we claim that $r^2 = 1$. We have $f(C) \subset rC$ so by equivariance $f^2(C) \subset r^2(C)$. But $f^2(C) \subset C$ so that $r^2(C) = C$. Hence $r^2 \in \Gamma_C \cap R$ and $r^2 = 1$ as required. It remains to show that elements $\omega \in \Gamma_C$ commute with r. Since Γ_C fixes C and rC, we have $\omega^{-1} r \omega C = rC$. But R is a normal subgroup of Γ so $\omega^{-1} r \omega \in R$. The unique element of R mapping C into rC is r so we have $\omega^{-1} r \omega = r$ as required.
(c) We have $f(C) = \sigma C$ for $\sigma \in \Sigma \setminus \Delta$. It follows from Lemma 7.11 that $f(C) = BC$ for some $B \in \mathcal{B}$, proving (c).

The final statement follows from the characterization of class I subgroups in Proposition 7.13 and the definition of class II subgroups. □

8.2. **Admissible subgroups of irreducible finite reflection groups.** In this subsection, we give a fairly explicit description of the class I and class II subgroups of an irreducible finite reflection group. We use this information to give a simple proof of Theorem 1.4 in the case when Γ is an irreducible finite reflection group.

Proposition 8.4. *Suppose that Γ is a finite reflection group acting irreducibly on $\mathbb{R}^n$, $n \geq 1$. The subgroup $\mathbf{1}$ is the unique subgroup of class I. If $-I \in \Gamma$, $\mathbb{Z}_2(-I)$ is the unique class II subgroup. If $-I \notin \Gamma$, then the class II subgroups are given by $\mathbb{Z}_2(\tau)$ where τ is a noncentral involution (that is, $\tau^2 = 1$, $\tau \neq \pm I$) such that $\mathbb{Z}_2(-\tau)$ is an isotropy subgroup of $\Gamma \oplus \mathbb{Z}_2(-I)$.*

Proof. Isotropy subgroups of finite reflection groups are themselves generated by reflections (Proposition 7.8). It follows from Proposition 7.13 that $\mathbf{1}$ is the only class I subgroup. Moreover, class II subgroups have order two.

Next observe that by irreducibility $C(\Gamma) = \{\pm I\}$ and so $\mathcal{B} = \{-I\}$. Suppose that $-I \in \Gamma$. Every class II subgroup is of the form $\Sigma = \mathbb{Z}_2(\tau)$, for some involution τ. It follows from Proposition 7.14 that $\Sigma_B = \mathbb{Z}_2(-\tau)$ is a class I subgroup. Hence $\mathbb{Z}_2(-\tau) = \mathbf{1}$ and so $\tau = -I$. Hence the only class II subgroup is $\mathbb{Z}_2(-I)$.

Finally, we apply Proposition 7.15 to obtain the class II subgroups when $B = -I \notin \Gamma$. Again $\Sigma = \mathbb{Z}_2(\tau)$ and $\Sigma_B = \mathbb{Z}_2(-\tau)$. If Σ_B is an isotropy subgroup, then we can take $J = \Sigma_B$ (note that $J \cap \Gamma = \mathbf{1}$, since $\tau \in \Gamma$, $-I \notin \Gamma$, implies that $-\tau \notin \Gamma$). Conversely, suppose that J is an isotropy subgroup of $\Gamma \oplus \mathbb{Z}_2(-I)$ such that $\Sigma_B \subset J$ and $J \cap \Gamma$ is reflection-free. Since $J \cap \Gamma$ is an isotropy subgroup of Γ we have $J \cap \Gamma = \mathbf{1}$. Hence J has at most two elements. It follows that $\Sigma_B = J$ as required. □

It is clear that the class I subgroup $\mathbf{1}$ is admissible and may be realized by a sink. The following lemma takes care of the class II subgroups.

Lemma 8.5. *Suppose that $n \geq 1$ and $\Sigma = \mathbb{Z}_2(\tau)$ is of class II (where $\tau \in \Gamma$ is not necessarily central). Then Σ is admissible for diffeomorphisms and may be realized by a period two sink.*

Proof. Let $x \in C$ be a point of trivial isotropy. Observe that $B\tau x \in C$. Since the set of points in C of trivial isotropy is path connected it follows form [7, Chapter 6, Theorem 3.1] that the map taking x to $B\tau x$ is Γ-equivariantly isotopic to the inclusion. Hence there is an extension to a Γ-equivariant diffeomorphism $\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ permuting x and $B\tau x$. Define $\phi = B \circ \phi_0$. It follows from our construction that

x is a periodic point of period two. The corresponding periodic orbit A consists of the points x and τx and is Σ-symmetric. The usual perturbation arguments allow us to assume that A is a sink. □

Corollary 8.6. *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a finite reflection group, $n \geq 1$. A subgroup $\Sigma \subset \Gamma$ is admissible for diffeomorphisms if and only if Σ is of class I or of class II. Moreover, admissibility can be realized by a sink or period two sink respectively.*

Proof. By Theorem 8.3, an admissible subgroup must be of class I or class II. The converse follows from Proposition 8.4 and Lemma 8.5. □

9. Attractors with symmetric connected components

Let $\Gamma \subset \mathbf{O}(n)$ be a finite group acting on $\mathbb{R}^n$, $n \geq 4$. Suppose that $\Sigma \subset \Gamma$ is of class I. By Theorem 1.5, Σ is admissible for diffeomorphisms and can be realized by a connected Axiom A attractor.

Suppose instead that A is a disconnected Σ-symmetric ω-limit set for a diffeomorphism ϕ. Let X be a closed and open nonempty proper subset of A. Define $\Delta_X = \{\sigma \in \Sigma \mid \sigma(X) = X\}$. If we set $Y = A \setminus X$, then $\Delta_X = \Delta_Y$ (since if X is fixed by $\sigma \in \Sigma$, so is Y). Clearly Δ_X is a normal subgroup of Σ_A.

