PATTERNS OF DESYNCHRONIZATION AND RESYNCHRONIZATION IN HETEROCLINIC NETWORKS

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ABSTRACT. We prove results that enable the efficient and natural realization of a large class of robust heteroclinic networks in coupled identical cell systems. We also propose some general conjectures that relate a natural and large class of robust heteroclinic networks that occur in networks modelled by equations of Lotka-Volterra type, and certain networks of symmetric systems, to robust heteroclinic networks in coupled cell networks.

Contents

1. Introduction

This work is about the realization of heteroclinic networks and cycles in networks of coupled dynamical systems. Since the definition of a heteroclinic network (or cycle) varies in the literature, we start by briefly reviewing some standard terminology and notation and then give the precise definition of a heteroclinic network that is appropriate for our intended applications.

Let M be a differential manifold and X be a smooth (at least C^1) vector field on M with associated flow Φ. Let $\mathbf{E} = \{\mathbf{p}_i \mid i \in I\}$ be a finite set of hyperbolic saddle points for X. For all $\mathbf{p} \in \mathbf{E}$, let $W^s(\mathbf{p}), W^u(\mathbf{p})$ respectively denote the (global) stable and unstable manifolds of $\mathbf{p}$. If $\mathbf{p}, \mathbf{q} \in \mathbf{E}$, a *connection* from $\mathbf{p}$ to $\mathbf{q}$ will be a Φ-trajectory $\phi : \mathbb{R} \rightarrow M$

such that $\phi(\mathbb{R}) \subset W^u(\mathbf{p}) \cap W^s(\mathbf{q})$. We denote the connection by $\mathbf{p} \overset{\phi}{\to} \mathbf{q}$. We always assume there are no self-connections (homoclinic loops) and so $W^s(\mathbf{p}) \cap W^u(\mathbf{p}) = \{\mathbf{p}\}$, all $\mathbf{p} \in \mathbf{E}$. If we let $\mathbf{C}^\star = \{\phi^\alpha \mid \alpha \in J^\star\}$ denote the set of all connections between equilibria in $\mathbf{E}$, then

$$\bigcup_{\mathbf{p},\mathbf{q} \in \mathbf{E}, \mathbf{p} \neq \mathbf{q}} W^u(\mathbf{p}) \cap W^s(\mathbf{q}) = \bigcup_{\alpha \in J^\star} \phi^\alpha(\mathbb{R}).$$

If $\mathbf{C} \subset \mathbf{C}^\star$, define $\Phi(\mathbf{C}) = \cup_{\phi \in \mathbf{C}}\phi(\mathbb{R}) \subset M$.

Definition 1.1. (Notation and assumptions as above.) A connected Φ-invariant subset Σ of M is a *heteroclinic network*, with *equilibrium set* $\mathbf{E}$ and *connection set* $\mathbf{C}$, if

(a) If $\mathbf{C} = \{\phi^\alpha \mid \alpha \in J\}$ is a subset of $\mathbf{C}^\star$.
(b) $\Sigma = \mathbf{E} \cup \Phi(\mathbf{C})$.
(c) For all ordered pairs $\mathbf{p}, \mathbf{q} \in \mathbf{E}$, there is a sequence

(1.1) $$\mathbf{p} \overset{\phi^{\alpha_0}}{\longrightarrow} \mathbf{s}_1 \overset{\phi^{\alpha_1}}{\longrightarrow} \mathbf{s}_2 \overset{\phi^{\alpha_3}}{\longrightarrow} \cdots \overset{\phi^{\alpha_{k-1}}}{\longrightarrow} \mathbf{s}_k \overset{\phi^{\alpha_k}}{\longrightarrow} \mathbf{q}$$

of connections, where $k = k(\mathbf{p}, \mathbf{q}) \geq 1$, the $\mathbf{s}_i$ are distinct equilibria in $\mathbf{E} \smallsetminus \{\mathbf{p}, \mathbf{q}\}$, and each connection $\phi^{\alpha_j} \in \mathbf{C}$.

If Σ contains equal numbers of equilibria and connections, then Σ is a *heteroclinic cycle*.

Remarks 1.2. (1) Taking $\mathbf{p} = \mathbf{q}$ in (1.1), we see that every equilibrium point $\mathbf{p} \in \mathbf{E}$ lies on a heteroclinic cycle. Consequently, every heteroclinic network is a union of heteroclinic cycles.

(2) If $\mathbf{p} \overset{\phi}{\longrightarrow} \mathbf{q}$, then $\mathbf{p} \neq \mathbf{q}$ by our standing assumption that there are no homoclinic loops. If $\dim(W^u(\mathbf{p})) = 1$, then $\phi(\mathbb{R})$ is a connected component of $W^u(\mathbf{p}) \smallsetminus \{\mathbf{p}\}$. If $\dim(W^u(\mathbf{p})) = 1$, for all $\mathbf{p} \in \mathbf{E}$, then Σ is a *simple* heteroclinic network (or cycle). Necessarily, Σ is compact and if $\mathbf{C} = \mathbf{C}^\star$, $\Sigma = \cup_{\mathbf{p} \in \mathbf{E}}W^u(\mathbf{p})$. In many applications, M will have boundary (for example, be a simplex) and $W^u(\mathbf{p}) \smallsetminus \{\mathbf{p}\}$ may only contain a single trajectory. If Σ is not simple, and $\mathbf{C}$ is not finite, then Σ need not be compact. ⊛

Definition 1.1 does not require that Σ contains *all* the connections between equilibria in $\mathbf{E}$. Define $\Sigma(\mathbf{E}) \subset M$ by

(1.2) $$\Sigma(\mathbf{E}) = \mathbf{E} \cup \Phi(\mathbf{C}^\star) = \bigcup_{\mathbf{p},\mathbf{q} \in \mathbf{E}} W^u(\mathbf{p}) \cap W^s(\mathbf{q}).$$

Obviously, $\Sigma(\mathbf{E})$ is a heteroclinic network if and only if there is a finite subset $\mathbf{C} \subset \mathbf{C}^\star$ for which $\mathbf{E} \cup \Phi(\mathbf{C})$ is a heteroclinic network. If $\Sigma(\mathbf{E})$ is a heteroclinic network, then it is the *maximal* heteroclinic network

with equilibrium set $\mathbf{E}$. Without further conditions, $\Sigma(\mathbf{E})$ may not be compact and the dynamics of $\Phi|\partial\Sigma(\mathbf{E})$ may be very complex.

If $\Sigma(\mathbf{E})$ is simple, then $\mathbf{C}^\star$ is finite, Σ is compact and $\Sigma = \cup_{\mathbf{p}\in\mathbf{E}}W^u(\mathbf{p})$. For general heteroclinic networks Σ, these conditions do not hold. We say that the heteroclinic network Σ is *clean* if Σ is compact and $\Sigma = \bigcup_{\mathbf{p}\in\mathbf{E}} W^u(\mathbf{p})$. If Σ is clean then

$$(1.3) \qquad \Sigma \ \subset \ \bigcup_{\mathbf{p}\in\mathbf{E}} W^s(\mathbf{p})$$

$$(1.4) \qquad \Sigma \ = \ \Sigma(\mathbf{E}) = \bigcup_{\mathbf{p},\mathbf{q}\in\mathbf{E},\mathbf{p}\neq\mathbf{q}} \overline{W^u(\mathbf{p}) \cap W^s(\mathbf{q})}.$$

It is easy to see that a heteroclinic network Σ can be realized as an asymptotically stable attractor if and only if Σ is clean.

We refer to [26, §2] for a general review of heteroclinic cycles and networks and their applications. Our interest here is in *robust* heteroclinic cycles and networks. On account of the Kupka-Smale theorem [51], heteroclinic cycles and networks can only be robust, that is persist under all sufficiently small C^1 perturbations of the associated vector field, if we work within a class of vector fields with additional structure. Invariably this structure is associated with the presence of invariant subspaces. Robust heteroclinic networks and cycles are well-known phenomena in models of population dynamics, ecology and game theory based on the Lotka-Volterra equations (for example, [44, 34, 35, 36, 37]). Typically, these systems are defined on a simplex, or the positive orthant $O_n = \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \cdots, n\}$ of $\mathbb{R}^n$, and have the 'extinction' hyperplanes $x_i = 0$ as invariant subspaces. Heteroclinic networks and cycles also occur robustly in differential equations which are *equivariant* with respect to a compact Lie group of symmetries (for example, [53, 46, 41, 27, 42, 40, 10, 21, 25]). In this case, robustness can occur because generic intersections of stable and unstable manifolds of equilibria in equivariant dynamics need not be transverse but can nonetheless be stable under sufficiently smooth equivariant perturbation of the underlying vector field [17, 18, 19].

If a heteroclinic network Σ contains a homoclinic loop (or self connection), then Σ cannot be robust by the Kupka-Smale theorem; this is the main reason we deny self-loops in definition 1.1.

In this article we make a careful exploration of the relationship between robust heteroclinic networks in semilinear feedback systems and coupled identical cell systems (see sections 2 and 3 for formal definitions). Semilinear feedback systems constitute a large class of network

models, including Lotka-Volterra systems and some equivariant systems, for which there is a substantial body of results on robust heteroclinic networks and cycles. From a mathematical and conceptual point of view, the theory is relatively elementary; in particular, it is straightforward to construct examples of clean heteroclinic networks. On the other hand, less is known about heteroclinic networks, or even heteroclinic cycles, in coupled systems of identical cells [57, 31, 30]. This is an unfortunate gap as coupled identical cell systems form a potentially attractive class of models (see below).

In previous work [26], it was shown that every strongly connected directed graph Γ with q edges and no self-loops could be realized as the graph of a robust heteroclinic network in a coupled system of $q + 1$ identical cells with 1-dimensional cell dynamics. In this paper, we show how this realization theorem is one case of a general correspondence between heteroclinic networks in semilinear feedback systems and coupled identical cell systems. Our main result, theorem 4.10, gives an efficient and natural realization of a large class of robust heteroclinic networks that occur in semilinear feedback systems as robust heteroclinic networks in coupled identical cell systems. These realizations are embedded in identical cell networks with close to the minimal possible number of identical cells, unlike in the realization theorem given in [26] (see also [11]). They have the attractive feature that each connection in the realization corresponds to a unique pattern of desynchronization and resynchronization along the connecting trajectory. We refer to [6, 5] for more examples and background on heteroclinic networks in coupled cell systems.

In [11], Ashwin and Postlethwaite consider the problem of realizing graphs as robust heteroclinic networks, though not in coupled identical cell networks. Their concept of a 'simplex network' realization is closely related to what we call an *edge* network (or cycle) in section 2 (see also [21]). In recent work [12], Ashwin and Postlethwaite have investigated the realization of graphs as robust heteroclinic networks in a class of networks with two cell types – though not a coupled cell network in the sense of Golubitsky *et al.* [30].

There has been significant recent interest in potential applications of robust heteroclinic networks to neural microcircuits and it has been suggested that they provide a useful model to explain the function of certain neural systems [52, 49]. Specifically, heteroclinic networks and cycles model "winnerless competition" where there is a local competition between different states but not necessarily a global winner. The models used in these works are based on systems (networks) of generalized Lotka-Volterra equations [2, 48] and part of the motivation for

the present article is to provide models based on a natural identical cell model. Our results go some way towards achieving this objective and identify some of the obstructions to obtaining realistic models based on fewer connections and more symmetry in an (additive) input structure [26]. We address some of these issues in more detail later and especially in the concluding comments. In general terms, robust heteroclinic phenomena seem particularly useful for explaining sequence generation and spatio-temporal encoding and have been found in rate-based [1] models, Hodgkin-Huxley-based models [33] and more general phase oscillator models [9] where they have been used to perform finite-state computations [8].

We conclude with a brief description of the contents of the work by section. In section 2 we review the theory of semilinear feedback systems – a large class of networks that includes Lotka-Volterra models – and describe the less well-known class of face heteroclinic cycles and networks [21]. In section 3, we review that part of the theory of coupled identical cell systems applicable to cells with asymmetric inputs and strongly connected networks without self loops. In section 4 we introduce the idea of a *synchronization transform.* This constitutes the basis of our method of going from heteroclinic face networks for a semilinear feedback system to a heteroclinic network for a coupled cell system. We pose two conjectures concerning the scope of the synchronization transform and prove a number of associated results, notably theorem 4.10. Section 5 is devoted to illustrative examples and includes part of the verification of the realization conjectures for identical cell networks containing three or four cells. We conclude the section with two examples illustrating the realization of robust heteroclinic networks in coupled identical cell systems. In section 6, we discuss a number of outstanding questions as well as comment on the possibility of obtaining more physically realistic models.

2. Semilinear Feedback systems

Semilinear Feedback systems (*SLF* systems) are a large, simple and transparent class of network models that naturally support many different types of robust heteroclinic cycles and networks. Well-known examples include Lotka-Volterra equations and some classes of equivariant systems. Parts of what we discuss here appear in [21, chapters 6 & 7] and [22] – though the terminology 'semilinear feedback system' was introduced later by the author (for example, [24]). Before we give our main definition, we need to establish some notational conventions.

Let $\mathbb{N}$ denote the natural numbers (the strictly positive integers) and $\mathbb{Z}_+$ denote the set of nonnegative integers. Let $\mathbb{R}$ denote the real numbers and $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$. Given $k \in \mathbb{N}$, define $\mathbf{k} = \{1, \dots, k\}$, and $\mathbf{k}^\bullet = \{0, 1, \dots, k\}$. For clarity, we sometimes use an overline, rather than brackets; for example, $\overline{\mathbf{k} - 1}$ rather than $(\mathbf{k} - 1)$ or $\mathbf{k} - \mathbf{1}$.

Definition 2.1. An SLF system is a network modelled by differential equations

$$(2.5) \qquad \dot{\mathbf{x}}_i = f_i(\mathbf{x}_i) + A_i(\mathbf{x}_i, F_i(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_k)), \ i \in \mathbf{k},$$

where $\mathbf{x}_i \in \mathbb{R}^n$, $A_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a bilinear form, and $f_i(0) = 0$, $i \in \mathbf{k}$.

Remark 2.2. The simplest examples of SLF systems, and those that we focus on in this article, occur when $n = 1$ and A_i is multiplication, $i \in \mathbf{k}$. That is,

$$(2.6) \qquad \dot{x}_i = f_i(x_i) + x_i F_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k), \ i \in \mathbf{k}.$$

Examples where $n > 1$ appear in [21]. $\qquad \qquad \qquad \qquad \qquad \qquad \circledast$

The equation $\dot{\mathbf{x}}_i = f_i(\mathbf{x}_i)$ defines the *intrinsic dynamics* of the *ith* node, $i \in \mathbf{k}$. If $f_1 = \dots = f_k$, the nodes have *identical* intrinsic dynamics. We usually assume identical intrinsic dynamics but that assumption is not needed for our main results. Systems of the form (2.6) often have additive input structure [26] – for example, if F_i is linear.

Henceforth assume $n = 1$. We regard (2.6) as equations for a network $\mathcal{N} = \{N_1, \dots, N_k\}$ consisting of k coupled nodes, $N_1, \dots, N_k$, where the node N_i has state x_i and phase space R_i where R_i is either $\mathbb{R}$ or $\mathbb{R}_+$ (we assume the same choice for all $i \in \mathbf{k}$). Let $\mathbf{R} = \prod_{i \in \mathbf{k}} R_i$ denote the *network phase space*. If the node phase space is $\mathbb{R}_+$, then $\mathbf{R}$ is the positive orthant O_k of $\mathbb{R}^k$. Observe that since $x_i = 0$ is flow-invariant for SLF systems, all $i \in \mathbf{k}$, so also is O_k.

For $i \in \mathbf{k}$, define $H_i = \{\mathbf{x} \in \mathbf{R} \mid x_i = 0\}$, and let $\mathcal{I}_k$ denote the set of all intersections $H_{i_1} \cap \dots \cap H_{i_p} \stackrel{\text{def}}{=} H_{i_1 \dots i_p}$. We assume $\mathbf{R} \in \mathcal{I}_k$ and note that $H_{1 \dots k} = \{(0, \dots, 0)\}$. Every $V \in \mathcal{I}_k$ is flow-invariant for an SLF system (since this is so for H_i, $i \in \mathbf{k}$).

Examples 2.3 (Examples of SLF systems). (1) A (generalized) Lotka-Volterra system is defined on the positive orthant $O_k = \mathbb{R}_+^k$ by

$$(2.7) \qquad \qquad \dot{x}_i = x_i G_i(\mathbf{x}) = f_i(x_i) + x_i F_i(\mathbf{x}), \ i \in \mathbf{k},$$

where $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$, $f_i(x_i) = x_i(a_i + b_i x_i)$, $F_i(\mathbf{x}) = \sum_{j \neq i} a_{ij} x_j$, and $a_i, b_i, a_{ij} \in \mathbb{R}$, $i, j \in \mathbf{k}$, We refer to [44, 34, 35, 36, 37] for explicit examples and applications.

(2) The cubic truncation of an equivariant system on $\mathbb{R}^k$ with symmetry group G satisfying $\mathbb{Z}_2^k \subset G \subset \mathbf{H}_k$, where $\mathbb{Z}_2^k$ is the group of orthogonal diagonal $k \times k$-matrices and $\mathbf{H}_k = \mathbb{Z}_2^k \rtimes S_k$ is the hyperoctahedral group of symmetries of the k-dimensional hypercube. For $i \in \mathbf{k}$, $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, we have

$$
\begin{aligned}
f_i(x_i) &= a_i x_i + b_i x_i^3, \\
F_i(\mathbf{x}) &= \sum_{j \neq i} a_{ij} x_j^2,
\end{aligned}
$$

We refer to [21, chapter 3] or [25, §4.5] for more details when G acts absolutely irreducibly on $\mathbb{R}^k$. ※

Remark 2.4. Every $\mathbb{Z}_2^k$-equivariant cubic truncation restricted to O_k can be transformed into a Lotka-Volterra system under the invertible transformation $x_i^2 \leftrightarrow u_i$ of the positive orthant O_k. In practice, for Lotka-Volterra systems, it is common to satisfy the constraint $u_1 + \ldots + u_k = 1$ and dynamics is naturally defined on the $(k-1)$-simplex in O_k defined by $u_1 + \ldots + u_k = 1$ using the replicator equation $\dot{u}_i = u_i(G_i(\mathbf{x}) - \overline{G}(\mathbf{x}))$, $i \in \mathbf{k}$, where $\overline{G}(\mathbf{x}) = \sum_{i \in \mathbf{k}} x_i G_i(\mathbf{x})$. Analogously, in the equivariant case, given a cubic truncation, dynamics is uniquely defined on the (spherical) simplex $\boldsymbol{\Delta}_{k-1} = S^{k-1} \cap O_k$ using the phase vector field (see [25, Chapters 4,5] and section 2.2 below). Dynamics on the simplex and spherical simplex $\boldsymbol{\Delta}_{k-1}$ are conjugate by the transformation $x_i^2 \leftrightarrow u_i$. Either viewpoint has its advantages; our preference is for working on the spherical simplex as this works well for bifurcation theory [25, Chapter 5]. ⊛

2.1. **The SLF model, bistability and heteroclinic cycles.** In figure 1 we show a single node with 1-dimensional dynamics given by

$$
\dot{x} = f(x) + xF(\mathbf{u}).
$$

If we regard $\mathbf{u}$ as a *control* variable, then feedback is linear in the node variable x. Using the control variable $\mathbf{u}$, we can easily switch the cell between two different attracting states. For example, suppose $f(x) = x - x^2$ and $F(\mathbf{u}) = \mathbf{u}$, $\mathbf{u} \in \mathbb{R}$. If $\mathbf{u} = 0$, then $x = 1$ is the unique asymptotically stable state of $\dot{x} = f(x) + xF(\mathbf{u})$. On the other hand, if $\mathbf{u} = -2$, then the unique asymptotically stable state of $\dot{x} = f(x) + xF(\mathbf{u})$ occurs at $x = 0$.

Now consider a network $\mathcal{N}$ built from nodes of this type. If we suppose the control variable for each node $N_i \in \mathcal{N}$ depends on the states of the remaining nodes in a network then it is easy to choose controls so that nodes repeatedly switch states in a fixed order.
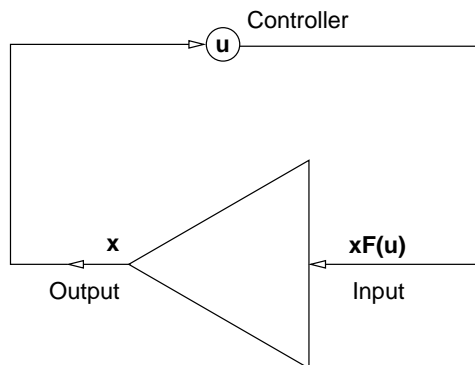
FIGURE 1. A single node with feedback loop

Example 2.5. Suppose that $k = 3$ and equations are given by

$$(2.8) \qquad \dot{x}_i = x_i(1 - x_i^2) - 3x_i x_{i+1}^2, \ i \in \mathbf{3},$$

where $x_{3+1} \overset{\text{def}}{=} x_1$ and $F_i(\mathbf{x}) = 3x_{i+1}^2$, $i \in \mathbf{3}$. The intrinsic dynamics of the ith node is $\dot{x}_i = x_i(1 - x_i^2)$ and $x_i = 1$ the unique asymptotically stable equilibrium in O_3. If we add in the inhibitory feedback term $-3x_i x_{i+1}^2$, then if $x_{i+1} = 1$, dynamics for the ith node is $\dot{x}_i = -2x_i - x_i^3$ and there is a unique asymptotically stable equilibrium at $x_i = 0$.

We regard a node N_i as being *active* if its state is close to 1, and *quiescent* if its state is close, but not equal, to zero. The node is growing if its state is increasing, and damped if its state is decreasing. Let $\mathbf{x} \in O_3$ be close to $(1, 0, 0)$ with $x_2, x_3 \neq 0$ (so N_1 is active, and N_2, N_3 are quiescent). Node N_2 will be be inhibiting N_1 weakly (small negative feedback), and node N_1 will be inhibiting N_3 strongly (x_1^2 is close to 1). On the other hand, N_2 will not be inhibited by N_3 (because N_3 is quiescent and growing smaller and the intrinsic dynamics will dominate the small negative feedback term). As N_2 grows it will start to inhibit N_1 strongly and eventually drive N_1 close to zero. At the same time N_2 grows towards 1. Once N_1 is close enough to zero, it stops inhibiting N_3 which can then grow and eventually inhibit N_2 and so on.

This mechanism leads to the formation of heteroclinic cycles. Indeed the system (2.8) has a simple asymptotically stable heteroclinic cycle Σ contained in a unique attractive flow invariant 2-sphere [25, Chapter 5, §2]. In the sequel we refer to this cycle as the *rock-paper-scissors*, or *RPS*, heteroclinic cycle. We refer to [44] where this heteroclinic cycle was first described (in the context of Lotka-Volterra equations) and to [53, 21, 32] for examples occurring in symmetric systems. ※

2.2. **Heteroclinic cycles and networks in SLF systems.** We start by defining the graph of a heteroclinic network with finite connection set. and then describe the theory of heteroclinic networks in SLF systems.

Heteroclinic networks: basic formalism and definitions. Suppose that Σ is a heteroclinic network with equilibrium set $\mathbf{E} = \{\mathbf{p}_i \mid i \in I\}$ and finite connection set $\mathbf{C} = \{\phi^\alpha \mid \alpha \in J\}$. The *directed graph* $\Gamma = \Gamma(\Sigma)$ of Σ is defined to have vertex set $\mathcal{V} = \{v_i \mid i \in I\}$ and edges $\mathcal{E} = \{e_\alpha \mid \alpha \in J\}$, where e_α is a directed edge $v_j{\rightarrow}v_i$ if ϕ^α is a connection from $\mathbf{p}_j$ to $\mathbf{p}_i$ (we allow multiple edges between two vertices). It follows from definition 1.1 that Γ is strongly connected and without self-loops.

If $v \in \mathcal{V}$, let d_v^{out} denote the *out-degree* of v: the number of edges in $\mathcal{E}$ which connect v to the remaining vertices in $\mathcal{V}$. If v corresponds to the equilibrium $\mathbf{p} \in \mathbf{E}$, define $d_{\mathbf{p}}^{\mathrm{out}} = d_v^{\mathrm{out}}$.

We will mainly be interested in heteroclinic networks $\Sigma \subset \mathbf{\Delta}_k$ for which $\mathbf{C} = \{\phi^\alpha \mid \alpha \in J\}$ is finite and

$$(2.9) \qquad d_{\mathbf{p}}^{\mathrm{out}} = \dim(W^u(\mathbf{p})), \text{ for all } \mathbf{p} \in \mathbf{E}.$$

If (2.9) holds and Σ is not simple, the set of connections $\mathbf{C}$ will typically be a *skeleton* or *framework* for the maximal heteroclinic network $\Sigma(\mathbf{E})$ (1.2). This viewpoint turns out to be natural for our discussion of heteroclinic networks in SLF systems. Of course, the way in which we choose $\mathbf{C}$ will be important. In many situations, a heteroclinic network will be robust only if the connection set $\mathbf{C}$ is finite (and well chosen).

2.3. **Reduction to a simplex.** We start by considering the well-studied [27, 25] class of SLF systems modelled by equations of the form

$$(2.10) \qquad \dot{x}_i = f_i(x_i) + x_i \left(\sum_{j \neq i} a_{ij} x_j^2 \right), \ i \in \mathbf{k},$$

where $f_i(x) = x - b_i x^3$, $b_i > 0$. The resulting system is $\mathbb{Z}_2^k$-equivariant with fundamental domain the positive orthant O_k. All our constructions will be within O_k (and so results relate directly to Lotka-Volterra systems by remark 2.4). Aside from working on O_k and using the invariance of the subspaces $\mathcal{I}_k$, we make no use of symmetry properties of the system. Indeed, it will generally be the case that there are no symmetries in the matrix $A = [a_{ij}]$ of coupling coefficients and no permutation symmetries of the network (no assumptions on the intrinsic

dynamics beyond $f_i(0) = 0$. We now use a simplification exactly analogous to the reduction of a Lotka-Volterra system to the simplex (see remark 2.4).

Write (2.10) in vector form as

$$(2.11) \qquad \dot{\mathbf{x}} = \mathbf{x} + \mathbf{Q}(\mathbf{x}), \ \mathbf{x} \in \mathbb{R}^k,$$

where the components Q_i of the vector field $\mathbf{Q}$ are given by $Q_i(\mathbf{x}) = -b_i x_i^3 + x_i(\sum_{j \neq i} a_{ij} x_j^2)$, $i \in \mathbf{k}$. Note that $\mathbf{Q} : \mathbb{R}^k \to \mathbb{R}^k$ is a homogeneous cubic polynomial map of $\mathbb{R}^k$. Associated to the system (2.11), we define a vector field $\mathcal{P}_Q$ on the unit sphere S^{k-1} in $\mathbb{R}^k$ by

$$\mathcal{P}_Q(\mathbf{u}) = \mathbf{Q}(\mathbf{u}) - (\mathbf{Q}(\mathbf{u}), \mathbf{u})\mathbf{u}, \ \mathbf{u} \in S^{k-1},$$

where $(\, , \,)$ denotes the standard inner product on $\mathbb{R}^n$. (The vector field $\mathcal{P}_Q$ is called the *phase vector field* in [20, 25].) The dynamics of $\mathcal{P}_Q$ encodes the nontrivial dynamics of the original system (2.11): every nonzero trajectory in $\mathbb{R}^k$ of (2.11) is the setwise lift of a $\mathcal{P}_Q$-trajectory on S^{k-1}. Moreover, by the invariant sphere theorem [20, 25], if we define $\mathbf{Q}_a(\mathbf{x}) = \mathbf{Q}(\mathbf{x}) - a\|\mathbf{x}\|^2\mathbf{x}$, $a \in \mathbb{R}$, then $\mathcal{P}_Q = \mathcal{P}_{Q_a}$ and, for sufficiently large a, $\dot{\mathbf{x}} = \mathbf{x} + \mathbf{Q}_a(\mathbf{x})$ has an invariant attracting $k - 1$-sphere $S(a)$ such that the dynamics on $S(a)$ is conjugate to the dynamics of $\mathcal{P}_Q$ on S^{k-1}. In particular, every robust heteroclinic cycle or network for $\mathcal{P}_Q$ will uniquely determine a robust heteroclinic cycle or network for $\dot{\mathbf{x}} = \mathbf{x} + \mathbf{Q}_a(\mathbf{x})$ for sufficiently large a. Conversely, every robust heteroclinic cycle or network for $\dot{\mathbf{x}} = \mathbf{x} + \mathbf{Q}(\mathbf{x})$ can be realized as a robust heteroclinic cycle or network for $\mathcal{P}_Q$.

More generally, suppose that Ψ is a smooth flow on S^{k-1} that preserves the invariant subspace structure induced by $\mathcal{I}_k$ (and so Ψ preserves the spherical simplex $\mathbf{\Delta}_{k-1} = S^{k-1} \cap O_k$). It is easy to show that there exists a smooth SLF system on $\mathbb{R}^k$, with flow Φ and the origin a source (as in (2.11)), such that S^{k-1} is Φ-invariant and attracting with $\Phi|S^{k-1}$ conjugate to Ψ. Of course, Φ need not be given by a cubic polynomial vector field.

As a consequence of these observations, we focus on the study of robust heteroclinic phenomena for smooth vector fields on S^{k-1} that preserve the invariant subspace structure determined by $\mathcal{I}_k$. Generally, we restrict to the spherical simplex $\mathbf{\Delta}_{k-1}$. This is no loss of generality if we have $\mathbb{Z}_2^k$-equivariance: $\mathbf{\Delta}_{k-1}$ is a fundamental domain for the $\mathbb{Z}_2^k$-action on S^{k-1}.

2.4. **Edge and face heteroclinic cycles and networks.** Assume $k \geq 3$. Let $r \in \overline{\mathbf{k-2}}^\bullet$. If $V \in \mathcal{I}_k$ is $r + 1$-dimensional, then $\mathbf{\Delta}_{k-1} \cap V$ is an *r-face* of $\mathbf{\Delta}_{k-1}$. In particular, if $r = 0$ then V is a coordinate

axis and $\mathbf{\Delta}_{k-1} \cap V$ is a *vertex*, and if $r = 1$, $\mathbf{\Delta}_{k-1} \cap V$ is an *edge*. We remark that if F is an r-face, then ∂F is a union of $r - 1$-faces. In particular, $\partial\mathbf{\Delta}_{k-1}$ is the union of all $k - 2$-faces. In coordinates, the vertices of $\mathbf{\Delta}_{k-1}$ are given by the set $\mathcal{V}(k) = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, where $\mathbf{v}_j$ is the positive unit vector along the coordinate axes x_j of $\mathbb{R}^k$, $j \in \mathbf{k}$. Every edge is uniquely determined by two vertices, every 2-face by three vertices and so on. It is easy to verify that there are $\binom{k}{r+1}$ r-faces, $r \in \overline{\mathbf{k-2}}^{\,\bullet}$. If $F = \mathbf{\Delta}_{k-1} \cap V$ is an r-face of $\mathbf{\Delta}_{k-1}$, define $\mathrm{Int}(F) = F \smallsetminus \partial F$ where ∂F is the boundary of F within the sphere $V \cap S^{k-1}$.

Definition 2.6 ([21]). Let $k \geq 3$, $r \in \overline{\mathbf{k-2}}$ and Φ be a smooth flow on $\mathbf{\Delta}_{k-1}$ which preserves the invariant subspace structure determined by $\mathcal{I}_k$. Suppose that Σ is a heteroclinic network for Φ with equilibrium set $\mathbf{E} = \{\mathbf{p}_i \mid i \in \boldsymbol{\ell}\}$, and finite connection set $\mathbf{C} = \{\phi^i \mid i \in \mathbf{q}\}$. We say that Σ is an *r-face heteroclinic network* if

>   (1) For each $i \in \boldsymbol{\ell}$, there exists an $r - 1$-face L_i of $\mathbf{\Delta}_{k-1}$, such that
>   >   (a) $\mathbf{p}_i \in \mathrm{Int}(L_i)$ (if $r = 1$, $\mathbf{p}_i$ is a vertex of $\mathbf{\Delta}_{k-1}$).
>   >   (b) If $i \neq j$, $L_i \neq L_j$.

>   (2) For each connection $\mathbf{p}_a \xrightarrow{\phi^i} \mathbf{p}_b \in \mathbf{C}$,
>   >   (a) there is an r-face F_i of $\mathbf{\Delta}_{k-1}$ such that $\phi^i(\mathbb{R}) \subset \mathrm{Int}(F_i)$.
>   >   (b) If $j \in \mathbf{q}$, $i \neq j$, then $F_i \neq F_j$.

>   (3) For all $\mathbf{p} \in \mathbf{E}$, $d_{\mathbf{p}}^{\mathrm{out}} = \dim(W^u(\mathbf{p}))$.

Remarks 2.7. (1) If $\ell = q$ in definition 2.6, we say that Σ is an *r-face heteroclinic cycle*. In this case, we label so that each vertex $\mathbf{p}_i$ is connected by ϕ^i to $\mathbf{p}_{i+1}$ (with the standard convention that $\mathbf{p}_{\ell+1} \overset{\text{def}}{=} \mathbf{p}_1$).
(2) If $r = 1$ in definition 2.6, we refer to Σ as an *edge network* (*edge cycle* if Σ is a heteroclinic cycle). If $r > 1$, we refer to Σ as a *face network* (*face cycle* if Σ is a heteroclinic cycle).
(3) If $\dim(W^u(\mathbf{p})) = 1$, $\mathbf{p} \in \mathbf{E}$, then $d_{\mathbf{p}}^{\mathrm{out}} = 1$ and Σ will be a simple heteroclinic cycle: since $\mathbf{p} \in \partial\mathbf{\Delta}_{k-1}$, there are no simple heteroclinic networks that are not heteroclinic cycles.
(4) If $\mathbf{p}$ lies in the interior of an $r - 1$-face L, then $d_{\mathbf{p}}^{\mathrm{out}} \leq k - r - 1$ (exactly $k - r$ r-faces contain L and so $\dim(W^u(\mathbf{p})) \leq k - r - 1$).   ⊛

Lemma 2.8. *If Σ is a r-face heteroclinic network, then $\overline{W^u(\mathbf{p})}$ is contained in a $d_{\mathbf{p}}^{out} + r - 1$-face of $\mathbf{\Delta}_{k-1}$.*

Proof. Using definition 2.6(3), the required face is spanned by the set of vertices of the faces given by definition 2.6(2b). □

Remark 2.9. Suppose Σ is an r-face heteroclinic network and $\Sigma(\mathbf{E})$ is clean ($\Sigma(\mathbf{E})$ is compact and $\Sigma(\mathbf{E}) = \bigcup_{\mathbf{p} \in \mathbf{E}} W^u(\mathbf{p})$). Let $\mathbf{p} \in \mathbf{E}$

and F be the $d_{\mathbf{p}}^{\text{out}} + r - 1$-face of $\boldsymbol{\Delta}_{k-1}$ given by lemma 2.8. Then $\partial W^u(\mathbf{p}) = \bigcup_{\mathbf{q} \in \mathbf{E}} W^s(\mathbf{q}) \cap \partial F$. This condition is often robust and easy to satisfy for edge networks. On the other hand, apart from simple cycles, clean r-face heteroclinic networks are not usually robust and many face heteroclinic networks will not have a clean realization.   ⊛