In [22, Theorem 4.6], it was shown that, under the additional assumption that A is Liapunov stable, Σ_X is not arbitrary. Indeed the quotient Σ/Σ_X is cyclic so that Σ is a cyclic extension of Σ_X. (If A is assumed to have finitely many connected components, then this result is elementary and the assumption that A is Liapunov stable is not required.)

In this section, we prove that the restriction in [22] is optimal.

Theorem 9.1. *Suppose that $n \geq 4$ and that $\Sigma \subset \Gamma$ is of class I. Suppose further that Σ is a cyclic extension of Δ with $\Sigma/\Delta \cong \mathbb{Z}_k$. Then there exists a Γ-equivariant diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with a Σ-symmetric Axiom A attractor consisting of k Δ-symmetric connected components.*

The proof of this result is similar to arguments in Section 4.1. The new ingredient is contained in the following proposition.

Proposition 9.2. *Suppose that $n \geq 3$. Let $\Sigma \subset \Gamma$ be of class I and suppose that Δ is a normal subgroup of Σ. Let $G_0 \subset \mathbb{R}^n$ be a smoothly embedded Σ-graph consisting of points of trivial isotropy. Let $\sigma \in \Sigma \setminus \Delta$. We may Δ-equivariantly perturb G_0 to a smoothly embedded Δ-graph G_1 such that there exists a Γ-equivariant diffeomorphism h of $\mathbb{R}^n$ mapping G_1 to σG_1.*

Proof. Let $\mathbb{R}^n_0$ denote the subset of $\mathbb{R}^n$ consisting of points of trivial Γ-isotropy. Since $n \geq 3$, we may find a Δ-equivariant isotopy $\phi_t : G_0 \to \mathbb{R}^n_0$, $t \in [0,1]$, such that if we set $G_t = \phi_t(G_0)$ then

(a) $\gamma G_t \cap G_t = \emptyset$ for all $\gamma \in \Gamma \setminus \Sigma$, $t \in [0,1]$.
(b) $\gamma G_1 \cap G_1 = \emptyset$ for all $\gamma \in \Gamma \setminus \Delta$.

We remark that (a) implies that the isotopy extends, as a Γ-equivariant map, to $\Gamma(G_0)$. Condition (b) implies that G_1 is a smoothly embedded Δ-graph. Since $\sigma \in N(\Delta)$, σG_1 is Δ-invariant. In particular, the isotopy ϕ_t induces a Δ-equivariant isotopy $\psi_t = \sigma\phi_t\sigma^{-1}$ moving G_0 to σG_1. Obviously ψ_t satisfies (a,b) and is supported in $\mathbb{R}^n_0$. Let $\rho_t = \psi_t \circ \phi_t^{-1}$. Then ρ_0 is the inclusion of G_0 in $\mathbb{R}^n$ and $\rho_1 : G_1 \to \sigma G_1$ is a Δ-equivariant diffeomorphism. Let h denote the Γ-equivariant extension of ρ_1 to $\Gamma(G_1)$. Since h is Γ-equivariantly isotopic to the inclusion map it follows that h extends to a Γ-equivariant diffeomorphism of $\mathbb{R}^n$. Obviously, $h(G_1) = \sigma G_1$. □

Proof of Theorem 9.1: Let $G_0 \subset \mathbb{R}^n$ be a smoothly embedded balanced Σ-graph consisting of points of trivial isotropy. Let $\sigma \in \Sigma \setminus \Delta$ generate Σ/Δ and $G_1 \subset \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the conclusions of Proposition 9.2. Let $f_1 : G_1 \to G_1$ be a smooth Δ-equivariant map satisfying condition (W). In particular, f_1 will be an expanding immersion. Set $G = \Sigma(G_1)$ and note that G is a smooth Σ-graph consisting of k connected components each of which is a smooth Δ-graph. The map $f : G_1 \to G_1$ extends Σ-equivariantly to G and so we may define $\chi = h \circ f : G \to G$. Clearly χ continues to satisfy condition (W). Passing to the inverse limit yields a (topologically transitive) Σ-symmetric solenoid $\mathcal{S}$ consisting of k connected Δ-symmetric solenoids. This will provide the model for our Axiom A attractor A.

Following Williams [23], we thicken G_1 to a tubular neighborhood U_1, extend f_1 to a map $f_1 : U_1 \to G_1$ and perturb to an embedding ϕ_1 of $U_1 \to U_1$, all the time preserving fibers of U_1 and equivariance. Extend ϕ_1 to a Γ-equivariant embedding $\phi_1 : U \to U$ where $U = \Gamma(U_1)$. As in the proof of Theorem 4.3, we can choose G_1 and f_1 so that ϕ_1 is Γ-equivariantly isotopic to the inclusion in $\mathbb{R}^n$. In addition, we may require that $h|U : U \to U$ and preserves fibers. Since ϕ_1 is Γ-equivariantly isotopic to the inclusion, we may extend ϕ_1 to a Γ-equivariant diffeomorphism ϕ_1 of $\mathbb{R}^n$. Define $\phi = h \circ \phi_1$. It follows just as in Section 4.1 that ϕ has an attractor A with the required properties. The dynamics on A is topologically conjugate to the shift dynamics on the solenoid $\mathcal{S}$. □

In Section 10 we shall require the following straightforward generalization of Theorem 9.1.

Lemma 9.3. *Suppose that $n \geq 4$ and that $\Delta \subset \Gamma$ is of class I, fixing a connected component $C \subset \mathbb{R}^n \setminus L$. Let $\Sigma \subset \mathbf{O}(n)$ be a finite cyclic extension of Δ that fixes C. (We no longer require that $\Sigma \subset \Gamma$.) Suppose that Σ/Δ has order k. Then there exists a Γ-equivariant diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with a $\Sigma \cap \Gamma$-symmetric Axiom A attractor consisting of k Δ-symmetric connected components.*

Proof. We can proceed along similar lines to the proof of Proposition 9.2 and Theorem 9.1. The only difference is that if G_0 is a smooth balanced Σ-graph, it may not be possible to Σ-equivariantly embed G_0 in $\mathbb{R}^n$. However, we may Δ-equivariantly embed G_0 as a Σ-symmetric subset of C and then we proceed just as before. $\square$

10. Symmetric Axiom A attractors for diffeomorphisms

Suppose that $\Gamma \subset \mathbf{O}(n)$ is a finite group acting on $\mathbb{R}^n$ with $n \geq 4$. If $\Sigma \subset \Gamma$ is of class I, then by Theorem 1.5 there exists a Γ-equivariant diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with a connected Σ-symmetric Axiom A attractor.