2.5. **Examples of edge and face cycles and networks.**

Examples 2.10. (1) In figure 2(a) we show a simple edge cycle Σ^e on $\boldsymbol{\Delta}_3$ and in figure 2(b) an edge network Σ^n on $\boldsymbol{\Delta}_3$ that is the union of two edge cycles Σ_1, Σ_2 which share the common edge $\mathbf{v}_1 \longrightarrow \mathbf{v}_2$. The
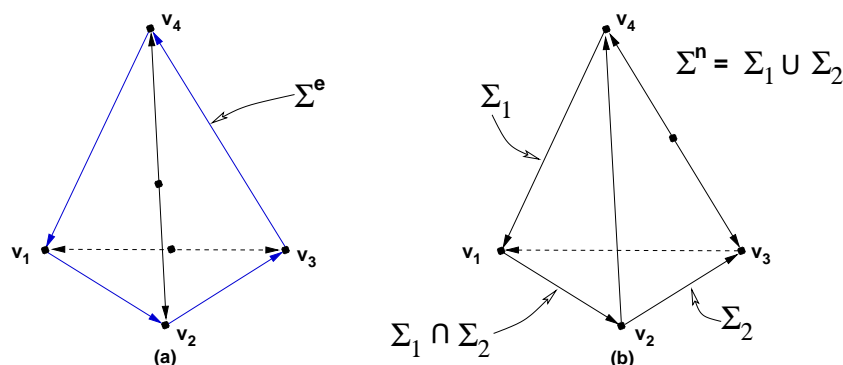
FIGURE 2. (a) Edge cycle and (b) edge network on $\boldsymbol{\Delta}_3$

edge cycle Σ^e shown in (a) was first studied in the setting of $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$-equivariant dynamics [28] and the edge network Σ^e was first analysed by Kirk and Silber [39] & Brannath [13]. The cycle Σ^e is clean but the network Σ^n is not clean. If we change dynamics so as to remove the equilibrium on the edge joining $\mathbf{v}_3, \mathbf{v}_4$ and instead have a connection $\mathbf{v}_3 \rightarrow \mathbf{v}_4$, then the resulting network, defined as the union of the unstable manifolds of the vertices of $\boldsymbol{\Delta}_3$, is clean and robust (see [13] for this and other variations). It is straightforward to realize either cycle or network using cubic maps as in (2.10) (see [28, 11]). More generally, any edge network can be realized using cubic polynomials [11]. We refer also to [21], where the equilibria at vertices are replaced by more complex dynamics. Finally, there are no face heteroclinic networks on $\boldsymbol{\Delta}_2$ – only heteroclinic edge cycles and no edge networks.
(2) The 2-face simple heteroclinic cycle on $\boldsymbol{\Delta}_3$ shown in figure 3 was first studied in [28] (we adopt the notational convention that $\mathbf{v}_{ij}$ is an equilibrium lying in the interior of the edge joining $\mathbf{v}_i$ to $\mathbf{v}_j$). The cycle can be realized using cubic polynomials [28]. For example, the

$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$-equivariant system

$$\dot{x}_i = x_i(1 + ax_i^2) + x_i \left(\sum_{j=1}^{3}(b_j + a)x_{i+j}^2 \right), \; i \in \mathbf{4},$$

admits a 2-face cycle provided that

$$a, \; 2a + b_2, \; 4a + b_1 + b_3, \; 4a + b_1 + b_2 + b_3 < 0,$$
$$b_1 b_3 > 0, \; (b_1^2 + b_3(b_2 - b_1)) \times (b_3^2 + b_1(b_2 - b_3)) < 0.$$

The cycle is clean. Moreover, there will be no equilibria in the interior of any 2-faces of $\mathbf{\Delta}_3$ (in the interior of $\mathbf{\Delta}_3$ there will be an equilibrium on $x_1 = \ldots = x_4$ and possibly a limit cycle, see [28]).
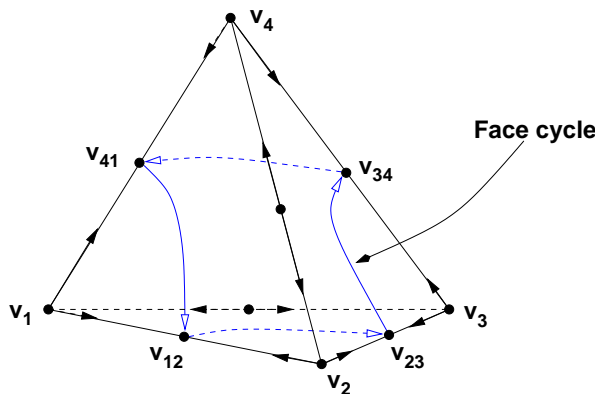
FIGURE 3. 2-face heteroclinic cycle on $\mathbf{\Delta}_3$

(3) Every 2-face heteroclinic network supported on $\mathbf{\Delta}_3$ must be a heteroclinic cycle since every 2-face heteroclinic network on $\mathbf{\Delta}_3$ is simple. We indicate how to construct 2-face heteroclinic networks supported on $\mathbf{\Delta}_4$. The spherical simplex $\mathbf{\Delta}_4$ has 10 2-faces and each edge lies on 3 2-faces. We can construct a smooth vector field X on O_5 which has a 2-face heteroclinic network Σ^a with 9 connections and 6 equilibria – see figure 4(a). Every vertex of $\Gamma(\Sigma^a)$ has degree 3 and each connection of Σ^a will lie in a unique 2-face of $\mathbf{\Delta}_4$ (there is no connection in the 2-face Δ_{134} spanned by the vertices $\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4$). We may choose X so that the stable and unstable manifolds of vertices only meet in the connections lying in 2-faces. The resulting network Σ^a will then be robust.

We cannot choose a vector field which has a clean (face) heteroclinic network with the same equilibrium set as Σ^a – $\partial W^u(\mathbf{v}_{23})$ will always
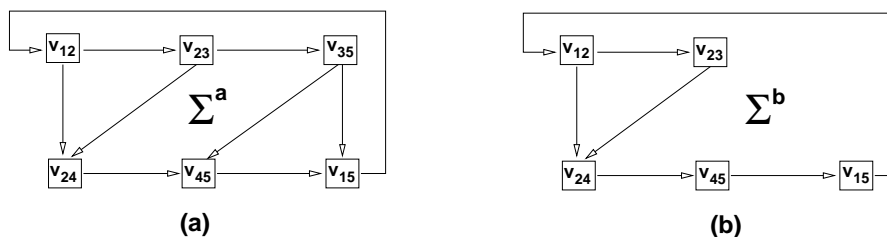
FIGURE 4. 2-face heteroclinic networks on $\mathbf{\Delta}_4$. (a) Robust network Σ^a with 9 connections; (b) A 6 connection network Σ^b that can be realized as a clean network.

contain points not in the unstable manifolds of the other equilibria. If we remove the connection $\mathbf{v}_{23} \longrightarrow \mathbf{v}_{35}$ and vertex $\mathbf{v}_{35}$, then we obtain the heteroclinic 2-face network Σ^b shown in figure 4(b) ($\dim W^u(\mathbf{v}) = 1$ unless $\mathbf{v} = \mathbf{v}_{12}$). We can construct a clean maximal network Σ^m with the same equilibrium set as Σ^b. However, Σ^m will not be robust: $W^u(\mathbf{v}_{12})$ will not intersect $W^s(\mathbf{v}_{24})$ transversally inside the 3-face spanned by the vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. If Σ^m is an asymptotically stable heteroclinic attractor, then there is the likelihood that vector field perturbations that preserve the invariant subspace structure but break the clean structure will result in dynamics for which Σ^b contains an essentially asymptotically stable attractor [45, 13, 39]. ※

Example 2.11 (Explicit construction using polynomials)**.** It is obvious that we can realize 2-face cycles on $\mathbf{\Delta}_k$, $k \geq 3$, if we work with general smooth vector fields of the form (2.5). Although we have not checked all the details, our expectation is every 2-face simple heteroclinic cycle can be realized on $\mathbf{\Delta}_k$, $k \geq 3$, using cubic polynomials of the form (2.10) (see also remark 2.13 below). It is not too difficult to find a cubic vector field of the form (2.11) which realizes the heteroclinic network Σ^a of examples 2.10(3). We indicate a few of the details for the reader interested in numerical experiments. Note that we implicitly use a method, based on Bézout's theorem [25, Chapter 4, §9], to show that 2-faces Δ_{ijk} which contain a face connection have no equilibria in the interior of the face. Consider the system

$$(2.12) \qquad \dot{x}_i = x_i + x_i \left(\sum_{j \neq i} \beta_{ij} x_j^2 - \alpha \|\mathbf{x}\|^2 \right), \; i \in \mathbf{5}.$$

Assume $\alpha \gg \max_{ij} |\beta_{ij}|$, so that the conditions of the invariant sphere theorem [25, Chapter 5, §1] apply, and the dynamics of (2.12) is asymptotic to an invariant 4-sphere. For each $i \in \mathbf{5}$, (2.12) has a unique strictly positive equilibrium $\mathbf{v}_i$ on the x_i-axis. The eigenvalue of the linearization of (2.12) at $\mathbf{v}_i$ in direction $\mathbf{v}_j$, $i \neq j$, is β_{ji}/α. Noting that $\alpha > 0$, we assume

$$\beta_{13}, \beta_{25}, \beta_{41}, \beta_{43} < 0$$

and that all other β_{ij} are strictly positive. Every connection of Σ^a is of the form $\mathbf{v}_{ij} \longrightarrow \mathbf{v}_{jk}$ and lies in the 2-face Δ_{ijk} with vertices $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$. For example, there is a connection $\mathbf{v}_{12} \longrightarrow \mathbf{v}_{23}$ contained in Δ_{123}. Hence the eigenvalue of linearization of (2.12) at $\mathbf{v}_{12}$ given by the eigendirection tangent to the connection at $\mathbf{v}_{12}$ is required to be strictly positive; that at $\mathbf{v}_{23}$ will be strictly negative. Noting that $\alpha \gg \max_{ij} |\beta_{ij}|$, we find (using [25, Chapter 4, §9]) that the required eigenvalue conditions on the connection in Δ_{123} hold if and only if

$$\frac{\beta_{31}}{\beta_{21}} + \frac{\beta_{32}}{\beta_{12}} > 1, \qquad \frac{\beta_{13}}{\beta_{23}} + \frac{\beta_{12}}{\beta_{32}} < 1$$

The remaining 16 conditions for the other 8 2-faces are obtained from these inequalities by trivial permutation arguments. We claim there is a non-empty open subset of the parameters β_{ij} for which the inequalities hold. We start by taking

$$\beta_{13}, \beta_{31}, \beta_{25}, \beta_{52}, \beta_{14}, \beta_{41} \beta_{34}, \beta_{43} = 0.$$

The first term in each of the 18 inequalities will then be zero. All 18 (strict) inequalities will then hold if and only if $\beta_{21} > \beta_{51}$, $\beta_{15} > \beta_{45} > \beta_{35}$, $\beta_{42} > \beta_{32} > \beta_{12}$, $\beta_{53} > \beta_{23}$ and $\beta_{54} > \beta_{24}$. Hence we can choose nonzero β_{ij} of the correct sign so that all inequalities hold. The vector field we have constructed has a heteroclinic network Σ containing Σ^a. We cannot exclude there being additional connections that do not lie in 2-faces, though Σ, Σ^a have the same equilibrium sets.   ※

Theorem 2.12. *Let $k \geq 2$, $0 < r \leq k - 1$. Then $\mathbf{\Delta}_k$ supports (clean) r-face heteroclinic cycles. If $r \leq k - 2$, $\mathbf{\Delta}_k$ supports (clean) r-face heteroclinic networks. In all cases, cycles and networks can be realized using smooth vector fields on O_{k+1} of form (2.5).*

Proof. We omit details of the routine construction of r-face heteroclinic cycles – see [21] for the case of heteroclinic face cycles. The existence of clean r-face heteroclinic networks is straightforward – see examples 2.10(3).   □

Remark 2.13. As we did in example 2.11, we can require in theorem 2.12 that the nodes have identical intrinsic dynamics defined by $f(x) =$

$x - \alpha x^3$, $\alpha > 0$. We conjecture that the conditions of the theorem can be satisfied using cubic vector fields of the form (2.10). A useful tool for the verification of the conjecture is that generically there is at most one equilibrium in the interior of each face. We refer to [25, Chapter 4, §9] for the general method which depends on Bezout's theorem and the invariant subspace structure.   ⊛

2.6. **Lattice structure on $\mathcal{I}_k$.** We conclude this section with a few brief remarks on the natural lattice structure on $\mathcal{I}_k$ defined by intersection and vector space sum.

Let $\prec$ denote the partial order on $\mathcal{I}_k$ defined by reverse inclusion: $V \prec W$ if $W \subset V$. Necessarily, $\mathbf{0} = H_{1...k}$ is the unique maximal element of $\mathcal{I}_k$ and $\mathbf{R}$ is the unique minimal element of $\mathcal{I}_k$. We use some standard definitions from lattice theory (see Davey and Priestly [14] for more details). We define the operations of *join* $\vee$ (least upper bound) and *meet* $\wedge$ (greatest lower bound) on $\mathcal{I}_k$ by

$$V \vee W \;\; = \;\; V \cap W,$$
$$V \wedge W \;\; = \;\; V + W \;\text{(vector space sum),}$$

where $V, W \in \mathcal{I}_k$. It is trivial to verify that $(\mathcal{I}_k, \vee, \wedge)$ has the structure of a (complete) lattice and that for all $V \in \mathcal{I}_k$ we have

$$V \vee \mathbf{0} = \mathbf{0}, \qquad V \wedge \mathbf{R} = \mathbf{R}.$$

When we relate SLF and coupled identical cell systems, we use a natural lattice structure on the set of synchrony subspaces. This structure will sometimes (not always) relate to the natural lattice structure on $\mathcal{I}_k$.

3. Coupled identical cell systems – asymmetric inputs

We refer to Stewart, Golubitsky and coworkers [57, 29, 31, 30] for general theory and background on coupled cell systems. Here we mainly review the formalism we use for networks of coupled identical cells. We use a 'flow-chart' formalism, similar to that used in electrical and computer engineering, that fits well with our intended applications of constructing networks with particular properties. We give necessary definitions, establish notational conventions and refer the reader to [3, 6, 4, 23] for more details, discussion and examples. We use the term coupled cell *network* to refer to the abstract object – a directed network graph codifying the connection structure with vertices corresponding to nodes – and generally use the term coupled cell *system* when we view the coupled cell network as a system of coupled differential equations [3]. We frequently abuse notation by letting $\mathcal{N}$ refer

to both the abstract network structure as well as a realization as a coupled cell system.

Let $\mathcal{N}$ be a coupled cell system consisting of $m \geq 2$ identical nodes (or 'cells') $N_1, \cdots, N_m$ each with phase space M. Denote the state variable for node N_i by $\mathbf{x}_i \in M$. Let $\mathbf{M} = M^m$ denote the network phase space. If each cell has p inputs, dynamics will be given by a system of differential equations of the form

$$(3.13) \qquad \dot{\mathbf{x}}_i = f(\mathbf{x}_i; \mathbf{x}_{I_i(1)}, \cdots, \mathbf{x}_{I_i(p)}), \ i \in \mathbf{m},$$

where $I_i : \mathbf{p} \rightarrow \mathbf{m}$, $i \in \mathbf{m}$. We refer to $f : M \times M^p \rightarrow TM$ as the *network map* and the corresponding vector field $\mathbf{F} : \mathbf{M} \rightarrow TM$ defined by (3.13) as the *network vector field*.

Under explicitly indicated to the contrary, we assume

  (1) There are no self loops: $i \notin I_i(\mathbf{p})$, all $i \in \mathbf{m}$.
  (2) Inputs are *asymmetric*: $f(\mathbf{x}; \mathbf{x}_1, \ldots, \mathbf{x}_p)$ is not symmetric in any subset of the variables $\mathbf{x}_1, \ldots, \mathbf{x}_p$.

Remark 3.1. The assumption of asymmetric inputs is a major simplification that often allows us to reduce proofs to the case where cells have a single input. From the application point of view (for example, in neuroscience) what seems to be most appropriate are asymmetric inputs that are not too far from symmetric. $\circledast$

Associated with the coupled cell network $\mathcal{N}$, there is the *network graph* $G(\mathcal{N})$. This consists of m-vertices v_i, corresponding to the nodes N_i, and mp directed edges e_α, $v_{I_i(s)} \rightarrow v_i$, $i \in \mathbf{m}$, $s \in \mathbf{p}$. We always assume that $\mathcal{N}$ (that is, $G(\mathcal{N})$) is connected. Usually, $\mathcal{N}$ will be strongly connected (every vertex pair v_i, v_j lies on a cycle). At this level of generality, the network $\mathcal{N}$ and graph $G(\mathcal{N})$ represent the same structure. Since inputs are asymmetric (condition (2)), the graph has p distinct edge types. In terms of the system, each cell has p *input types*. If $\alpha \in \mathbf{p}$, $i \in \mathbf{m}$, let $d_{\alpha,i}^{\mathrm{in}}$ denote the *in-degree* of vertex i for inputs of type α. Since inputs are asymmetric, $d_{\alpha,i}^{\mathrm{in}} = 1$ for all $\alpha \in \mathbf{p}$, $i \in \mathbf{m}$.

3.1. **Synchrony classes and synchrony subspaces.** Let $\mathcal{P}(m)$ denote the set of partitions of $\mathcal{N}$. If $\mathcal{X} = \{X^j \mid j \in \boldsymbol{\ell}\} \in \mathcal{P}(m)$, let $s(j)$ be the number of cells in X^j, $j \in \boldsymbol{\ell}$. The partition is *nontrivial* if $\ell < m$ (at least one $s(j)$ is strictly bigger than 1). Label cells in X^j as $N_{i_1^j}, \cdots, N_{i_{s(j)}^j}$, where $i_1^j < \ldots < i_{s(j)}^j$, and set $J^j = \{i_1^j, \cdots, i_{s(j)}^j\}$. We have $\cup_{j \in \boldsymbol{\ell}} J^j = \mathbf{m}$.

Now view $\mathcal{N}$ as a coupled cell system. If $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m) \in \mathbf{M}$ denotes the state of the network, we may group states according to the

partition $\mathcal{X}$ and write $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^\ell)$, where $\mathbf{x}^j = (\mathbf{x}_{i_1^j}, \ldots, \mathbf{x}_{i_{s(j)}^j}) \in M^{s(j)}$ will denote the state of the $s(j)$ cells in X^j. Define

$$\mathbf{\Delta}_j = \{\mathbf{x}^j \mid \mathbf{x}_{i_1^j} = \cdots = \mathbf{x}_{i_{s(j)}^j}\} \subset M^{s(j)}, \ \ j \in \boldsymbol{\ell},$$

and let

$$\mathbf{\Delta}(\mathcal{X}) = \prod_{j \in \boldsymbol{\ell}} \mathbf{\Delta}_j = \{\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^\ell) \mid \mathbf{x}^j \in \mathbf{\Delta}_j, \ \ j \in \boldsymbol{\ell}\}.$$

denote the corresponding *polydiagonal* subspace.

Definition 3.2 ([6, 3], cf [57, 31, 30])**.** The partition $\mathcal{X}$ is a *synchrony class* for the coupled cell network $\mathcal{N}$ if the subspace $\mathbf{\Delta}(\mathcal{X})$ is dynamically invariant for every realization of $\mathcal{N}$ as a coupled cell system. In terms of coupled cell systems, if $\mathcal{X}$ is a synchrony class then the invariant subspace $\mathbf{\Delta}(\mathcal{X}) \subset \mathbf{M}$ is a *synchrony subspace* (a *polysynchronous* subspace in the terminology of [30]).

Let $\mathcal{D}(m) = \{\mathbf{\Delta}(\mathcal{X}) \mid \mathcal{X} \in \mathcal{P}(m)\}$ denote the set of all polydiagonal subspaces of $\mathbf{M}$. Obviously $\mathcal{D}(m)$ is indepenendent of M and $\mathcal{D}(m) \approx \mathcal{P}(m)$.

If $\mathcal{X} = \{\{N_1, \ldots, N_m\}\}$, then $\mathcal{X}$ is always a synchrony class: the *maximal synchrony class* $\mathfrak{s}_0$. The associated invariant space is the diagonal $\mathbf{S}_0 \overset{\text{def}}{=} \mathbf{\Delta}(\mathbf{M})$ and is referred to as the *minimal synchrony subspace*. It is the synchrony subspace of minimal dimension for every realization of a coupled cell network. If $\mathcal{T} = \{\{N_1\}, \ldots, \{N_m\}\}$ is the trivial partition of $\mathbf{m}$, then $\mathcal{T}$ defines the *null synchrony class* $\mathfrak{s}_\infty$ and $\mathbf{M} = \mathbf{\Delta}(\mathcal{T}) \overset{\text{def}}{=} \mathbf{S}_\infty$ defines the *null* or *trivial* synchrony subspace.

We give a simple and very useful criterion for synchrony subspaces (see [26], [30, §7] for greater generality).

Proposition 3.3. *Let $\mathcal{N}$ be a coupled cell network as above. Suppose that $\mathcal{X} = \{X^j \mid j \in \boldsymbol{\ell}\}$ is a partition of $\mathcal{N}$. Then $\mathcal{X}$ is a synchrony class iff for all $i, j \in \mathbf{m}$ and every input type $\alpha \in \mathbf{p}$, either no cell in X^i receives an input of type α from a cell in X^j or else every cell in X^i receives exactly one input of type α from a cell in X^j.*

Remark 3.4. Let $d_{\alpha,i}^{\text{in}}(j)$ denote the in-degree at node i for inputs of type α originating from cells in X^j. Proposition 3.3 implies that $\mathcal{X}$ is a synchrony class iff for every input type α and all $j, k \in \boldsymbol{\ell}$, $d_{\alpha,\cdot}^{\text{in}}(j)$ is constant on X^k.   ⊛

As a straightforward consequence of proposition 3.3 and remark 3.4, we have a useful result that allows us to combine synchrony subspaces.

Proposition 3.5 ([6, Theorem 3.8]). *Let $\mathcal{N}$ be a coupled cell network with asymmetric inputs. Suppose that the partitions $\mathcal{X} = \{X_i \mid i \in I\}$, $\mathcal{Y} = \{Y_j \mid j \in J\}$ both define synchrony classes of $\mathcal{N}$. Then the intersection partition $\mathcal{X} \cap \mathcal{Y} = \{X_i \cap Y_j \mid i \in I, j \in J\}$ defines a synchrony class of $\mathcal{N}$.*

Remark 3.6. Proposition 3.5 fails if cells have symmetric inputs – see [6, §7] (or later this section). We refer to Stewart [56] for the general theory when there are symmetric inputs (see also [7]).   ⊛

Let $\mathcal{S} = \mathcal{S}(\mathcal{N})$ denote the set of all synchrony subspaces of the coupled cell system $\mathcal{N}$. We have a partial order $\prec$ on $\mathcal{S}$ defined by reverse inclusion.

Proposition 3.7. *$\mathcal{S}$ has the natural structure of a complete lattice $(\mathcal{S}, \vee, \wedge)$ with join and meet defined by*

   (1) *$V \vee W = V \cap W$.*
   (2) *$V \wedge W$ is the synchrony subspace defined by the intersection of the partitions defining V and W.*

The maximal element of $(\mathcal{S}, \vee, \wedge)$ is $\mathbf{S}_0 = \mathbf{\Delta}(\mathbf{M})$ (the minimal synchrony subspace). The minimal element of $\mathcal{S}$ is $\mathbf{S}_\infty = \mathbf{M}$ (corresponding to the null synchrony class $\mathfrak{s}_\infty$).

Proof. Immediate from proposition 3.5.   □

Remark 3.8. The meet and join operations are naturally defined on the set $\mathcal{D}(m) \approx \mathcal{P}(m)$ of all polydiagonal subspaces of $\mathbf{M}$ and so $(\mathcal{P}(m), \vee, \wedge)$ has the structure of a complete lattice. The set $\mathcal{S}(\mathcal{N})$ of all synchrony subspaces of an coupled identical cell system with m nodes is a sublattice of $(\mathcal{P}(m), \vee, \wedge)$ provided that cells have asymmetric inputs [56]. In what follows we usually omit reference to the null synchrony subspace $\mathbf{S}_\infty$.   ⊛

Notation for synchrony subspaces. It is useful to introduce some simplified notation for synchrony subspaces. Let $\mathcal{X} = \{X^j \mid j \in \boldsymbol{\ell}\}$ be a nontrivial partition of $\mathcal{N}$. After relabelling we may assume that for some $q \leq \ell$, we have $s(1), \ldots, s(q) > 1$, $s(i) = 1$, $i > q$. For $j \in \mathbf{q}$, we have $\mathbf{\Delta}_j = \{\mathbf{x}^j \mid \mathbf{x}_{i_1^j} = \ldots = \mathbf{x}_{i_{s(j)}^j}\}$. With these conventions, we write

$$\Delta(\mathcal{X}) = (i_1^1 i_2^1 \ldots i_{s(1)}^1 \| \ldots \| i_1^q \ldots i_{s(q)}^q).$$

Examples 3.9. (1) $\mathbf{S}_0 = \mathbf{\Delta}(\mathbf{M}) = (12 \ldots m)$.
(2) The notation is naturally compatible with the meet and join operations. For example, if $(125\|89)$, $(127\|48)$ and (46) are synchrony subspaces of an identical cell network, then

(1) $(125\|89) \vee (127\|48) = (1257\|489)$.
(2) $(125\|89) \wedge (127\|48) = (12)$.
(3) $(125\|89) \vee (46) = (125\|89\|46)$, $(127\|48) \vee (46) = (127\|468)$.
(4) $(125\|89) \wedge (46) = (127\|48) \wedge (46) = \mathbf{S}_\infty$.   ※

Let $(\mathcal{S}, \vee, \wedge)$ be a finite lattice with maximal element $\mathbf{S}_0$ and minimal element $\mathbf{S}_\infty$. If $\mathcal{G}$ is a nonempty subset of the finite lattice $(\mathcal{S}, \vee, \wedge)$, we define $< \mathcal{G} >$ to be the subset of $\mathcal{S}$ generated from $\mathcal{G}$, using the operations $\vee, \wedge$, together with the minimal element $\mathbf{S}_\infty$.

Definition 3.10. (Notation and assumptions as above.) Let $\mathcal{G}$ be a subset of the finite lattice $(\mathcal{S}, \vee, \wedge)$ and suppose that $\mathbf{S}_\infty \notin \mathcal{G}$. The set $\mathcal{G}$ is a *generating set* for $(\mathcal{S}, \vee, \wedge)$ if $< \mathcal{G} > = \mathcal{S}$. If every other generating set for $(\mathcal{S}, \vee, \wedge)$ contains at least as many elements as $\mathcal{G}$, then $\mathcal{G}$ is a *basis* for $\mathcal{S}$.

Example 3.11. The lattice $(\mathcal{D}(m), \vee, \wedge)$ has basis $\mathcal{B} = \{(1, j) \mid j = 2, \ldots, m\} \cup \{(23 \ldots m)\}$. To see this, observe that $(1j_1 \ldots j_s) \in <\mathcal{B}>$, $2 \le j_1 < j_2 < \ldots j_s \le m$, using only the $\vee$ operation on $\{(1, j) \mid j = 2, \ldots, m\}$. Hence, $(j_1 \ldots j_s) = (1j_1 \ldots j_s) \wedge (2 \ldots m) \in <\mathcal{B}>$, $2 \le j_1 < j_2 < \ldots j_s \le m$. We easily obtain the remaining polydiagonal subspaces using only the $\vee$ operation and so $\mathcal{B}$ is a generating set. We leave it to the reader to verify that every generating set has at least m elements and so $\mathcal{B}$ is a basis (in this case, if $m > 2$, $\mathbf{S}_\infty$ lies in the set generated from $\mathcal{B}$ using the operations $\vee, \wedge$).   ※

Next we consider when a coupled identical m cell network $\mathcal{N}$ has $\mathcal{S}(\mathcal{N}) = \mathcal{D}(m)$.

Lemma 3.12. *Let $\mathcal{N}$ be a coupled identical cell network with $m \ge 3$ cells and lattice of synchrony subspaces $\mathcal{S}$. A necessary condition for $\mathcal{S} = \mathcal{D}(m)$ is that $\mathcal{N}$ has self-loops.*

Proof. By remark 3.4, it is enough to consider the case where cells have just one input type. Suppose $\mathcal{S} = \mathcal{D}(m)$ and $\mathcal{N}$ is connected. By proposition 3.3, there exists $j \in \mathbf{m}$ such that there is a connection $j{\to}i$, for all $i \in \mathbf{m}$. Hence N_j has a self-loop. If $\mathcal{N}$ is not connected, then every connection will be a self-loop. $\square$

Example 3.13. It follows from lemma 3.12 that a strongly connected m cell network without self loops has $\mathcal{S} \subsetneq \mathcal{D}(m)$. For example, if $m = 3$, there are at most three synchrony subspaces (see [6] and also below). If we allow self-loops, then an m-cell identical cell network can be strongly connected and have $\mathcal{S} = \mathcal{D}(m)$, provided that cells have at least m inputs. See figure 5(a) for the case $m = 3$. If we allow symmetric inputs, it is easy to construct an m-cell identical cell
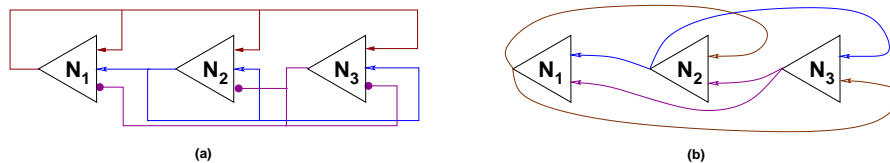
FIGURE 5. 3 cell connected networks with the maximal number of synchrony subspaces: (123), (12), (13) and (23): (a) Asymmetric inputs; (b) Symmetric inputs.

network with $\mathcal{S} = \mathcal{D}(m)$, provided that cells have at least $m-1$ inputs. See figure 5(b) for case $m = 3$. From our perspective, the networks of figure 5 are not particularly interesting as neither can support robust heteroclinic cycles.   ※

In [6] it was shown that (up to network equivalence) there were exactly two strongly connected identical three cell networks, asymmetric inputs, supporting robust heteroclinic cycles (one of these networks had self-loops). Subsequently, it was shown [26] that every heteroclinic network $\mathcal{N}$ with q connections could be realized as a robust heteroclinic network in a strongly connected identical cell network $\mathcal{P}_{q+1}$ consisting of $q + 1$ cells, each with q asymmetric inputs. Rather than restate the general result, we give an example that embeds the RPS heteroclinic 3-cycle in $\mathcal{P}_4$.

Example 3.14. In figure 6, we show the network $\mathcal{P}_4$ constructed in [26]. Every synchrony subspace of $\mathcal{P}_4$ can be written as a join of
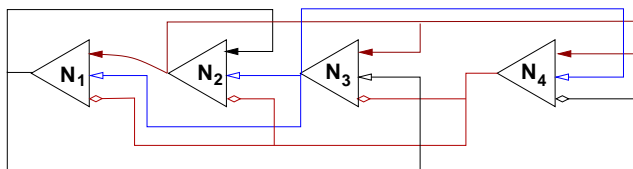

FIGURE 6. A 4 cell network with synchrony subspaces generated by (12), (13) and (14).

generating synchrony subspaces. Assuming cells have 1-dimensional dynamics, with phase space $\mathbb{R}$, the heteroclinic 3 cycle $a \to b \to c \to a$ can be realized as a robust heteroclinic cycle Σ in $\mathcal{P}_4$. The result follows from the main theorem in [26]: the equilibria $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \Sigma$ will lie on the synchrony subspace (1234); the connection $\mathbf{a} \to \mathbf{b}$ will lie in (123); the connection $\mathbf{b} \to \mathbf{c}$ will lie in (124); and the connection $\mathbf{c} \to \mathbf{a}$

will lie in (134). Each equilibrium will have a 1-dimensional unstable manifold and along each connection, one of the nodes desynchronizes from the other three (synchronized) nodes. Note that Σ is not simple according to the strict definition we gave in section 1: Σ only contains one component of $W^u(\mathbf{p}) \smallsetminus \{\mathbf{p}\}$ for each equilibrium point $\mathbf{p} \in \Sigma$. Subsequently, our approach will be to work on a flow-invariant 'fundamental domain' $D_+ \subset \mathbb{R}^4$ for the coupled cell system so that each $W^u(\mathbf{p}) \smallsetminus \{\mathbf{p}\}$ has just one component in D_+. For $\mathcal{P}_4$, define $D_+ = \{\mathbf{x} \in \mathbb{R}^4 \mid x_i \geq x_1, i = 2,3,4\}$ and note that we can construct Σ so that $\Sigma \subset D_+$ and Σ is simple if we restrict dynamics to D_+. Henceforth, we use the term simple heteroclinic cycle (or network) in this restricted sense (we give a formal definition in section 5).   ※

In example 3.14, the RPS heteroclinic cycle lying in $\Delta_2 \subset O_3$ is realized as a heteroclinic 3 cycle in a four identical cell system. In the remaining sections, we describe a far reaching generalization of this simple result and prove that a heteroclinic r-face network Σ in an SLF system on O_k can be realized (in many ways) as a robust heteroclinic network in a strongly connected coupled identical cell system with $k+1$ cells, each with k asymmetric inputs, and no self loops. Each connection will be associated to a unique pattern of desynchronization and resynchronization (at the equilibria at the end points); these patterns correspond precisely to the connection structure for Σ viewed as an SLF system.

We conclude this section with an example illustrating some of the issues that arise if we allow symmetric inputs (see also the concluding comments at the end of the article).

Examples 3.15. (1) It is often the case that the presence of many invariant subspaces can lead to the existence of robust heteroclinic cycles and networks. However, if we assume cells have symmetric inputs, this intuition may fail. There are two problems: (a) 'too many' invariant subspaces – this may lead to multiplicities in eigenvalues; (b) symmetry in the inputs leads to fewer free parameters and this can make it harder to obtain specified linearizations at equilibria. For example, consider the network shown in figure 5(b) and assume 1-dimensional node dynamics. The synchrony subspaces $(12), (13)$ and (23) form a pencil of planes containing the line (123). Since the synchrony subspaces are all flow-invariant, this suggests that if $\mathbf{p} \in (123)$ is an equilibrium of the network vector field $\mathbf{F}$, then the two eigenvalues of $D\mathbf{F}(\mathbf{p})$ corresponding to eigendirections transverse to (123) should be equal. A straightforward computation verifies that if the network vector field $\mathbf{F}$ is given by the map $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, $(x,y,z) \mapsto f(x;y,z)$ (symmetric in y, z)

and we let $\alpha = \frac{\partial f}{\partial x}(\mathbf{p})$, $\beta = \frac{\partial f}{\partial y}(\mathbf{p}) = \frac{\partial f}{\partial z}(\mathbf{p})$, then $D\mathbf{F}(\mathbf{p})$ has eigenvalues $\alpha - \beta$ (multiplicity 2) and $\alpha + 2\beta$. It follows there can be no heteroclinic loops lying in two of the planes $(12), (13)$ and (23) connecting saddle equilibria on (123). Moreover, there are only two free parameters α, β and so it is not possible to choose three eigenvalues independently at an equilibrium on (123). Similar remarks and comments apply to the $(m-1)$-symmetric input generalization of figure 5(b) to an m cell network, $m \geq 3$.

(2) Notwithstanding the previous example, it is certainly possible to find robust heteroclinic cycles in identical cell networks with *some* symmetric inputs. We give an example based on [23],[6, §7.1]. Consider the 6 identical cell system $\mathcal{N}$ with network equations

$$\dot{x}_1 = f(x_1; x_3, x_6), \ \dot{x}_2 = f(x_2; x_4, x_5), \ \dot{x}_3 = f(x_2; x_5, x_1),$$
$$\dot{x}_4 = f(x_4; x_5, x_1), \ \dot{x}_5 = f(x_5; x_3, x_2), \ \dot{x}_6 = f(x_6; x_3, x_2).$$

If cells have symmetric inputs – $f(x; y, z) = f(x; z, y)$ – there are fourteen synchrony subspaces:

$$\begin{aligned}
\mathbf{S}_0 &= (123456), \ \mathbf{S}_1 = (12\|3456), \ \mathbf{S}_2 = (34\|1256), \\
\mathbf{S}_3 &= (12\|34\|56), \ \mathbf{S}_4 = (34\|256), \mathbf{S}_5 = (34\|56), \ \mathbf{S}_6 = (34), \\
\mathbf{S}_7 &= (56), \ \mathbf{S}_8 = (12\|35\|46), \ \mathbf{S}_9 = (134\|56), \ \mathbf{S}_{10} = (1234\|56), \\
\mathbf{S}_{11} &= (13\|56), \ \mathbf{S}_{12} = (25\|34), \ \mathbf{S}_{13} = (134\|256).
\end{aligned}$$

(For asymmetric inputs, $\mathbf{S}_8, \ldots, \mathbf{S}_{13}$ are not synchrony subspaces.) The network $\mathcal{N}$ also has a $\mathbb{Z}_2$-symmetry generated by the cell permutation $N_1 \leftrightarrow N_2$, $N_3 \leftrightarrow N_5$, $N_4 \leftrightarrow N_6$. Proposition 3.5 fails if inputs are symmetric: take $\mathcal{X}$ to be the partition defined by $\mathbf{S}_8$ and $\mathcal{Y}$ the partition defined by $\mathbf{S}_3$.

Under the assumption of symmetric inputs, we show that we can choose network dynamics determined by $f : \mathbb{R}^3 \to \mathbb{R}$ such that there is a robust, simple (in the sense of example 3.14) and asymptotically stable heteroclinic cycle Σ contained in $\mathbf{S}_6 \cup \mathbf{S}_7$ with equilibria $\mathbf{p}, \overline{\mathbf{p}} \in \mathbf{S}_5 = \mathbf{S}_6 \cap \mathbf{S}_7$.