In this section we prove the analogous result for subgroups Σ of class II. Recall that Σ has an index two subgroup Δ which is of class I and fixes a connected component $C \subset \mathbb{R}^n \setminus L$. Let B be the Γ-equivariant involution satisfying $BC = \sigma C$ for $\sigma \in \Sigma \setminus \Delta$.

Theorem 10.1. *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a finite group and $n \geq 4$. Let $\Sigma \subset \Gamma$ be of class II. Then there exists a Γ-equivariant diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with a Σ-symmetric Axiom A attractor. The attractor has two connected components, one of which lies in C, the other in BC.*

Proof. Let Σ_B denote the group $\Sigma_B = \Delta \cup B(\Sigma \setminus \Delta) \subset \mathbf{O}(n)$. Observe that Σ_B is a subgroup of Γ if and only if $B \in \Gamma$ but in any case Σ_B fixes the connected component C. Moreover, Δ is an index two subgroup of Σ_B so that the hypotheses of Lemma 9.3 are satisfied with $k = 2$. It follows from the lemma that there is a Γ-equivariant diffeomorphism $\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ with an Axiom A attractor A consisting of two connected components, both of which are Δ-symmetric. If A_0 denotes one of the components then the other component is $B\sigma A_0$, where σ is any element of $\Sigma \setminus \Delta$. Let $\phi = B \circ \phi_0$. Then ϕ is a Γ-equivariant diffeomorphism and $A = A_0 \cup \sigma A_0$ is a Σ-symmetric Axiom A attractor. $\square$

Remark 10.2. A simpler proof of Theorem 10.1 that does not rely on Lemma 9.3 is possible when $B \in \Sigma$. In this case $\Sigma = \Delta \oplus \mathbb{Z}_2(B)$. Since Δ is of class I, it follows from Theorem 4.3 that we can find a Γ-equivariant diffeomorphism $\phi_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with a connected Δ-symmetric Axiom A attractor $A_0 \subset C$. We define $\phi = B \circ \phi_0$. Since A_0 is topologically transitive for $\phi^2 = \phi_0^2$, it follows that $A = A_0 \cup BA_0$ is an Axiom A attractor for the Γ-equivariant diffeomorphism ϕ. Moreover $\Sigma_A = \Sigma$.

Remark 10.3. For certain class II subgroups Σ, we can relax the condition that $n \geq 4$.

(i) Suppose that $\Sigma = \mathbb{Z}_2(\tau)$. By Lemma 8.5, Σ is admissible for diffeomorphisms and may be a realized by a period two sink for all $n \geq 1$.

(ii) Suppose that $B \in \Sigma$ and Δ is cyclic. Combining the construction above for the case $B \in \Sigma$ with Lemma 4.5, we see that the conclusion of Theorem 10.1 is valid for $n = 3$.

(iii) Again, suppose that Δ is cyclic and that $n \geq 3$. Suppose further that $\Sigma \setminus \Delta$ contains an element σ lying in the center of Γ. Applying Lemma 4.5 once more, we may construct a Γ-equivariant diffeomorphism ϕ_0 with a Δ-symmetric Axiom A attractor A_0. Define $\phi = \sigma \circ \phi_0$. Since σ is central, ϕ is Γ-equivariant. Moreover, just as in Section 9, the Σ-symmetric set $A = A_0 \cup \sigma A_0$ is an Axiom A attractor for ϕ. (Here, σ plays the role of the extension h.)

11. Admissibility for diffeomorphisms in low dimensions

We continue to assume that $\Gamma \subset \mathbf{O}(n)$ is a finite group. In Section 8 we proved that admissible subgroups are of class I or class II for all $n \geq 1$. To this point we have proved the converse statement for $n \geq 4$ and for some special cases when $n \leq 3$. In this section, we give a complete treatment of the cases $n \leq 3$ and so complete the proof of Theorem 1.4. Our approach is to compute the subgroups of class I and class II explicitly (using the methods of 7.3) and to verify admissibility on a case by case basis.

11.0.1. $n = 1$. The only cases to consider are $\Gamma = \mathbf{1}$ and $\Gamma = \mathbb{Z}_2$ both of which are irreducible finite reflection groups. We can apply the results of 8.2 but it is clear anyhow that $\mathbf{1}$ is of class I in both cases and that $\mathbb{Z}_2$ is a class II subgroup of $\Gamma = \mathbb{Z}_2$. Moreover all subgroups are realized by (periodic) sinks.

11.0.2. $n = 2$. The finite subgroups of $\mathbf{O}(2)$ are $\mathbb{Z}_m$ and $\mathbb{D}_m$, $m \geq 1$. The subgroups of class I and class II are listed in Table 1. We note that all these subgroups are cyclic and can be realizable by periodic sinks. When $\Gamma = \mathbb{D}_m$, the relevant subgroups are of order one and two and we can apply Lemma 8.5. When $\Gamma = \mathbb{Z}_m$ and $\Sigma = \mathbb{Z}_k$, any generator of $\mathbb{Z}_k$ is a $\mathbb{Z}_m$-equivariant diffeomorphism with $\mathbb{Z}_k$-symmetric periodic orbits. Using a standard perturbation argument, we can ensure that one of these periodic orbits is a periodic sink. Hence Theorem 1.4 is valid when $n = 2$.

Next we verify the entries in Table 1. The group $\mathbb{Z}_m$ contains no reflections and hence all subgroups are of class I. The group $\mathbb{D}_m$ is generated by reflections so $\mathbf{1}$ is the unique subgroup of class I. Subgroups of class II must have order two. The groups $\mathbb{D}_1$ and $\mathbb{D}_2$ are abelian and all order two subgroups are of class II. When $m = 3$, $\mathbb{D}_m$ is an irreducible finite reflection group and we may apply Proposition 8.4. When m is even, $-I \in \Gamma$ and the conclusion is immediate. When m is odd, there is up to conjugacy a single order two subgroup $D_1 = \mathbb{Z}_2(\tau) \subset \mathbb{D}_m$. The element τ is a reflection and it follows (since we are working in $\mathbb{R}^2$) that $-\tau$ is a reflection. In particular, $\mathbb{Z}_2(-\tau)$ is an isotropy subgroup of the group $\mathbb{D}_m \oplus \mathbb{Z}_2(-I)$ (or $\mathbb{D}_{2m}$) as required.

$\mathbb{Z}_m$, $m \geq 1$	Class I	$\mathbb{Z}_k$
$\mathbb{D}_m$, $m \geq 1$ odd	Class I	$\mathbf{1}$
	Class II	$\mathbb{D}_1$
$\mathbb{D}_m$, $m \geq 4$ even	Class I	$\mathbf{1}$
	Class II	$\mathbb{Z}_2$
$\mathbb{D}_2$	Class I	$\mathbf{1}$
	Class II	$\mathbb{D}_1$, $\mathbb{Z}_2$

TABLE 1. Admissible subgroups when $n = 2$ (k divides m)

11.0.3. $n = 3$. In this subsection we compute the class I and class II subgroups of the finite subgroups of $\mathbf{O}(3)$. It follows from the degenerate construction in Section 5 that the class I subgroups are admissible for diffeomorphisms. We verify that the class II subgroups are admissible and hence complete the proof of Theorem 1.4. We will assume some familiarity with the notation for subgroups of $\mathbf{O}(3)$ employed in [5, 9, 16, 19]. (However note that class I, II (and III) subgroups have a completely different meaning in these references.) The group $\mathbb{Z}_2(-I)$ is denoted by $\mathbb{Z}_2^c$.

The subgroups of $\mathbf{O}(3)$ can be divided into the so called *planar* subgroups and *exceptional* subgroups. The planar subgroups are denoted by

$$\mathbb{D}_{2m}^d,\ \mathbb{D}_m^z,\ \mathbb{D}_m \oplus \mathbb{Z}_2^c,\ \mathbb{D}_m,\ \mathbb{Z}_{2m}^-,\ \mathbb{Z}_m \oplus \mathbb{Z}_2^c,\ \mathbb{Z}_m,$$

where $m \geq 1$. We note that certain pairs of subgroups are conjugate, namely $\mathbb{D}_2^d$ and $\mathbb{D}_2^z$, $\mathbb{D}_1^z$ and $\mathbb{Z}_2^-$, $\mathbb{D}_1 \oplus \mathbb{Z}_2^c$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2^c$, $\mathbb{D}_1$ and $\mathbb{Z}_2$. The *exceptional* subgroups are

$$\mathbb{I},\ \mathbb{O},\ \mathbb{T},\ \mathbb{I} \oplus \mathbb{Z}_2^c,\ \mathbb{O} \oplus \mathbb{Z}_2^c,\ \mathbb{T} \oplus \mathbb{Z}_2^c,\ \mathbb{O}^-.$$

The exceptional subgroups are easier to work with and we describe our results for these groups first. The results appear in Table 2. We begin with some general observations. Since each exceptional group Γ acts absolutely irreducibly, the centralizer $C(\Gamma)$ consists always of $\pm I$. The groups $\mathbb{I}$, $\mathbb{O}$ and $\mathbb{T}$ contain no reflections so that all subgroups are of class I. At the other extreme, the groups $\mathbb{I} \oplus \mathbb{Z}_2^c$, $\mathbb{O} \oplus \mathbb{Z}_2^c$ and $\mathbb{O}^-$ are generated by reflections so that there are nontrivial subgroups of class I. The first two of these groups contain $-I$ so by Proposition 8.4 the only subgroup of class II is $\mathbb{Z}_2^c$. In the case of $\mathbb{O}^-$ we must search for subgroups of the form $\mathbb{Z}_2(\tau)$ such that $\mathbb{Z}_2(-\tau)$ is an isotropy subgroup of $\mathbb{O} \oplus \mathbb{Z}_2^c$. The latter is a finite reflection group, so this condition is equivalent to asking that $-\tau$ is a reflection. Hence $\mathbb{Z}_2(\tau) \subset \mathbb{O}^- \cap \mathbf{SO}(3) = \mathbb{T}$. Up to conjugacy, there is a single subgroup of order two in $\mathbb{T}$, denoted simply by $\mathbb{Z}_2$ in the references listed above.

The one remaining exceptional subgroup is $\Gamma = \mathbb{T} \oplus \mathbb{Z}_2^c$. The reflection-free isotropy subgroups of Γ are $\mathbf{1}$, $\mathbb{Z}_3$ and hence these are the subgroups of class I. By the general remarks above, $\mathcal{B}$ consists of $-I$. Taking products with the class I subgroups yields the class II subgroups $\mathbb{Z}_2^c$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_2^c$. It remains to rule out further subgroups of class II by virtue of Proposition 7.14. It is enough to observe that there are no class I subgroups of even order, hence no candidates for Σ_B.

Finally, we observe that all subgroups of class II in Table 2 fall into type (i) or (ii) in Remark 10.3 and hence are admissible (even by an Axiom A attractor). Hence Theorem 1.4 is valid for Γ an exceptional subgroup of $\mathbf{O}(3)$.

Next we consider the planar subgroups of $\mathbf{O}(3)$. The details are quite tedious, and we restrict ourselves here to a sketch of the general procedure to be followed. The full details for the group $\Gamma = \mathbb{D}_{2m}^d$ are given in an appendix.