Let $\mathbf{p} = (a_1, a_2, b, b, c, c) \in \mathbf{S}_5$ and define

$$
\begin{aligned}
\frac{\partial f}{\partial x}(a_1; b, c) &= \alpha_1, & \frac{\partial f}{\partial y}(a_1; b, c) = \frac{\partial f}{\partial z}(a_1; b, c) &= \beta \\
\frac{\partial f}{\partial x}(a_2; b, c) &= \alpha_2, & \frac{\partial f}{\partial y}(a_2; b, c) = \frac{\partial f}{\partial z}(a_2; b, c) &= \gamma \\
\frac{\partial f}{\partial x}(b; c, a_1) &= \alpha_3, & \frac{\partial f}{\partial y}(b; c, a_1) = \frac{\partial f}{\partial z}(b; c, a_1) &= \delta \\
\frac{\partial f}{\partial x}(c; b, a_2) &= \alpha_4, & \frac{\partial f}{\partial y}(c; b, a_2) = \frac{\partial f}{\partial z}(c; b, a_2) &= \eta.
\end{aligned}
$$

The linearization of the network vector field $\mathbf{F}$ determined by f at $\mathbf{p}$ is

$$
(3.14) \qquad J(\mathbf{p}) =
\begin{pmatrix}
\alpha_1 & 0 & \beta & 0 & 0 & \beta \\
0 & \alpha_2 & 0 & \gamma & \gamma & 0 \\
\delta & 0 & \alpha_3 & 0 & \delta & 0 \\
\delta & 0 & 0 & \alpha_3 & \delta & 0 \\
0 & \eta & \eta & 0 & \alpha_4 & 0 \\
0 & \eta & \eta & 0 & 0 & \alpha_4
\end{pmatrix}.
$$

Noting that $\mathbf{S}_5$ is $J(\mathbf{p})$-invariant, the matrix of $J(\mathbf{p})|\mathbf{S}_5 = J_{BC}$ is

$$
(3.15) \qquad J_{BC} =
\begin{pmatrix}
\alpha_1 & 0 & \beta & \beta \\
0 & \alpha_2 & \gamma & \gamma \\
\delta & 0 & \alpha_3 & \delta \\
0 & \eta & \eta & \alpha_4
\end{pmatrix}.
$$

We have similar expressions for $J_B = J(\mathbf{p})|\mathbf{S}_6$ and $J_C = J(\mathbf{p})|\mathbf{S}_7$. For a simple cycle, all eigenvalues of $J_{BC}(\mathbf{p})$ must have strictly negative real part and the eigenvalues corresponding to the eigenlines not contained in $\mathbf{S}_5$ must be real, nonzero and of opposite sign. Since $\mathbf{S}_6 \supset \mathbf{S}_5$ and both spaces are $J(\mathbf{p})$-invariant, we see that the eigenvalue of J_B with eigenline transverse to $\mathbf{S}_5$ must be α_4 – the sum of the eigenvalues of $J_{BC}(\mathbf{p})$ is $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, while the sum of the eigenvalues of J_B is $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4$. Similarly the eigenvalue of J_C with eigenline transverse to $\mathbf{S}_5$ must be α_3. Hence for a simple cycle with saddle at $\mathbf{p}$ we must have $\alpha_3 \alpha_4 < 0$ and all eigenvalues of J_{BC} strictly negative. It is not hard to choose $\alpha_1, \ldots, \alpha_4, \beta, \gamma, \delta, \eta$ to achieve this. For example, if we take $\alpha_1 = \alpha_2 = \alpha_4 = -1$, $\alpha_3 = 1$, $\beta = 0$, $\gamma = 2$, $\delta = 1$ and $\eta = -2$, then the characteristic equation of J_{BC} is $\lambda^3 + \lambda^2 + 6\lambda + 1 = 0$ and so by the Routh-Hurwitz criterion all roots of the characteristic equation have strictly negative real parts. With a little more work, we may also require that the weakest contracting eigenvalue dominates the expanding eigenvalue. This is sufficient to guarantee asymptotic stability of the cycle we construct.

The previous arguments show there is no obstruction to constructing a network vector field which has a hyperbolic saddle point on $\mathbf{S}_5$ with 1-dimensional (punctured) stable manifold lying in either $\mathbf{S}_6 \smallsetminus \mathbf{S}_5$ or $\mathbf{S}_7 \smallsetminus \mathbf{S}_5$. We briefly sketch how to construct a network vector field with connecting trajectories giving a heteroclinic cycle between two hyperbolic saddle points on $\mathbf{S}_5$. Pick $\mathbf{p} = (a_1, a_2, b, b, c, c)$, $\overline{\mathbf{p}} = (\bar{a}_1, \bar{a}_2, \bar{b}, \bar{b}, \bar{c}, \bar{c}) \in \mathbf{S}_5$ with $\{a_1, a_2, b, c\} \cap \{\bar{a}_1, \bar{a}_2, \bar{b}, \bar{c}\} = \emptyset$. Choose hyperbolic linear maps $A, \overline{A}$ at $\mathbf{p}, \overline{\mathbf{p}}$ so that unstable eigendirections are 1-dimensional and lie in $\mathbf{S}_6$, $\mathbf{S}_7$ respectively. Using $A, \overline{A}$ we construct the network vector field on a neighbourhood of $\mathbf{S}_5$ in $\mathbf{M}$ with index 1 saddle points at $\mathbf{p}, \overline{\mathbf{p}}$. Next choose smooth connections $\mathbf{p} \rightarrow \overline{\mathbf{p}} \subset \mathbf{S}_6$, $\overline{\mathbf{p}} \rightarrow \mathbf{p} \subset \mathbf{S}_7$ which match with the eigenlines, transverse to $\mathbf{S}_5$, near the equilibria $\mathbf{p}, \overline{\mathbf{p}}$. Using the method of [26], perturb connections so that the network vector field is well defined on the connections – regard the connections as subsets of $\mathbf{S}_6 = \mathbf{S}_7 = \mathbb{R}^5$. Extend the network vector field smoothly to all of $\mathbb{R}^3$.   ※

4. The Synchronization Transform

In the next two section we describe a general method for transforming invariant subspaces and heteroclinic networks of SLF networks to synchrony subspaces and heteroclinic networks of coupled identical cell networks.

Henceforth we always assume

(1) Identical cells with asymmetric inputs.
(2) The network graph is strongly connected and without self loops.
(3) If the network has k cells, then each cell has at most $k - 1$ asymmetric inputs.
(4) Node dynamics for coupled cell systems is 1-dimensional (phase space $\mathbb{R}$).

Suppose that $k \geq 2$ and let $\mathcal{D}(k)$ denote the lattice of all proper polydiagonal subspaces of $\mathbf{R} = \mathbb{R}^k$ with minimal element $\mathbf{S}_\infty = \mathbf{R}$ and maximal element (minimal synchrony subspace) $\mathbf{S}_0 = \mathbf{\Delta}(\mathbf{R})$. Let $\mathcal{L}(k)$ be the sublattice of $\mathcal{I}_k$ generated by the subspaces $\{H_i \mid i \in \mathbf{k}, \ i \neq 1\}$. The maximal element $\mathbf{V}_0$ of $\mathcal{L}(k)$ is the subspace $H_{2\ldots k}$ – the x_1-axis. The minimal element is $\mathbf{R}$.

Definition 4.1. (Notation and assumptions as above.) A *synchronization transform* of *weight* s is a triple $(T, \mathcal{B}, \mathcal{S})$ where

(1) $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is a linear isomorphism: the *synchronization map*.
(2) $\mathcal{S}$ is a complete sublattice of $\mathcal{D}(k)$ with maximal element $\mathbf{S}_0$, and minimal element $\mathbf{S}_\infty$.

(3) $\mathcal{B} = \{W_1, \ldots, W_s\}$ is a subset of $\mathcal{L}(k)$ and $\{T(W_i) \mid i \in \mathbf{s}\}$ is a basis of $\mathcal{S}$.

(4) $T(\mathbf{V}_0) = \mathbf{S}_0 \in \mathcal{S}$.

(5) If $V \in \mathcal{D}(k) \smallsetminus \mathcal{S}$, then $T^{-1}(V) \notin \mathcal{L}(k)$.

A coupled k identical cell network $\mathcal{N}$ supports (or has) a synchronization transform if we can find a synchronization transform $(T, \mathcal{B}, \mathcal{S})$ such that $T : \mathbb{R}^k \to \mathbb{R}^k$ and $\mathcal{S} = \mathcal{S}(\mathcal{N})$.

Remarks 4.2. (1) If $(T, \mathcal{B}, \mathcal{S})$ is a synchronization transform then $\mathbf{S}_0 \in \mathcal{S}$ by (4). In particular, $s \geq 1$ and either $\mathbf{V}_0 \in \mathcal{B}$ or $\mathbf{V}_0$ lies in the sublattice generated by $\mathcal{B}$.

(2) Although $\mathbf{S}_\infty \in \mathcal{S}$ (see the preamble to definition 3.10), we do not require that $\mathbf{S}_\infty$ can be written in terms of $T(W_1), \ldots, T(W_s)$ using the $\vee, \wedge$ operations. Of course, since T is a linear isomorphism, $T(\mathbf{R}) = \mathbf{S}_\infty$. In what follows we invariably omit reference to the minimal classes $\mathbf{R}, \mathbf{S}_\infty$ in our descriptions of lattices and sublattices.

(3) It it neither required, nor true, that T induces a lattice isomorphism between $\mathcal{S}$ and the sublattice of $\mathcal{L}(k)$ generated by $\mathcal{B}$. ⊛

Examples 4.3 (Networks supporting synchronization transforms).
(1) Suppose $k = 3$. Up to permutation of coordinates, there are three nonempty subsets of $\mathcal{L}(3)$ containing $\mathbf{V}_0$:

$$\mathcal{B}_0 = \{\mathbf{V}_0, H_2, H_3\} = \mathcal{L}(3), \ \mathcal{B}_1 = \{\mathbf{V}_0, H_2\}, \ \mathcal{B}_\infty = \{\mathbf{V}_0\}.$$

We remark that $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_\infty$ are sublattices of $\mathcal{L}(3)$. If $(T, \mathcal{B}_\alpha, \mathcal{S}_\alpha)$ is a synchronization transform, $\alpha \in \{0, 1, \infty\}$, then (up to permutation of coordinates) we have

(1) $\mathcal{S}_0 = \{(123), (12), (13)\}$ ($s = 2$).
(2) $\mathcal{S}_1 = \{(123), (12)\}$ ($s = 2$).
(3) $\mathcal{S}_\infty = \{(123)\}$ ($s = 1$).

The synchronization map for $\alpha = 0$ is given by the linear isomorphism T_0 satisfying $T_0(H_2) = (12)$, $T_0(H_3) = (13)$ and $T_0(\mathbf{V}_0) = (123)$. In matrix form,

$$T_0 = \begin{pmatrix} 1 & b & c \\ 1 & b_2 & c \\ 1 & b & c_3 \end{pmatrix},$$

where $(b_2 - b)(c_3 - c) \neq 0$. For example, we may take $b = c = 0$, $b_2 c_3 \neq 0$. Finding explicit synchronization maps for the other two cases is a simple computation. Note that $(12) \wedge (13) = \mathbf{S}_\infty$ for $\mathcal{S}_0$ but that $\mathbf{S}_\infty$ cannot be expressed in terms of basis elements for $\mathcal{S}_1, \mathcal{S}_\infty$. In all cases, T will induce a lattice isomorphism $T^\star : \mathcal{B}_\alpha \to \mathcal{S}_\alpha$.

We have already given a 3 identical cell network that realizes $\mathcal{S}_0$ – see figure 5(b). In figure 7 we show identical cell networks that realize $\mathcal{S}_1$ and $\mathcal{S}_\infty$. Note that for (a) we need *two* input types and at least

(a)   **(b)**

FIGURE 7. Three node identical cell system with nontrivial synchrony subspaces (a) $(12), (123)$, and (b) (123).

one cell will have a duplicated input – in this case N_3. For (b) we only need *one* input – shown by the unbroken line in figure 7(b). The second set of inputs (broken line in the figure) can be filled using any configuration (denying self loops).

(2) Take $k = 5$, $s = 3$, $\mathcal{B} = \{H_{23}, H_{45}, H_{235}\}$ and require $T(H_{23}) = (12\|34)$, $T(H_{45}) = (15\|23)$ and $T(H_{235}) = (12\|345)$. We have

   (1) $\mathbf{V}_0 = H_{23} \cap H_{45}$.
   (2) $\mathcal{S} = \{(12\|34), (15\|23), (12\|345), (12345)\}$.

We may realize a synchronization map with the matrix

$$T = [t_{ij}] = \begin{pmatrix} 1 & a & b & c & d \\ 1 & a_2 & b_2 & c & d \\ 1 & a_2 & b_2 & c_3 & d_3 \\ 1 & a_4 & b_4 & c_3 & d_3 \\ 1 & a & b & c_3 & d_5 \end{pmatrix}.$$

where $(c - c_3)(d_5 - d_3), [(a_2 - a)(b_4 - b_2) - (b_2 - b)(a_4 - a_2)] \neq 0$ are required for T to be nonsingular and

$$a_2 - a_4, a - a_2, a_4 - a_2, b - b_2, b_2 - b_4, b - b_4, d - d_5, d - d_3 \neq 0$$

are needed to ensure $\mathcal{S} = \{(12\|34), (15\|23), (12\|345), (12345)\}$ and (3c) of definition 4.1 are satisfied. The coupled cell system shown in figure 8 has exactly these synchrony subspaces.

Note that T does not induce a lattice isomorphism between $\mathcal{S}$ and the sublattice of $\mathcal{L}(k)$ generated by $\mathcal{B}$ under the join and meet operations on $\mathcal{I}_k$. Indeed, $H_{45} \wedge H_{235} = H_5$ and $T(H_5) \notin \mathcal{S}$.

(3) If $k > 2$, it is not possible to choose a synchronization transform $(T, \mathcal{B}, \mathcal{S})$ such that $\mathcal{S} = \mathcal{D}(k)$ since the maximal number of elements in $\mathcal{S}$ is 2^{k-1} (the cardinality of $\mathcal{L}(k)$). Since we deny self loops, and
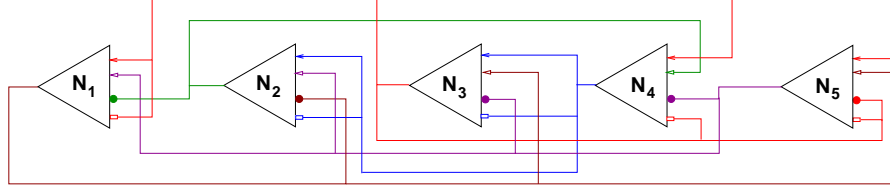
FIGURE 8. A five node identical cell system with nontrivial synchrony subspaces: $(12\|34), (15\|23), (12\|345), (12345)$.

$k > 2$, there are no strongly connected coupled cell networks for which every polydiagonal subspace is a synchrony subspace (lemma 3.12).
(4) If $k = 4$ and we take the sublattice $\mathcal{S}$ of $\mathcal{D}(4)$ generated by $(123), (124), (12\|34)$, then there is no synchronization transform. It is straightforward to verify that there is no strongly connected network $\mathcal{N}$, *without self loops*, for which $\mathcal{S}(\mathcal{N}) = \mathcal{S}$. ※

Remark 4.4. We conjecture that if $T : \mathbb{R}^k \to \mathbb{R}^k$ and $(T, \mathcal{B}, \mathcal{S})$ is a synchronization transform of weight s, then (a) $\mathcal{S}$ contains at most 2^{k-1} synchrony classes and (b) that $s \leq k-1$. We note that (a) is immediate if T induces a lattice isomorphism between the lattice generated by $\mathcal{B}$ and $\mathcal{S}$. ⊛

Theorem 4.5. *Let $k \geq 2$.*

(1) *If $\mathcal{B} = \{\mathbf{V}_0\}$, there exists a strongly connected coupled identical k cell network $\mathcal{N}$ without self loops for which $\mathcal{S}(\mathcal{N}) = \{(12\ldots k)\}$. It suffices that cells have $s \leq 2$ input types.*

(2) *Suppose that $(a_i, b_i) \in \mathcal{D}(k)$, $i \in \overline{\mathbf{k-1}}$. Let $\mathcal{S} \subset \mathcal{D}(k)$ be the sublattice generated by $\{(a_i, b_i) \mid i \in \overline{\mathbf{k-1}}\}$ and suppose that $\vee_{i \in \overline{\mathbf{k-1}}}(a_i, b_i) = \mathbf{S}_0$. Then there exists a strongly connected coupled identical k cell network $\mathcal{N}$ without self loops for which $\mathcal{S}(\mathcal{N}) = \mathcal{S}$. Each cell will have $k - 1$ input types.*

In both cases, there are synchronization transforms $(T, \mathcal{B}, \mathcal{S})$ for which $\mathcal{S}$ is generated by $\{T(W_i) \mid W_i \in \mathcal{B}\}$.

Proof. (1) Let $\mathcal{N}_1$ be the k identical cell network with one input type and connections $N_2 \to N_1, \ldots, N_k \to N_{k-1}$, $N_1 \to N_k$. If k is prime, the only synchrony subspace will be $(12\ldots k)$. If k is not prime, then it is straightforward exercise, based on proposition 3.3, to show that every synchrony subspace of $\mathcal{N}_1$ is of the form

$$\mathbf{S}_{PQ} = (1\,Q{+}1\ldots(P{-}1)Q{+}1\|2\ldots(P{-}1)Q{+}2\|\ldots\|Q\,2Q\ldots PQ),$$

where $k = PQ$ is a proper factorization of k (note that $S_{PQ} \neq S_{QP}$ unless $P = Q$). Choose a second input type and connections $N_{j+1} \rightarrow N_j$, $j > 1$ and $N_3 \rightarrow N_1$ so as to define the network $\mathcal{N}_2$ with two input types. The only synchrony subspace of $\mathcal{N}_2$ is $(12 \ldots k)$.