First, recall [5] that the groups $\mathbb{D}_m$, $\mathbb{Z}_m$, $\mathbb{Z}_m \oplus \mathbb{Z}_2^c$ (m odd) and $\mathbb{Z}_{2m}^-$ (m even) contain no reflections. Hence all subgroups are of class I. In Table 3 we exhibit the remaining planar subgroups as a semi-direct product of the subgroup R generated by reflections and a maximal refection-free isotropy subgroup Γ_C, cf Proposition 7.13. The class I subgroups are then the subgroups of Γ_C.

Next we observe that $\mathbb{R}^3$ splits into two irreducible subspaces ($\mathbb{R}^2$ and $\mathbb{R}$) under the action of most of the planar subgroups Γ. (This is

$\mathbb{O}^-$	Class I	$\mathbf{1}$
	Class II	$\mathbb{Z}_2$
$\mathbb{I} \oplus \mathbb{Z}_2^c$	Class I	$\mathbf{1}$
	Class II	$\mathbb{Z}_2^c$
$\mathbb{O} \oplus \mathbb{Z}_2^c$	Class I	$\mathbf{1}$
	Class II	$\mathbb{Z}_2^c$
$\mathbb{T} \oplus \mathbb{Z}_2^c$	Class I	$\mathbb{Z}_3$, $\mathbf{1}$
	Class II	$\mathbb{Z}_3 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2^c$
$\mathbb{I}$	Class I	all subgroups
$\mathbb{O}$	Class I	all subgroups
$\mathbb{T}$	Class I	all subgroups

TABLE 2. Admissible subgroups of the exceptional subgroups of $\mathbf{O}(3)$

Γ	R	Γ_C	Class I
$\mathbb{D}_{2m}^d, m$ odd	$\mathbb{D}_{2m}^d$	$\mathbf{1}$	$\mathbf{1}$
$\mathbb{D}_{2m}^d, m$ even	$\mathbb{D}_m^z$	$\mathbb{D}_1$	$\mathbb{D}_1$, $\mathbf{1}$
$\mathbb{D}_m^z$	$\mathbb{D}_m^z$	$\mathbf{1}$	$\mathbf{1}$
$\mathbb{Z}_{2m}^-, m$ odd	$\mathbb{Z}_2^-$	$\mathbb{Z}_m$	$\mathbb{Z}_k$
$\mathbb{D}_m \oplus \mathbb{Z}_2^c, m$ odd	$\mathbb{D}_m^z$	$\mathbb{D}_1$	$\mathbb{D}_1$, $\mathbf{1}$
$\mathbb{D}_m \oplus \mathbb{Z}_2^c, m$ even	$\mathbb{D}_m \oplus \mathbb{Z}_2^c$	$\mathbf{1}$	$\mathbf{1}$
$\mathbb{Z}_m \oplus \mathbb{Z}_2^c, m$ even	$\mathbb{Z}_2^-$	$\mathbb{Z}_m$	$\mathbb{Z}_k$

TABLE 3. Class I subgroups for the planar subgroups of $\mathbf{O}(3)$ that contain reflections (k divides m)

true for all the planar subgroups that contain at least one reflection, except for the subgroups $\mathbb{D}_2^z$, $\mathbb{D}_2 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2^-$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2^c$, which must be treated individually.) Each irreducible subspace yields a unique equivariant involution, and the product of these yields a third, $-I$. Hence $\mathcal{B} = \{-I, \pm J\}$ where J is the involution coming from the two-dimensional irreducible. It is now an easy matter to decide which involutions lie in Γ and to compute the corresponding class II subgroups of the form $\Delta \oplus \mathbb{Z}_2(B)$ where $B \in \mathcal{B} \cap \Gamma$. This information is contained in Table 4. Again we observe that the class II subgroups are of type (i) or (ii) in Remark 10.3 so that these subgroups are indeed admissible. (Note that when k is odd, $\mathbb{Z}_{2k}^-$ is isomorphic to $\mathbb{Z}_m \oplus \mathbb{Z}_2^-$.)

In Table 5 we list the planar subgroups that contain class II subgroups with $B \notin \Sigma$. The subgroups where $B \in \Gamma$ and $B \notin \Gamma$ are listed

Γ	$\mathcal{B} \cap \Gamma$	Σ
$\mathbb{D}^d_{2m}, m \geq 3$ odd	$-J$	$\mathbb{Z}_2^-$
$\mathbb{D}^d_{2m}, m \geq 2$ even	J	$\mathbb{D}_2$, $\mathbb{Z}_2$
$\mathbb{D}^z_m, m \geq 3$ odd	—	—
$\mathbb{D}^z_m, m \geq 4$ even	J	$\mathbb{Z}_2$
$\mathbb{D}_m \oplus \mathbb{Z}_2^c, m \geq 3$ odd	$-I$	$\mathbb{D}_1 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2^c$
$\mathbb{D}_m \oplus \mathbb{Z}_2^c, m \geq 4$ even	$-I$, $\pm J$	$\mathbb{Z}_2^-$, $\mathbb{Z}_2^c$, $\mathbb{Z}_2$
$\mathbb{Z}^-_{2m}, m \geq 3$ odd	$-J$	$\mathbb{Z}^-_{2k}$
$\mathbb{Z}_m \oplus \mathbb{Z}_2^c, m \geq 4$ even	$-I$, $\pm J$	$\mathbb{Z}^-_{2k}, k$ odd, $\mathbb{Z}_k \oplus \mathbb{Z}_2^c$
$\mathbb{D}^z_2$		$\mathbb{Z}_2^-$, $\mathbb{Z}_2$
$\mathbb{D}_2 \oplus \mathbb{Z}_2^c$		$\mathbb{Z}_2^-$, $\mathbb{Z}_2^c$, $\mathbb{Z}_2$
$\mathbb{Z}_2^-$		$\mathbb{Z}_2^-$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2^c$		$\mathbb{Z}_2^-$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2^c$