(2) We proceed by induction on k. The result is true trivially if $k = 2$ and follows by examples 4.3(1) if $k = 3$. Suppose the result is proved for $k < n$. We prove for $k = n$. By a simple counting argument, there exists $i \in \mathbf{n}$ such that either a_i or b_i occurs just once in $\{a_1, b_1, a_2, \ldots, a_{n-1}, b_{n-1}\}$. Relabelling, suppose that $i = n - 1$, $b_{n-1} = n$ occurs just once and $a_{n-1} = 1$. Then $\{(a_i, b_i) \mid i \in \overline{\mathbf{n} - \mathbf{2}}\}$ satisfies the hypotheses of the proposition for $k = n-1$ and so there exists a strongly connected coupled identical $n-1$ cell network $\mathcal{N}_{n-1}$ without self loops for which $\mathcal{S}(\mathcal{N}_{n-1})$ is generated by $(a_1, b_1), \ldots, (a_{n-2}, b_{n-2})$. Change cells in $\mathcal{N}_{n-1}$ by adding one new input type and form the n cell network $\mathcal{N}$ by adding one new cell N_n to $\mathcal{N}_{n-1}$, and

(1) connecting N_n to the $n - 1$-input of cells $N_1, \ldots, N_{n-1}$,
(2) connecting N_1 to the $n - 1$-input of cell N_n,
(3) connecting N_j to the $j - 1$-input of cell N_n, $j = 2, \ldots, n - 1$.

It is straightforward to check that $\mathcal{S}(\mathcal{N})$ is equal to the sublattice generated by $\{(a_i, b_i) \mid i \in \overline{\mathbf{n} - \mathbf{1}}\}$.

Finally, it is easy to construct synchronization transforms $(T, \mathcal{B}, \mathcal{S})$ for which $\mathcal{S}$ is generated by $\{T(W_i) \mid W_i \in \mathcal{B}\}$. In case (2) T will map $x_{j+1} = 0$ to $x_{a_j} = x_{b_j}$, $j \in \overline{\mathbf{k} - \mathbf{1}}$. $\square$

Remark 4.6. If we take generating set $\{(1, j) \mid j = 2, \ldots, k\}$, then the network given by the proposition will be the network $\mathcal{P}_k$ of [26]. Even though the sublattices of theorem 4.5(2) are always isomorphic to the sublattice generated by $\{(1, j) \mid j = 2, \ldots, k\}$, the associated networks are generally *not* linearly or dynamically equivalent [16, 3] (allowing for permutation of cells). ⊛

Suppose that $\mathcal{N} = \{N_1, \ldots, N_k\}$ is an identical cell network and that each cell has $p \geq 1$ asymmetric inputs. We construct a new identical cell network $\widehat{\mathcal{N}} = \{\widehat{N}_1, \ldots, \widehat{N}_{k+1}\}$ such that each cell $\widehat{N}_j$ has $p + 1$ asymmetric inputs and $\mathcal{N}$ is naturally embedded in $\widehat{\mathcal{N}}$. To do this, we start with the network $\mathcal{N}$ and add one input type to each cell N_j to obtain a new cell $\widehat{N}_j$ with $p + 1$ input types. Add a new cell $\widehat{N}_{k+1}$. It remains to fill the inputs of type $p + 1$ for $\widehat{N}_1, \ldots, \widehat{N}_k$ and the inputs of type $1, \ldots, p + 1$ of $\widehat{N}_{k+1}$. The output of $\widehat{N}_{k+1}$ will go to the input of type $p + 1$ for each of the cells $\widehat{N}_1, \ldots, \widehat{N}_k$. We fill all the inputs of $\widehat{N}_{k+1}$ with the output of $\widehat{N}_1$. This defines the identical cell network $\widehat{\mathcal{N}}$.

Lemma 4.7. *(Notation and assumptions as above.) Let $\mathcal{N}$ be a strongly connected identical cell network without self loops and with cells having $p \geq 1$ asymmetric inputs.*

(1) *$\widehat{\mathcal{N}}$ is a strongly connected identical cell network without self loops.*

(2) *$\mathcal{S}(\widehat{\mathcal{N}}) = \mathcal{S}(\mathcal{N}) \cup \{\mathbf{S}_0\}$, where $\mathbf{S}_0$ is the maximal element of $\mathcal{S}(\widehat{\mathcal{N}})$.*

(3) *$\mathcal{N}$ has a synchronization transform $(T, \mathcal{B}, \mathcal{S}(\mathcal{N}))$ of weight s if and only if $\widehat{\mathcal{N}}$ has a synchronization transform $(\widehat{T}, \widehat{\mathcal{B}}, \mathcal{S}(\widehat{\mathcal{N}}))$ of weight $s + 1$.*

Proof. The first statement is trivial. Concerning (2), it suffices to show that the only synchrony subspace for $\widehat{\mathcal{N}}$ containing N_{k+1} is $(1 \ldots k+1)$. Let $(A_1 \| \ldots \| A_s)$ be a synchrony subspace for $\widehat{\mathcal{N}}$ containing N_{k+1} as a synchronized node. Without loss of generality, we may suppose that $k + 1 \in A_1$. Since there are connections $\widehat{N}_{k+1} \to \widehat{N}_j$, $j \in \mathbf{k}$, and $\widehat{N}_1 \to \widehat{N}_{k+1}$, all to inputs of type $p + 1$, we must have $1 \in A_1$. Hence for all $i \in \mathbf{k}$, $j \in \mathbf{k} \cap A_1$, $\ell \in \mathbf{p}$, if there is a connection from $\widehat{N}_i$ to the ℓ-input of $\widehat{N}_j$, then $i \in A_1$ (since there is a connection from $\widehat{N}_1$ to the ℓ-input of $\widehat{N}_{k+1}$. It follows by the strong connectivity of $\widehat{\mathcal{N}}$ that $A_1 = (1 \ldots k+1)$, proving (2). Finally, (3) is a routine computation. $\qquad\square$

Examples 4.8. (1) Let $k = 4$. Up to permutation of nodes, theorem 4.5 gives three strongly connected identical cell networks with synchrony subspaces generated by (a) (1234), (b) $(12), (13), (14)$, and (c) $(12), (23), (34)$. Using the results for $k = 3$, lemma 4.7 yields three additional 4 cell identical network with synchrony subspaces generated by (d) $(12), (123), (1234)$, (e) $(123), (1234)$, (f) $(12), (23), (1234)$. However, this list is far from complete – see the concluding remarks 4.2 at the end of the section and the tables at the end of section 5.

(2) For $k \geq 2$, there is a coupled identical k cell network with synchrony subspaces $(12), (123), (1234), \ldots, (12 \ldots k)$. This follows by applying lemma 4.7 a total of $k - 2$ times to the 2 identical cell strongly connected network with one input type. All networks constructed support a synchronization transform. ※

Realization Conjecture, part I

Let $k \geq 2$ and $\mathcal{S}$ be a complete sublattice of $\mathcal{D}(k)$. If there exists a synchronization transform $(T, \mathcal{B}, \mathcal{S})$, then there is a strongly connected coupled identical k cell network $\mathcal{N}$, without self loops and with lattice of synchrony subspaces $\mathcal{S}(\mathcal{N}) = \mathcal{S}$. Cells of $\mathcal{N}$ have (at most) $k - 1$ asymmetric inputs.

Remarks 4.9. (1) The conjecture is not interesting if we allow self loops (see example 3.13(1)) or do not require strong connectivity (see examples 4.3(3)). In particular, the converse to the conjecture – that a synchronization transform is necessary – is obviously false if we do not require strong connectivity and no self-loops. Later we give an example (see example 4.11) where the converse can fail if the network equations have symmetry which acts non trivially on $\mathcal{S}$.

(2) The conjecture is true for $k = 3, 4$ (for $k = 3$, see examples 4.3(1); the case $k = 4$ may be shown on a case-by-case basis, see the end of section 5).   ⊛

When the Realization Conjecture holds, it seems to be usually true that the coupled cell network $\mathcal{N}$ can be chosen so that

- (A) Generically the linearizations of network vector fields at equilibria on synchrony subspaces have no multiple eigenvalues.
- (B) Given $\mathbf{U} \in \mathcal{S}$, $m \in \mathbf{k}^{\bullet}$, $\mathbf{p} \in \mathbf{U}$, there exists a network vector field which has a hyperbolic equilibrium at $\mathbf{p}$ of index m.

In order to verify (A), it suffices to show that equilibria on the minimal synchrony subspace $\mathbf{S}_0$ generically do not have multiple eigenvalues. More generally, if generically there are no multiple eigenvalues for equilibria on a synchrony subspace $\mathbf{U} \in \mathcal{S}$, the same will be true for all $\mathbf{T} \in \mathcal{S}$ such that $\mathbf{T} \supset \mathbf{U}$. If there is a synchronization transform, we have found no examples where the presence of multiple eigenvalues impacts the existence of robust heteroclinic networks or cycles (see also the last paragraph of example 4.11). Concerning (B), this is easy to verify if $m = 0, k$. Obviously if (B) holds for m, (B) holds for $k - m$.

Theorem 4.10. *Part I of the Realization Conjecture is true for all identical cell networks satisfying the conditions of theorem 4.5(2). Moreover, conditions (A,B) above hold.*

Proof. It is straightforward and easy to verify that any k identical cell network $\mathcal{N}$ satisfying the conditions of theorem 4.5(2) has a synchronization transform $(T, \mathcal{B}, \mathcal{S}(\mathcal{N}))$ of weight $k - 1$. By construction, the cells in $\mathcal{N}$ have $k - 1$ inputs. For condition (A) use remarks 4.9(3). Finally, for (B) it follows by induction on k, that we can reduce to the case $(12) \in \mathcal{S}$, $\mathbf{p} \in (12)$, This case is easy to check.   □

Realization Conjecture, part II

Let $k \geq 2$ and $\mathcal{N}$ be a k identical cell network with lattice of synchrony subspaces $\mathcal{S}$. Let $G \subset S_k$ denote the group of permutation symmetries of $\mathcal{N}$. A synchronization transform $(T, \mathcal{B}, \mathcal{S})$ exists if and only if G acts as the identity on $\mathcal{S}$. In particular, if $G_{\mathcal{S}}$ is the subgroup of S_k

preserving $\mathcal{S}$ then a sufficient condition for a synchronization transform is that $G_{\mathcal{S}}$ acts as the identity on $\mathcal{S}$.

Example 4.11 (Scope and converse to the realization conjectures). Suppose that $k = 4$, $s = 3$, $\mathcal{B} = \{H_{23}, H_{24}, H_{34}\}$, and we require $T(H_{23}) = (12\|34)$, $T(H_{24}) = (13\|24)$ and $T(H_{34}) = (14\|23)$. We have $\mathcal{S} = \{(12\|34), (13\|24), (14\|23), (1234)\}$. We can realize a synchronization map with the matrix

$$
T = [t_{ij}] = \begin{pmatrix} 1 & a & b & c \\ 1 & a_2 & b_2 & c \\ 1 & a_2 & b & c_3 \\ 1 & a & b_2 & c_3 \end{pmatrix}.
$$

where $a \neq a_2, b \neq b_2, c \neq c_3$. The coupled cell system shown in figure 9(a) has synchrony subspaces $\mathcal{S}$. A new feature is that $\mathcal{S}$, and the
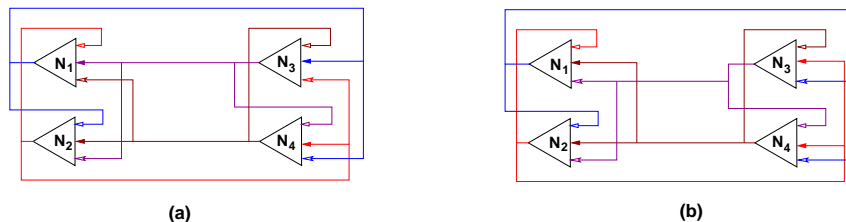

(a)   (b)

FIGURE 9. Four identical cell networks with nontrivial synchrony subspaces (a) $(12\|34), (13\|24), (14\|23), (1234)$, (b) $(12), (34), (13\|24), (12\|34), (1234)$

network realizing these synchrony subspaces, have a nontrivial symmetry group – the Klein four-group K – generated by the involutions $(12)(34)$, $(13)(24)$, $(14)(23)$. However, $\mathcal{S}$ is *pointwise fixed* by K and there is no action induced on $\mathcal{B}$ – this is compatible with part II of the realization conjecture.

If instead we take $\mathcal{S} = \{(12), (34), (13\|24), (12\|34), (1234)\}$, generated by $(12), (34), (13\|24)$, then it is *not* possible to find a synchronization transform even though there is a coupled identical 4 cell network $\mathcal{N}$ that realizes these synchrony subspaces (see figure 9(b)). Observe that the symmetry group of $\mathcal{S}$ (or $\mathcal{N}$) contains the involution $\sigma = (24)(13)$. Although σ preserves $\mathcal{S}$ it does not act trivially – $\sigma((12)) = (34)$. As a consequence, if there is a synchronization map $T : \mathbb{R}^4 \to \mathbb{R}^4$, there is a nontrivial $\mathbb{Z}_2$ action induced on $\mathcal{B} \subset \mathcal{L}(4)$. This gives an extra constraint on T which cannot be satisfied. That is, we cannot define generators for $\mathcal{S}$ in terms of elements of $\mathcal{L}(4)$. Parts I & II of the realization conjecture hold.

In a related direction, suppose

$$\mathcal{S} = \{(12), (12\|34), (13\|24), (14\|23), (1234)\}.$$

A basis for $\mathcal{S}$ contains *four* elements. The symmetry group of $\mathcal{S}$ contains the involution $\sigma = (12)$ and acts nontrivially on $\mathcal{S}$. It easy to verify directly that we cannot find $\mathcal{B} = \{W_1, W_2, W_3, W_4\}$ and a synchronization map T such that $T(W_1), \ldots, T(W_4)$ generate $\mathcal{S}$ – the conditions force T to be singular. A case by case analysis of the one input connection structures that support $(12\|34), (13\|24), (14\|23)$ and (12) shows that (34) will always be a synchrony subspace and so it is not possible to add a second input type that allows (12) and denies (34) (self loops do not help here). Although we can find a 4 identical cell network with synchrony subspaces $\mathcal{S}'$ generated by $(12), (34), (13\|24), (14\|23)$, the resulting network is not strongly connected if we deny self loops. Note that the involution $(13)(24)$ is a symmetry of $\mathcal{S}'$ which does not act trivially on $\mathcal{S}'$. Again, all this is consistent with part II of the realization conjecture.

Finally suppose that $\mathcal{S}$ is generated by $(12), (34), (1234)$. The involution $\sigma = (13)(24)$ is a nontrivial symmetry of $\mathcal{S}$ However, it is not possible to find a coupled cell network $\mathcal{N}$ with $\mathcal{S}(\mathcal{N}) = \mathcal{S}$ such that σ is a symmetry of $\mathcal{N}$. Here there exist coupled cell networks $\mathcal{N}$ with $\mathcal{S}(\mathcal{N}) = \mathcal{S}$, and which all have associated synchronization transforms, but σ is never a symmetry of $\mathcal{N}$. Moreover, equilibria in $(1234) \smallsetminus (12\|34)$ will have multiple eigenvalues. This provides an example where condition (A) is not satisfied.   ※

4.1. **Necessary conditions for $\mathcal{S}$ to be a lattice of synchrony subspaces.** Fix $k \geq 4$ and let $\mathcal{S}$ be a complete sublattice of $\mathcal{D}(k)$. We are interested in giving structural conditions on $\mathcal{S}$ that imply $\mathcal{S}$ cannot be the lattice of synchrony subspaces of a strongly connected k identical cell network without self loops.

Suppose that A is a subset of $\mathbf{k}$. Let $|A|$ denote the number of elements in A. In what follows we assume $3 \leq |A| < k$ and set $|A| = p$. Relabelling, suppose that $A = \{1, \ldots, p\}$. If $\mathfrak{s} = (A_1\| \ldots \|A_r) \in \mathcal{S}$, define $\text{supp}(\mathfrak{s}) = \cup_{i \in \mathbf{r}} A_i$ and set $\mathcal{S}_A = \{\mathfrak{s} \in \mathcal{S} \mid \text{supp}(\mathfrak{s}) \subset A\}$.

Definition 4.12. (Notation and assumptions as above.) The set $\mathcal{S}_A$ is a *synchrony substructure* of $\mathcal{S}$ if

(1) $(1 \ldots p) \in \mathcal{S}_A$.
(2) $\mathcal{S}_A$ is a sublattice of $\mathcal{S}$.

The synchrony substructure $\mathcal{S}_A$ is *indecomposable* if given $\mathfrak{s}, \mathfrak{t} \in \mathcal{S}_A$ there is a chain $\mathfrak{s} = \mathfrak{s}_1, \ldots, \mathfrak{s}_q = \mathfrak{t}$ of synchrony subspaces lying in $\mathcal{S}_A$ such that $\text{supp}(\mathfrak{s}_i) \cap \text{supp}(\mathfrak{s}_{i+1}) \neq \emptyset$, $i \in \overline{\mathbf{q}-\mathbf{1}}$.

Remarks 4.13. (1) If $\mathcal{S}_A$ a synchrony substructure, then $\mathcal{S}_A$ naturally embeds as a sublattice of $\mathcal{D}(p)$ with maximal element $(1\ldots p)$.

(2) More generally, if $B = (B_1\|\ldots\|B_r) \in \mathcal{S}$ and $\mathrm{supp}(B) \cap A = \emptyset$, we define $\mathcal{S}_{A,B}$ to consist of all $\mathfrak{s} \in \mathcal{S}$ such that $\mathfrak{s} = (A_1\|\ldots\|A_r\|B)$, $A_i \subset A$, $i \in \mathbf{r}$. The set $\mathcal{S}_{A,B}$ will be a *B-relative synchrony substructure* if $(1\ldots p\,\|B) \in \mathcal{S}_{A,B}$ and $\mathcal{S}_{A,B}$ is a sublattice of $\mathcal{S}$. Indecomposability is defined in the obvious way.   ⊛

Lemma 4.14. *(Notation and assumptions as above.) Let $\mathcal{N}$ be an identical k cell network with lattice of synchrony subspaces $\mathcal{S}(\mathcal{N}) = \mathcal{S}$. Assume cells have q asymmetric inputs. Suppose that $A \subset \mathbf{k}$ and $\mathcal{S}_A$ is a synchrony substructure of $\mathcal{S}$. For each input type $j \in \mathbf{q}$ one of the following conditions holds*

    (1) *There exists $\ell \in \mathbf{k} \smallsetminus A$ such that for all $i \in A$, there is a connection $N_\ell \rightarrow N_i$ to input j of N_i.*

    (2) *For all $i \in A$, there exists $\ell = \ell(i) \in A$, such that there is a connection $N_\ell \rightarrow N_i$ to input j of N_i.*

In particular, $\mathcal{S}_A$ determines an identical cell network $\mathcal{N}_A$, with cells $\{N_i \mid i \in A\}$ defined by deleting all inputs to, and outputs from, cells $N_i \in \mathcal{N}_A$ which connect to cells N_k, $k \notin A$.