TABLE 4. Class II subgroups satisfying $B \in \Sigma$ for the planar subgroups of $\mathbf{O}(3)$ that contain reflections (k divides m)

Γ	$B \in \Gamma$	$B \notin \Gamma$
$\mathbb{D}^d_{2m}, m \geq 3$ odd		$\mathbb{D}^z_1$, $\mathbb{D}_1$
$\mathbb{D}^d_{2m}, m \geq 2$ even		$\mathbb{D}^z_1$
$\mathbb{D}^z_m, m \geq 3$ odd		$\mathbb{D}^z_1$
$\mathbb{D}_m \oplus \mathbb{Z}_2^c, m \geq 3$ odd	$\mathbb{D}^z_1$	
$\mathbb{Z}_m \oplus \mathbb{Z}_2^c, m \geq 4$ even	$\mathbb{Z}^-_{2k}, k$ even	

TABLE 5. Class II subgroups with $B \notin \Sigma$ for the planar subgroups of $\mathbf{O}(3)$ (k divides m)

separately and are obtained by applying Propositions 7.14 and 7.15 respectively. The class II subgroups are of type (i) or (iii) in Remark 10.3 and hence are admissible. This completes the proof of Theorem 1.4.

The information in Tables 3, 4 and 5 is combined to produce in Table 6 a list of the subgroups of class I and class II for the planar subgroups of $\mathbf{O}(3)$.

12. Strong admissibility of class II subgroups

Let Σ be a subgroup of a finite group $\Gamma \subset \mathbf{O}(n)$. Recall that Σ is *strongly admissible for diffeomorphisms* if there exists a Γ-equivariant diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with a connected Liapunov stable Σ-symmetric

$\mathbb{D}_{2m}^d, m \geq 3$ odd	Class I Class II	$\mathbf{1}$ $\mathbb{D}_1^z$, $\mathbb{Z}_2^-$, $\mathbb{D}_1$
$\mathbb{D}_{2m}^d, m \geq 2$ even	Class I Class II	$\mathbb{D}_1$, $\mathbf{1}$ $\mathbb{D}_1^z$, $\mathbb{D}_2$, $\mathbb{Z}_2$
$\mathbb{D}_m^z, m \geq 3$ odd	Class I Class II	$\mathbf{1}$ $\mathbb{D}_1^z$
$\mathbb{D}_m^z, m \geq 4$ even	Class I Class II	$\mathbf{1}$ $\mathbb{Z}_2$
$\mathbb{D}_2^z$	Class I Class II	$\mathbf{1}$ $\mathbb{Z}_2^-$, $\mathbb{Z}_2$
$\mathbb{D}_m \oplus \mathbb{Z}_2^c, m \geq 3$ odd	Class I Class II	$\mathbb{D}_1$, $\mathbf{1}$ $\mathbb{D}_1^z$, $\mathbb{D}_1 \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_2^c$
$\mathbb{D}_m \oplus \mathbb{Z}_2^c, m \geq 2$ even	Class I Class II	$\mathbf{1}$ $\mathbb{Z}_2^-$, $\mathbb{Z}_2^c$, $\mathbb{Z}_2$
$\mathbb{D}_m, m \geq 2$	Class I	$\mathbb{D}_k$, $\mathbb{Z}_k$
$\mathbb{Z}_{2m}^-, m \geq 1$ odd	Class I Class II	$\mathbb{Z}_k$ $\mathbb{Z}_{2k}^-$
$\mathbb{Z}_{2m}^-, m \geq 2$ even	Class I	$\mathbb{Z}_{2k}^-$ (m/k odd), $\mathbb{Z}_k$
$\mathbb{Z}_m \oplus \mathbb{Z}_2^c, m \geq 3$ odd	Class I	$\mathbb{Z}_k \oplus \mathbb{Z}_2^c$, $\mathbb{Z}_k$
$\mathbb{Z}_m \oplus \mathbb{Z}_2^c, m \geq 2$ even	Class I Class II	$\mathbb{Z}_k$ $\mathbb{Z}_{2k}^-$, $\mathbb{Z}_k \oplus \mathbb{Z}_2^c$
$\mathbb{Z}_m, m \geq 1$	Class I	$\mathbb{Z}_k$

TABLE 6. Admissible subgroups of the planar subgroups of $\mathbf{O}(3)$ (k divides m)

ω-limit set. By Theorem 1.5, subgroups Σ of class I are strongly admissible for diffeomorphisms provided $n \geq 3$. It remains to determine which subgroups of class II are strongly admissible.

Proposition 12.1. *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a finite group acting on $\mathbb{R}^n$, $n \geq 3$, and that $\Sigma \subset \Gamma$ is a subgroup of class II. Then Σ is strongly admissible for diffeomorphisms if and only if $\Sigma = \Delta \oplus \mathbb{Z}_2(\tau)$ where Δ is of class I and τ is a central reflection.*

Proof. Let Σ be a class II subgroup. Then Σ has an index two class I subgroup Δ fixing a connected component C of $\mathbb{R}^n \setminus L$. In addition, there is a Γ-equivariant involution $B \in \mathcal{B}$ such that $BC = \sigma C$ for all $\sigma \in \Sigma \setminus \Delta$.

Now suppose that Σ is strongly admissible and that A is a connected Liapunov stable Σ-symmetric ω-limit set. We show that Σ contains a reflection τ lying in the center of Γ. (In particular, we can take $B = \tau$.)

First observe that A intersects two connected components of $\mathbb{R}^n \setminus L$ and hence $A \cap L \neq \emptyset$. In particular, $A \cap \mathrm{Fix}(\tau) \neq \emptyset$ for some reflection $\tau \in \Sigma$. By Proposition 6.2, Σ contains the reflection τ. It follows from Theorem 8.3(a) that $\Sigma = \Delta \oplus \mathbb{Z}_2(\tau)$. Since Σ contains no further reflections, the connected components C and τC are adjacent. It follows from Lemma 7.6 that τ is central in R. By Theorem 8.3(a), τ commutes with elements of Γ_C. It follows from Proposition 7.9 that τ lies in the center of Γ.