Proof. Relabelling cells, we may assume $A = \mathbf{p}$. Since $(1\ldots p) \in \mathcal{S}_A$, the result is immediate from lemma 3.3.   □

Proposition 4.15. *Let $k \geq 4$ and $\mathcal{S}$ be a complete sublattice of $\mathcal{D}(k)$.*

    (1) *A necessary condition for there to exist a strongly connected k identical cell network $\mathcal{N}$, without self loops, with $\mathcal{S}(\mathcal{N}) = \mathcal{S}$, is that for every indecomposable synchrony substructure $\mathcal{S}_A$, with $|A| = p$, there exists a p identical cell network $\mathcal{N}_A$, without self loops, and with $\mathcal{S}(\mathcal{N}_A) = \mathcal{S}_A$.*

    (2) *A necessary condition for there to exist a strongly connected k identical cell network $\mathcal{N}$, without self loops, with a synchronization transform $(T, \mathcal{B}, \mathcal{S})$, is that for every indecomposable synchrony substructure $\mathcal{S}_A$, with $|A| = p$, there exists a synchronization transform $(T_A, \mathcal{B}_A, \mathcal{S}_A)$.*

Proof. Both statements follows easily from lemma 4.14; we omit the routine details.   □

Remark 4.16. Proposition 4.15 can easily be generalized to indecomposable *B*-relative synchrony substructures.   ⊛

Example 4.17. In figure 10(a) we show a 5 cell network $\mathcal{N}$ with $\mathcal{S}(\mathcal{N}) \supset \{(12), (13), (123)\}$. Taking $A = \mathbf{3}$, it is easy to see that $\mathcal{S}_A$
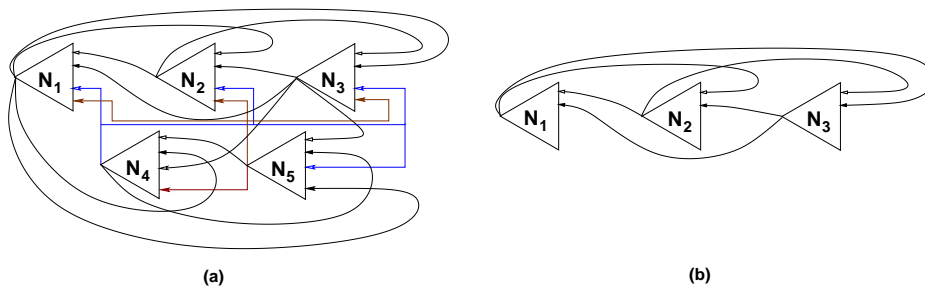
FIGURE 10. A 5 identical cell network with synchrony substructure $\mathcal{S_3}$.

is a synchrony substructure. The associated network $\mathcal{N}_A$ is shown in figure 10(b). ※

Examples 4.18. (1) Suppose that $k = 6$ and $\mathcal{S}$ is generated by

$$\{(12), (34), (13\|24), (14\|23), (1256)\}.$$

It follows from example 4.11 and proposition 4.14 that it is not possible to find a strongly connected 6 identical cell network $\mathcal{N}$, without self loops, such that $\mathcal{S}(\mathcal{N}) = \mathcal{S}$.
(2) Again, using example 4.11 and proposition 4.14, it is not possible to find a synchronization transform for the 6 cell network with synchrony subspaces generated by $\{(12), (34), (13\|24), (2356)\}$. Note that these synchrony spaces cannot be realized by a strongly connected 6 cell network without self loops. ※

4.2. **Summary.** The realization conjectures suggest a way to identify a large class of coupled identical cell networks which have synchrony subspaces closely related to invariant subspaces of SLF systems. In examples 4.8(1), example 4.11 we have identified eight coupled cell networks where the realization conjectures apply. Parts 1 and 2 (where applicable) of the conjectures also hold when $\mathcal{S}$ has generating sets (i) $\{(12), (13\|24)\}$, (j) $\{(12), (134)\}$, (k) $\{(12), (12\|34), (1234)\}$, (l) $\{(12\|34), (13\|24)\}$, and (m) $\{(12), (34), (1234)\}$. We discuss some of these examples further in the next section where our main focus is on investigating what this relationship between coupled cell networks and SLF systems implies about heteroclinic cycles and networks in coupled cell networks.

5. TRANSFORMING HETEROCLINIC NETWORKS

In this section we investigate ways of proving the existence of heteroclinic cycles and networks in coupled identical cell systems using

known heteroclinic cycles and networks in SLF systems. Specifically, suppose that $(T, \mathcal{B}, \mathcal{S})$ is a synchronization transform, with synchronization map $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$, and that there is a strongly connected k identical cell network, without self loops, which has lattice of synchrony subspaces $\mathcal{S}(\mathcal{N}) = \mathcal{S}$ – that is, part I of the Realization Conjecture holds. We show how we can use the synchronization map to transform robust heteroclinic cycles and networks for SLF systems, with k nodes, into corresponding results on robust heteroclinic cycles and networks for a coupled k identical cell network $\mathcal{N}$ with $\mathcal{S}(\mathcal{N}) = \mathcal{S}$. Our focus will be on examples that illustrate the approach; we present general conjectures in the next section.

5.1. **Simple heteroclinic networks and cycles for coupled identical cell systems.** As we indicated in example 3.14, heteroclinic cycles in coupled identical cell systems will generally not be simple (as defined in section 1) unless we restrict domains (just as we do for equivariant maps and generalized Lotka-Volterra systems).

Let $(T, \mathcal{B}, \mathcal{S})$ be a synchronization transform with $\mathcal{B} = \{H_i \mid i \geq 2\}$, $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$ and $\mathcal{S}$ generated by $T^\star(H_i) = (a_i, b_i)$ as in the statement of theorem 4.5(2). We have $T(H_i) = \{\mathbf{y} \in \mathbb{R}^k \mid y_{a_i} = y_{b_i}\}$. Let $\mathcal{N}$ denote the associated coupled cell network given by theorem 4.5(2).

Define the domain $D_+(T) = D_+ \subset \mathbb{R}^k$ by

$$D_+ = \{T(\mathbf{x}) \mid x_i \geq 0, i \geq 2\}.$$

Observe that D_+ is flow-invariant for coupled cell dynamics on $\mathbb{R}^k$, $\partial D_+ = D_+ \cap \bigcup_{i \geq 2} T(H_i)$, $D_+ \supset \mathbf{\Delta}(\mathbb{R}^k)$, and there are 2^{k-1} different choices for D_+ depending on the signs of the matrix entries for T. Up to homeomorphism, $D_+ = \mathbb{R} \times O_{k-1}$. A heteroclinic cycle Σ will be a simple heteroclinic cycle for network dynamics on $\mathcal{N}$, if we can choose T so that $\Sigma \subset D_+$ and Σ is simple in the sense of section 1 for dynamics restricted to D_+. The definition extends without difficulty to the case when $\mathcal{B}$ is a proper subset of $\{H_i \mid i \geq 2\}$.

Example 5.1 (A five cell system). The coupled cell network shown in figure 11 has lattice $\mathcal{S}(\mathcal{N})$ of synchrony subspaces generated by $\{(12), (23), (34), (25)\}$. Assuming 1-dimensional node dynamics, the equations for the system of figure 11 are

$$\dot{x}_1 = f(x_1; x_3, x_4, x_5, x_2), \quad \dot{x}_2 = f(x_2; x_3, x_4, x_5, x_1),$$

$$\dot{x}_3 = f(x_3; x_2, x_4, x_5, x_1), \quad \dot{x}_4 = f(x_4; x_2, x_3, x_5, x_1),$$
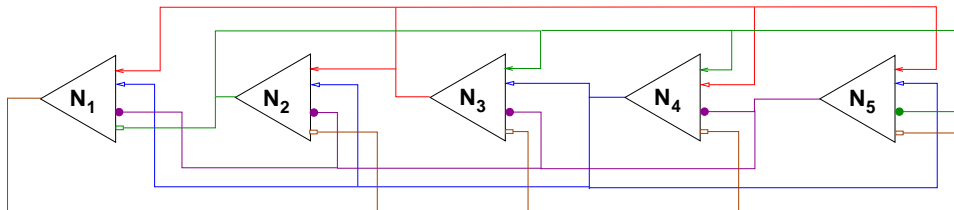
$$\dot{x}_5 = f(x_5; x_3, x_4, x_2, x_1).$$

FIGURE 11. A 5 node identical cell network $\mathcal{N}$ with synchrony subspaces $\mathcal{S}(\mathcal{N})$: (12), (23), (34), (25), (123), (125), (234), (235), (12‖34), (25‖34), (1234), (1235), (2345), (125‖34), and (12345).

We claim that we can choose f so that the identical cell system $\mathcal{N}$ has a robust simple heteroclinic cycle Σ

$$\cdots \to (235) \xrightarrow{23} (123) \xrightarrow{12} (12\|34) \xrightarrow{34} (25\|34) \xrightarrow{25} (235) \to \cdots$$

Note that vertices and connections are labelled by synchrony type. Observe that one cell desynchronizes along each connection of Σ and there is a resynchronization at the end point of the connection (to a different cluster of synchronized cells from those at the initial point of the connection).

It follows from theorem 4.5(2) that there is an associated synchronization transform $(T, \mathcal{B}, \mathcal{S})$. The synchronization map T will map hyperplanes as follows

$$\begin{aligned}
H_2 &\longrightarrow \quad x_1 = x_2, \quad H_3 \longrightarrow x_3 = x_4 \\
H_4 &\longrightarrow \quad x_2 = x_5, \quad H_5 \longrightarrow x_2 = x_3.
\end{aligned}$$

We use the synchronization map T for a formal derivation of Σ from the heteroclinic 2-face cycle $\Sigma_F \subset \mathbf{\Delta}_4$

$$\cdots \to \mathbf{v}_{23} \to \mathbf{v}_{34} \to \mathbf{v}_{45} \to \mathbf{v}_{25} \to \mathbf{v}_{23} \to \cdots .$$

This heteroclinic cycle is supported on the 3-face $\Delta_{2345} \approx \mathbf{\Delta}_3$ of $\mathbf{\Delta}_4$ (see figure 4: map $\mathbf{v}_{ij}$ to $\mathbf{v}_{i-1\,j-1}$).

We have $T(\mathbf{v}_{23}) = (u, V, V, w, V) \in (235)$, $T(\mathbf{v}_{34}) \in (123)$, $T(\mathbf{v}_{45}) \in (12\|34)$, and $T(\mathbf{v}_{25}) \in (25\|34)$. Each connection for the face cycle Σ_F will be mapped to the corresponding connection for Σ.

We assumed above that the heteroclinic face cycle Σ_F was a subset of $x_1 = 0$ – although this makes it easier to visualize the face cycle, the assumption is unnecessary and a little misleading. If we assume that the x_1-coordinates of all of the vertices of the cycle are non-zero, then Σ_F will be a 3-face heteroclinic cycle lying in $\mathbf{\Delta}_4$. Since the synchrony map is a linear isomorphism it preserves the dimension of invariant subspaces. For example, the vertex $\mathbf{v}_{123} \in \text{Int}(\Delta_{123})$, and

T embeds $\mathrm{Int}(\Delta_{123})$ into the 3-dimensional synchrony subspace space $(235) = T(H_{45})$. The connection from $\mathbf{v}_{123}$ lies in the 3-face Δ_{1234} spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ and the interior of this face gets embedded by T in the 4-dimensional synchrony space $(23) = T(H_5)$. If we take $x_1 = 0$, then $\mathbf{v}_{23}$ lies on the edge spanned by $\mathbf{v}_2, \mathbf{v}_3$. The interior of the edge still gets mapped into (235) by T.

What is essential is that the number of zero coordinates of a vertex on the face cycle, not counting the x_1-coordinate, is the same for all vertices on the cycle. In our example, there are two zero coordinates, not counting the x_1-coordinate. We remark that when we have connections between fully synchronous equilibria, there is no choice: we always take equilibria on the x_1-axis $\mathbf{V}_0$.

Along similar lines, we can realize

$$\cdots \to (2345) \overset{25\|34}{\to} (125\|34) \overset{12\|34}{\to} (1234) \overset{123}{\to} (1235) \overset{235}{\to} (2345) \to \cdots$$

as a robust heteroclinic cycle Σ_e. Using the synchronization map T, Σ_e can be derived from the heteroclinic edge cycle (cf. figure 2(a))

$$\cdots \to \mathbf{v}_2 \to \mathbf{v}_5 \to \mathbf{v}_4 \to \mathbf{v}_3 \to \mathbf{v}_2 \to \cdots$$

Equivalently, Σ_e may be derived from the 2-face cycle

$$\cdots \to \mathbf{v}_{12} \to \mathbf{v}_{15} \to \mathbf{v}_{14} \to \mathbf{v}_{13} \to \mathbf{v}_{12} \to \cdots,$$

Having identified two potential heteroclinic cycles via the above mechanism, it is easily verified by direct computation that network dynamics can be chosen so that either cycle is realized as a robust heteroclinic cycle in the coupled cell network architecture $\mathcal{N}$.

Summarizing, we propose a method for constructing robust heteroclinic cycles and networks for coupled identical cell networks. If part I of the Realization Conjecture is true, and we can satisfy conditions (A,B) where required, then, modulo geometric obstructions to making connections, the method will always work to yield robust heteroclinic cycles and networks. We give precise conjectures in the final section of the paper.   ※

Remark 5.2. If we assume that the first and second inputs of cells are symmetric in the previous example, then it can be shown that it is still possible to construct the robust simple cycle Σ. However, because of multiple eigenvalues, it is no longer possible to construct a robust simple cycle Σ_e. Essentially, synchrony transforms of face heteroclinic cycles and networks work better than edge cycles if we want to allow for some symmetry in the input structure to cells.   ✳

Example 5.3 (A four cell SLF system)**.** Next we consider two ways of deriving a heteroclinic cycle between synchronized states from the RPS heteroclinic 3 cycle Σ shown in figure 12.
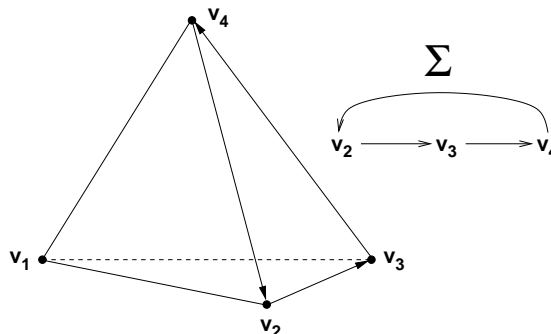

FIGURE 12. An SLF system with heteroclinic 3 cycle.

We start by realizing Σ in the coupled cell network of figure 6 – the coupled cell network $\mathcal{P}_4$ of [26]. This network has lattice of synchrony subspaces $\mathcal{S}$ generated by $(12), (13)$, and (14) and so satisfies the conditions of theorem 4.5(2). Hence there is an associated synchronization transform $(T, \mathcal{B}, \mathcal{S})$. The synchronization map T will map hyperplanes as follows

$$
\begin{aligned}
x_2 = 0 &\longrightarrow x_1 = x_2 \\
x_3 = 0 &\longrightarrow x_1 = x_3 \\
x_4 = 0 &\longrightarrow x_1 = x_4
\end{aligned}
$$

The heteroclinic cycle Σ transforms to the heteroclinic cycle $(134) \overset{13}{\rightarrow} (123) \overset{12}{\rightarrow} (124) \overset{14}{\rightarrow} (134)$ which can be realized as a robust simple heteroclinic cycle in the network $\mathcal{P}_4$. We briefly indicate some of the straightforward computations. With node phase space $\mathbb{R}$, network dynamics on $\mathcal{P}_4$ is given by the system

$$
\begin{aligned}
\dot{x}_1 = f(x_1; x_2, x_3, x_4), \quad \dot{x}_2 = f(x_2; x_1, x_3, x_4), \\
\dot{x}_3 = f(x_3; x_2, x_1, x_4), \quad \dot{x}_4 = f(x_4; x_2, x_3, x_1).
\end{aligned}
$$

If $\mathbf{t} = (a, a, a, b) \in (123)$ is an equilibrium of the network vector field $\mathbf{F}$, then the eigenvalues of the linearization $D\mathbf{F}(\mathbf{t})$ are

$$
\begin{aligned}
\lambda_{1,4} &= \frac{1}{2}\left((\bar{\alpha}_1 + \sigma) \pm \sqrt{(\bar{\alpha}_1 + \sigma)^2 - 4(\bar{\alpha}_1\sigma - \alpha_4\bar{\sigma})}\right) \\
\lambda_2 &= \alpha_1 - \alpha_2, \\
\lambda_3 &= \alpha_1 - \alpha_3,
\end{aligned}
$$

where $\bar{\mathbf{t}} = (b, a, a, a)$, then $\alpha_i = \frac{\partial f}{\partial x_i}(\mathbf{t})$, $\bar{\alpha}_i = \frac{\partial f}{\partial x_i}(\bar{\mathbf{t}})$, $i \in \mathbf{4}$, $\sigma = \sum_{i \in \mathbf{3}} \alpha_i$, $\bar{\sigma} = \sum_{i \in \mathbf{3}} \bar{\alpha}_i$ and we have set $\bar{\mathbf{t}} = (b, a, a, a)$. Consequently, we can choose $\mathbf{F}$ near $\mathbf{t}$ so that $\mathbf{t}$ is of index 1 with $W^u(\mathbf{t}) \subset (13)$. Similar computations hold for equilibria in (134) and (124). It follows there are no local obstructions to realizing the cycle in $\mathcal{P}_4$. Using the method of [26, §4.5], it is easily shown there are no geometric obstructions to realizing Σ as a robust (simple) heteroclinic cycle in $\mathcal{P}_4$.