Next we prove the converse. Suppose that Σ is of class II and has the form $\Delta \oplus \mathbb{Z}_2(\tau)$ where τ is a central reflection and $\Delta \subset \Gamma_C$ is of class I. We describe the construction of a connected Liapunov stable ω-limit point set. Our construction involves minor modifications to the "ribboned graph" construction of Section 5. First, note that the complete smooth Σ-graph $G(\Sigma)$ cannot be embedded in $\mathbb{R}^n$: each point in $G(\Sigma)$ has trivial isotropy but the embedded graph must intersect $\mathrm{Fix}(\tau)$. Instead, we start with the Δ-graph $G(\Delta)$. Choose any nontrivial element $\delta \in \Delta$ and introduce a new vertex v of degree two bisecting the edge J_δ that joins 1 to δ. If we consider all points $\{v_i\}$ in the orbit $\Delta \cdot v$ to be vertices then the resulting graph $G(\Delta)_0$ is a balanced smooth Δ-graph.

The idea is to embed the Δ-graph $G(\Delta)_0$ in $\mathbb{R}^n$ as a set A_0 with the image of the vertices $\{v_i\}$ lying in $\mathrm{Fix}(\tau)$ and the remaining points in the connected component C. Since τ is central, it follows from Lemma 7.6 that $\dim(C \cap \mathrm{Fix}(\tau)) = n-1$ and hence that $G(\Delta)_0$ can be embedded in the required manner. We identify the image of each vertex under the embedding with the vertex itself. In particular, $v_i \in A_0$ for each i. Construct the ribboned graph $\mathcal{Z} = G(\Delta)_0 \times (-1,1)$ as in Section 5. As before, the embedding of $G(\Delta)_0$ in $\overline{C}$ extends to an immersion of $\mathcal{Z}$ in $\overline{C}$. If E_i is the edge of $G(\Delta)$ containing v_i then we require that the ribbon $\mathcal{Z}_i = E_i \times (-1,1)$ is embedded so that $\mathcal{Z}_i \setminus \{v_i\} \subset C$ and so that $\mathcal{Z}_i$ has infinite order of tangency with $\mathrm{Fix}(\tau)$ at v_i.

Now we proceed as in Section 5 to define a vector field X on $\overline{C}$ with A_0 as a connected Δ-symmetric Liapunov stable ω-limit set consisting entirely of equilibria. Since X together with all of its derivatives approaches zero at the boundary of C, X extends to a smooth vector field on $\mathbb{R}^n$. Passing to the time one map as in the previous subsection yields a Γ-equivariant diffeomorphism ϕ_0 with A_0 as a Liapunov stable ω-limit set. Moreover A_0 is also a Liapunov stable ω-limit set for ϕ_0^2. It follows that the connected Σ-symmetric set $A = A_0 \cup \tau A_0$ is a Liapunov stable ω-limit set for the composition $\phi = \tau \circ \phi_0$. Since τ is central, ϕ is a Γ-equivariant diffeomorphism. □

Remark 12.2. We have found only degenerate constructions of connected ω-limit sets for subgroups of class II regardless of the size of n. This raises the question of whether there exist Γ-structurally stable equivariant diffeomorphisms which have connected ω-limit sets or attractors with class II symmetry groups. One way of attempting to construct such diffeomorphisms would be to allow the presence of Γ-transversal non-transversal intersections of invariant manifolds in the ω-limit sets. However, such intersections are likely to lead to moduli (see [13]). On account of this, we think it unlikely that there exist Γ-structurally stable equivariant diffeomorphisms which have connected attractors with class II symmetry groups. On the other hand, it should be noted that Labarca & Pacifico have constructed an example of a structurally stable non-Axiom A vector field on a manifold with boundary [20]. It follows easily from their construction that there exists a $\mathbb{Z}_2$-structurally stable vector field which is not Axiom A.

Whatever the situation for structural stability, it is simple to show that connected Axiom A attractors do not exist for subgroups of class II. Indeed, suppose that A is a connected Axiom A attractor (or more generally a connected basic set) for the equivariant diffeomorphism ϕ. Then A is topologically mixing under ϕ. In particular, A is transitive under all powers of ϕ. Suppose that A is Σ-symmetric, where Σ is of class II and contains the central reflection τ, then A intersects C and τC. Obviously, $f^2(A \cap C) \subset A \cap C$ contradicting transitivity.

Remark 12.3. Proposition 12.1 is stated for $n \geq 3$, but the restriction on strongly admissible subgroups of class II holds for all $n \geq 1$. In the reverse direction, there are some anomalous cases when $n = 1$ and $n = 2$. When $n = 1$ and $\Gamma = \mathbb{Z}_2$, then $\mathbb{Z}_2$ is of class II but is not strongly admissible for diffeomorphisms. (Note that diffeomorphisms of the line have ω-limit sets consisting of at most two points. Of course, the origin does not satisfy our definition for a $\mathbb{Z}_2$-symmetric set as there are no points with nontrivial isotropy.)

The remaining anomalous case arises when $n = 2$ and $\Gamma = \mathbb{Z}_m$. Since Γ contains no reflections, all subgroups are of class I. However, only the subgroups $\mathbb{Z}_m$ and $\mathbf{1}$ are strongly admissible (admissibility being realized by an irrational rotation on a normally hyperbolic circle and by a sink). As was the case for flows, Corollary 5.3, there is a topological obstruction for strong admissibility of the remaining subgroups $\mathbb{Z}_k$, $1 < k < m$, k dividing m.

Appendix A. Appendix

In Section 11, we sketched the derivation of the class I and class II subgroups for the planar subgroups of $\mathbf{O}(3)$. As promised we give the details when $\Gamma = \mathbb{D}^d_{2m}$.

The group $\mathbb{D}^d_{2m}$ is generated by $-I \cdot \pi/m$ and κ where π/m, $\kappa \in \mathbf{SO}(3)$ have the matrix representations

$$\pi/m = \begin{pmatrix} \cos\pi/m & -\sin\pi/m & 0 \\ \sin\pi/m & \cos\pi/m & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \kappa = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The index two subgroup generated by $2\pi/m$ and $-I \cdot \pi/m \cdot \kappa$ is denoted by $\mathbb{D}^z_m$ and acts as $\mathbb{D}_m$ on the (x, y)-plane and trivially on the z-axis. In particular, $\mathbb{D}^z_m$ is generated by reflection. It follows that $\mathbb{D}^d_{2m}$ is generated by reflections if and only if $\mathbb{D}^d_{2m} \setminus \mathbb{D}^z_m$ contains a reflection. Excluding elements of $\mathbf{SO}(3)$ the remaining elements of $\mathbb{D}^d_{2m}$ are given by $-I \cdot j\pi/m$ for j odd. Such an element is a reflection only for $j = m$. hence if m is odd, $\mathbb{D}^d_{2m}$ is generated by reflections and there are no nontrivial class I subgroups. If m is even, the subgroup $R \subset \mathbb{D}^d_{2m}$ generated by reflections is $\mathbb{D}^z_m$.

Next observe that $\mathbb{D}^d_{2m}$ is isomorphic to $\mathbb{D}_{2m}$ and hence contains up to conjugacy three subgroups of order two. These are generated by $(-I)^m \cdot \pi$, $-I \cdot \pi/m \cdot \kappa$ and κ. the first subgroup is denoted by $\mathbb{Z}^-_2$ if m is odd and by $\mathbb{Z}_2$ if m is even. The remaining subgroups are $\mathbb{D}^z_1$ and $\mathbb{D}_1$ respectively. Although some of these subgroups are conjugate in $\mathbf{O}(3)$, they are not conjugate in $\mathbb{D}^d_{2m}$.

The reflection-free subgroup $\mathbb{D}_1$ fixes the y-axis in $\mathbb{R}^3$ and is easily checked to be an isotropy subgroup when m is even. Hence for m even, the class I subgroups are given by $\mathbb{D}_1$ and $\mathbf{1}$. This completes the verifications of the entries in Table 3.

Since $m \geq 2$ we have the set of equivariant involutions $\mathcal{B} = \{-I, \pm J\}$ where $J = \pi$. The unique central involution is $(-1)^m J$. The subgroups of class II in Table 4 are easily computed as direct sums of the form $\Delta \oplus \mathbb{Z}_2((-1)^m J)$.

Finally, we verify the entries in Table 5. First we show using Proposition 7.14 that there are subgroups of class II with $B \in \Gamma$. This is immediate when m is odd since there are no class I subgroups of even order. When m is even, we are forced to choose $B = \pi$ and $\Sigma_B = \mathbb{D}_1 = \mathbb{Z}_2(\kappa)$. Hence $\Sigma = \mathbb{Z}_2(\pi\kappa)$ which is conjugate to the class I subgroup $\mathbb{D}_1$.

It remains to compute the subgroups of class II for which $B \notin \Gamma$ using Proposition 7.15. For all choices of B we have $\Gamma \oplus \mathbb{Z}_2(B) = \mathbb{D}_{2m} \oplus \mathbb{Z}^c_2$.

The nontrivial isotropy subgroups $J \subset \mathbb{D}_{2m} \oplus \mathbb{Z}_2^c$ are listed for example in [5] and are

$$\mathbb{D}_{2m} \oplus \mathbb{Z}_2^c,\ \mathbb{D}_m^z,\ \mathbb{D}_2^d,\ \mathbb{D}_1^z,\ \mathbb{Z}_2^-.$$

We need only consider those isotropy subgroups J that contain no reflections in $\mathbb{D}_{2m}^d$, hence we can discard $\mathbb{D}_{2m} \oplus \mathbb{Z}_2^c$ and $\mathbb{D}_m^z$. The subgroups $\mathbb{D}_2^d$ and $\mathbb{Z}_2^-$ contain the reflection $-I \cdot \pi$ which lies in $\mathbb{D}_{2m}^c$ when m is odd. Hence, when m is odd we must take $\Sigma_B = \mathbb{D}_1^z = \mathbb{Z}_2(-\kappa)$. The involutions $B \in \mathcal{B}$ that do not lie in $\mathbb{D}_{2m}^d$ are $B = -I$ and $B = \pi$. The corresponding subgroups $\Sigma = (\Sigma_B)_B$ are

$$\Sigma = \mathbb{Z}_2(\kappa) = \mathbb{D}_1, \qquad \Sigma = \mathbb{Z}_2(-\pi\kappa) = \mathbb{D}_1^z.$$

This yields the required subgroups of class II.

It is easily checked that the isotropy subgroups $J = \mathbb{D}_1^z$ and $J = \mathbb{Z}_2^-$ yield no new class II subgroups when m is even. This leaves the case $J = \mathbb{D}_2^d$ which is generated by $-I \cdot \pi$ and κ.Since we have already discarded the possibilities $\Sigma_B = \mathbb{D}_1^z$ and $\Sigma_B = \mathbb{Z}_2^-$ the only subgroups $\Sigma_B \subset J$ to consider are

$$\Sigma_B = \mathbb{D}_2^d, \qquad \Sigma_B = \mathbb{D}_1 = \mathbb{Z}_2(\kappa).$$

In the second case the choices $B = -I$ and $B = -I \cdot \pi$ yield $\Sigma = \mathbb{Z}_2(-\kappa)$ and $\Sigma = \mathbb{Z}_2(-\pi\kappa)$ both of which are conjugate to $\mathbb{D}_1^z$. When $\Sigma_B = \mathbb{D}_2^d$ we must take $\Delta = \mathbb{Z}_2(\kappa) = \mathbb{D}_1$. If $B = -I$, Σ is generated by κ and π so we obtain $\Sigma = \mathbb{D}_1$. Nothing is obtained by taking $B = -I \cdot \pi$.

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