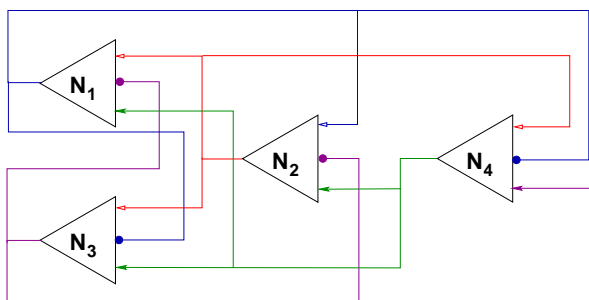Next we indicate how we can realize Σ in the coupled cell network shown in figure 13.

FIGURE 13. A 4 identical cell network with synchrony subspaces generated by $(12), (13), (34)$.

Just as in the previous example, theorem 4.5(2) applies and there exists an associated synchronization transform $(T, \mathcal{B}, \mathcal{S})$. The synchronization map T maps the hyperplanes according to

$$\begin{aligned}
x_2 = 0 &\longrightarrow x_1 = x_2 \\
x_3 = 0 &\longrightarrow x_3 = x_4 \\
x_4 = 0 &\longrightarrow x_1 = x_3.
\end{aligned}$$

Using the synchronization map T, Σ transforms to the heteroclinic cycle

$$\cdots \to (134) \overset{34}{\to} (12\|34) \overset{12}{\to} (123) \overset{134}{\to} (134) \to \cdots$$

The heteroclinic cycle may be realized as a robust heteroclinic network in the identical cell network of figure 13 which has equations

$$\begin{aligned}
\dot{x}_1 &= f(x_1; x_2, x_3, x_4), \quad \dot{x}_2 = f(x_2; x_1, x_3, x_4), \\
\dot{x}_3 &= f(x_3; x_2, x_1, x_4), \quad \dot{x}_4 = f(x_4; x_2, x_1, x_3).
\end{aligned}$$

Local computations are similar to those given above and omitted. ※

The previous examples all concern strongly connected identical cell networks which have the maximal number of synchrony subspaces. We

next look at examples with smaller lattices of synchrony subspaces. For simplicity, we restrict to 4 cell networks.

5.2. 4 **cell networks.**

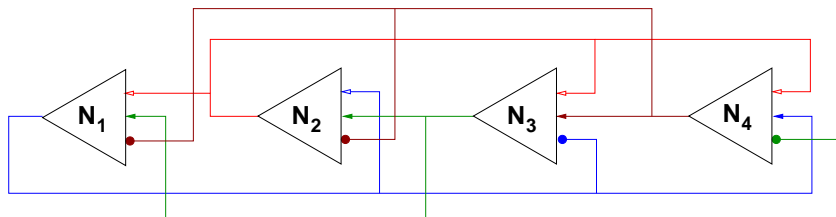**Example 5.4.** Consider the 4 cell network $\mathcal{N}$ shown in figure 14.

FIGURE 14. A 4 identical cell network $\mathcal{N}$ with lattice of synchrony subspaces $\mathcal{S} = \{(12), (134), (1234)\}$.

The network of figure 14 is associated to the synchronization transform $(T, \mathcal{B}, \mathcal{S})$, where the synchronization map T maps hyperplanes according to

$$H_2 \;\longrightarrow\; x_1 = x_2$$
$$H_{34} \;\longrightarrow\; x_1 = x_3 = x_4.$$

The nontrivial synchrony subspaces have different dimensions and so we do not expect to see heteroclinic cycles that are transforms of edge or face cycles.

For this example, the only possible *robust* heteroclinic cycles for $\mathcal{N}$ must have equilibria on $(1234) = \mathbf{S}_0$ and so we expect to use the synchrony transform T to transform a heteroclinic cycle with equilibria on $\mathbf{V}_0$ to a robust heteroclinic cycle for $\mathcal{N}$ with equilibria on $\mathbf{S}_0$. The heteroclinic cycle with equilibria on $\mathbf{V}_0$ cannot be a face heteroclinic cycle – the equilibria would have to lie on the same face.

It is not hard to check that an SLF system in $\mathbb{R}^4$ can support a robust heteroclinic cycle with equilibria on $\mathbf{V}_0$. Connections in H_2 (respectively, H_{34}) will then be transformed into connections in (12) (respectively, (134)). Given $\mathbf{p}, \mathbf{q} \in \mathbf{S}_0$, we search for a network map f such that the equations for the network of figure 14,

$$\dot{x}_1 = f(x_1; x_2, x_3, x_4), \quad \dot{x}_2 = f(x_2; x_1, x_3, x_4),$$
$$\dot{x}_3 = f(x_3; x_2, x_4, x_1), \quad \dot{x}_4 = f(x_4; x_2, x_1, x_3),$$

support a robust heteroclinic cycle Σ with equilibria $\mathbf{p}, \mathbf{q} \in \mathbf{S}_0$, and connections $\mathbf{p} \overset{12}{\to} \mathbf{q}$ and $\mathbf{q} \overset{134}{\to} \mathbf{p}$. Observe that *two* nodes desynchronize along the connection $\mathbf{p} \to \mathbf{q}$.

Assuming node phase space $\mathbb{R}$, the eigenvalues $\lambda_i(\mathbf{s})$ of $D\mathbf{F}(\mathbf{s})$ at an equilibrium $\mathbf{s} \in \mathbf{S}_0$ of the network vector field $\mathbf{F}$ are

$$\lambda_1(\mathbf{s}) = \sum_{i \in \mathbf{4}} \alpha_i, \ \lambda_2(\mathbf{s}) = \alpha_1 - \alpha_2,$$

$$\lambda_{3,4}(\mathbf{s}) = \alpha_1 - \frac{1}{2}\left((\alpha_3 + \alpha_4) \pm \imath\sqrt{3}(\alpha_3 - \alpha_4)\right),$$

where $\alpha_i = \frac{\partial f}{\partial x_i}(\mathbf{s})$, $i \in \mathbf{4}$. The eigenvalue λ_1 corresponds to dynamics on $\mathbf{S}_0$; the eigenvalues λ_1, λ_2 to dynamics on (134); and the eigenvalues $\lambda_1, \lambda_3, \lambda_4$ to dynamics on (12). Since λ_3, λ_4 are complex conjugate, we cannot find a simple heteroclinic cycle with equilibria at $\mathbf{p}, \mathbf{q}$.

One possibility is to require $\lambda_1(\mathbf{p}), \lambda_2(\mathbf{p}) < 0$, $\mathrm{Re}(\lambda_{3,4}(\mathbf{p})) > 0 > \mathrm{Re}(\lambda_{3,4}(\mathbf{q})), \lambda_1(\mathbf{q})$ and $W^u(\mathbf{p}) \smallsetminus \{\mathbf{p}\} \subset W^s(\mathbf{q}) \subset (12)$. If, in addition, we require that $\lambda_2(\mathbf{q}) > 0$, and there is a connection from $\mathbf{q}$ to $\mathbf{p}$ in (134), we obtain a robust heteroclinic cycle with equilibria at $\mathbf{p}, \mathbf{q}$ (there will be a continuum of connections from $\mathbf{p}$ to $\mathbf{q}$).   ※

Example 5.5. The network of figure 15 is associated to the synchronization transform $(T, \mathcal{B}, \mathcal{S})$, where T maps hyperplanes according to

$$H_{23} \longrightarrow x_1 = x_2, x_3 = x_4$$
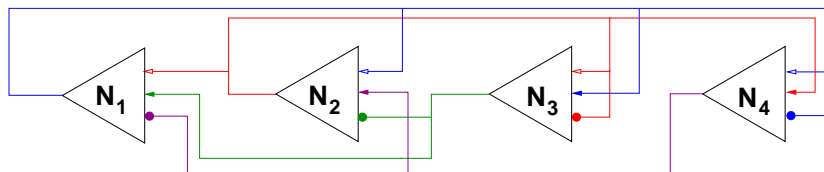$$H_{34} \longrightarrow x_1 = x_3, x_2 = x_4.$$

FIGURE 15. A 4 identical cell network with synchrony subspaces $\mathcal{S} = \{(12\|34), (13\|24), (1234)\}$.

The equations for the network are

$$\dot{x}_1 = f(x_1; x_2, x_3, x_4), \quad \dot{x}_2 = f(x_2; x_1, x_4, x_3),$$
$$\dot{x}_3 = f(x_3; x_2, x_1, x_2), \quad \dot{x}_4 = f(x_4; x_1, x_2, x_1).$$

It is straightforward to verify that the network structure supports a heteroclinic cycle with vertices $\mathbf{p}, \mathbf{q} \in \mathbf{S}_0$ and connections $\mathbf{p} \overset{12\|34}{\to} \mathbf{q}$ and $\mathbf{q} \overset{13\|24}{\to} \mathbf{p}$. This cycle is derived from a cycle for an SLF system on $\mathbb{R}^4$ (but not from a face heteroclinic cycle – see the discussion for

example 5.4). The eigenvalues of the linearization of the network vector field an an equilibrium $\mathbf{s} \in \mathbf{S}_0$ are

$$\lambda_1 = \sum_{i \in \mathbf{4}} \alpha_i, \ \lambda_2 = \alpha_1 - \alpha_3 - \alpha_4, \ \lambda_3 = \alpha_1 - \alpha_3 - \alpha_4, \ \lambda_4 = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4,$$

where $\alpha_i = \frac{\partial f}{\partial x_i}(\mathbf{s})$, $i \in \mathbf{4}$. The eigenvalue λ_1 corresponds to dynamics on $\mathbf{S}_0$; the eigenvalues λ_1, λ_2 to dynamics on $(12\|34)$; and the eigenvalues λ_1, λ_3 to dynamics on $(13\|24)$.

Similarly, we predict and find that the 4 cell identical system of figure 9(a), with synchrony subspaces $(12\|34)$, $(13\|24)$, $(14\|23)$, and (1234), supports a robust heteroclinic cycle Σ connecting three equilibria $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbf{S}_0$ and with three connections, each lying in one of the synchrony subspaces $(12\|34)$, $(13\|24)$, $(14\|23)$. We remark the example gives an interesting variation on the standard representation of the RPS heteroclinic 3 cycle.

The 4-cell network of figure 9(b), which does not have a synchronization transform, does not support robust heteroclinic cycles with equilibria on $\mathbf{S}_0$. The network does support a robust heteroclinic cycle with equilibria on $(12\|34)$ and connections in (12) and (34). ※

5.3. **Summary of the results for** 4 **cell networks.** Assume networks are strongly connected, without self loops and that cells have asymmetric inputs. By simple heteroclinic cycle we mean here that unstable manifolds of equilibria are one-dimensional. In every case, a simple heteroclinic cycle will extend to a heteroclinic network which is simple in the sense of section 1 and has the same set of equilibria.

In table 1, we list the 4 identical cell networks which do not support robust heteroclinic cycles. These networks all have synchronization transforms and satisfy parts I and II of the realization conjecture as well as conditions (A,B).

In table 2, we list the 4 cell networks that support robust heteroclinic cycles.

Remarks 5.6. (1) The network $\mathcal{V}_1$ does not support robust simple heteroclinic cycles and does not satisfy condition (B). We refer to example 5.4. The networks $\mathcal{V}_2$, $\mathcal{V}_4$ support both simple and not simple robust heteroclinic cycles. All three networks show the phenomenon of two cells desynchronizing along one of the connections.
(2) All networks have a synchronization transform and satisfy the realization conjectures. Other than $\mathcal{V}_1$, all networks satisfy (B). ⊛

Finally, in table 3, we list the 4 cell networks that support robust heteroclinic cycles but do not satisfy condition (A).

TABLE 1. 4 cell networks which do not support robust heteroclinic cycles

	Network equations	Generators of $\mathcal{S}(\mathcal{N})$	Multiple eigenvalues
$\mathcal{U}_1$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_3, x_1, x_4)$ $\dot{x}_3 = f(x_3; x_4, x_4, x_2)$ $\dot{x}_4 = f(x_4; x_1, x_3, x_1)$	(1234)	No
$\mathcal{U}_2$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_4, x_2, x_4)$ $\dot{x}_4 = f(x_4; x_3, x_1, x_2)$	(12), (1234)	No
$\mathcal{U}_3$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_3, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_1, x_2, x_4)$ $\dot{x}_4 = f(x_4; x_1, x_1, x_1)$	(123), (1234)	No
$\mathcal{U}_4$	$\dot{x}_1 = f(x_1; x_2, x_4, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_4, x_1, x_2)$ $\dot{x}_4 = f(x_4; x_3, x_2, x_2)$	(12‖34), (1234)	No
$\mathcal{U}_5$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_2, x_4)$ $\dot{x}_4 = f(x_4; x_3, x_1, x_3)$	(12), (123), (1234)	No

Remarks 5.7. (1) $\mathcal{W}_1$ has a synchronization transform. Although there is no robust heteroclinic cycle joining equilibria on (1234), the network does support robust heteroclinic cycles joining equilibria on (12‖34). (2) $\mathcal{W}_2$ does not support a synchronization transform. Although there is no robust heteroclinic cycle joining equilibria on (1234), $\mathcal{W}_2$ does support robust heteroclinic cycles joining equilibria on (12‖34).   ⊛

We have identified 16 inequivalent networks which are associated to a synchronization transform and for which the realization conjectures hold. There is a network $\mathcal{W}_2$ – figure 9(b) – which is not associated to a synchronization transform and is consistent with part II of the realization conjecture. Of the 16 networks which are associated to a synchronization transform, 11 support robust heteroclinic cycles which in every case are related to heteroclinic cycles in an SLF system.

5.4. **Heteroclinic networks.** We conclude with an example of a robust heteroclinic network which can be realized in coupled cell networks and is the transforms of an SLF face network.

Example 5.8. In figure 16 we show a schematic for a robust 3-face heteroclinic network Σ_N supported in $\boldsymbol{\Delta}_5 \subset \mathbb{R}^6$. Referring to figure 16,

TABLE 2. 4 cell networks which support robust hetero-
clinic cycles and satisfy condition (A).

	Network equations	Generators of $\mathcal{S}(\mathcal{N})$	Multiple eigenvalues	Robust heteroclinic cycles	Simple heteroclinic cycles
$\mathcal{V}_1$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_2, x_4)$ $\dot{x}_4 = f(x_4; x_3, x_1, x_3)$	(12), (134)	No	Yes, but see remarks	No: complex eigenvalues
$\mathcal{V}_2$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_4, x_1, x_2)$ $\dot{x}_4 = f(x_4; x_3, x_3, x_2)$	(12), (13‖24)	No	Yes	Yes, but complex eigen- values possible
$\mathcal{V}_3$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_4, x_3)$ $\dot{x}_3 = f(x_3; x_2, x_1, x_2)$ $\dot{x}_4 = f(x_4; x_1, x_2, x_1)$	(12‖34), (13‖24)	No	Yes	Yes
$\mathcal{V}_4$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_4, x_1, x_2)$ $\dot{x}_4 = f(x_4; x_3, x_2, x_2)$	(12), (12‖34), (1234)	No	Yes	Yes, but complex eigen- values possible
$\mathcal{V}_5$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_1, x_4)$ $\dot{x}_4 = f(x_4; x_1, x_1, x_1)$	(12), (13), (1234)	No	Yes	Yes
$\mathcal{V}_6$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_2, x_4)$ $\dot{x}_4 = f(x_4; x_1, x_2, x_3)$	(12), (123), (12‖34)	No	Yes	Yes
$\mathcal{V}_7$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_4, x_3)$ $\dot{x}_3 = f(x_3; x_4, x_2, x_1)$ $\dot{x}_4 = f(x_4; x_3, x_1, x_2)$	(12‖34), (13‖24) (14‖23)	No	Yes	Yes
$\mathcal{V}_8$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_2, x_4)$ $\dot{x}_4 = f(x_4; x_1, x_3, x_1)$	(123), (124)	No	Yes	Yes
$\mathcal{V}_9$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_1, x_4)$ $\dot{x}_4 = f(x_4; x_2, x_3, x_1)$	(12), (13), (14)	No	Yes	Yes
$\mathcal{V}_{10}$	$\dot{x}_1 = f(x_1; x_2, x_3, x_4)$ $\dot{x}_2 = f(x_2; x_1, x_3, x_4)$ $\dot{x}_3 = f(x_3; x_2, x_1, x_4)$ $\dot{x}_4 = f(x_4; x_2, x_1, x_3)$	(12), (13), (34)	No	Yes	Yes

$\mathbf{v}_{145}$ is an equilibrium on the 2-face Δ_{145} spanned by vertices $\mathbf{v}_1$, $\mathbf{v}_4$, $\mathbf{v}_5$. Similarly for the other equilibria. Each connection between equilibria lies in a 3-face of $\mathbf{\Delta}_5$. For example, $\mathbf{v}_{145} \rightarrow \mathbf{v}_{135}$ lies in the 3-face Δ_{1345}.

TABLE 3. 4 cell networks which support robust hetero-
clinic cycles but do not satisfy condition (A).

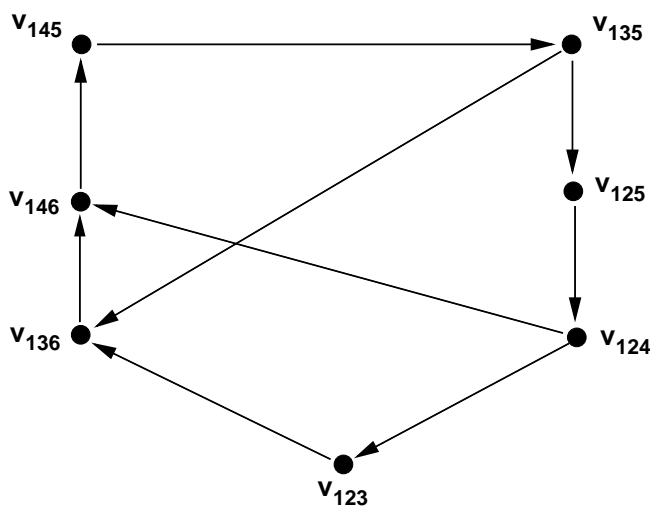|	Network equations	Generators of $\mathcal{S}(\mathcal{N})$	Multiple eigenvalues	Robust heteroclinic cycles	Simple heteroclinic cycles
$\mathcal{W}_1$	$\dot{x}_1 = f(x_1; x_2, x_4, x_3)$ $\dot{x}_2 = f(x_2; x_1, x_4, x_3)$ $\dot{x}_3 = f(x_3; x_4, x_2, x_2)$ $\dot{x}_4 = f(x_4; x_3, x_2, x_2)$	(12), (34), (1234)	Yes: (1234)	Yes: (12‖34)	Yes
$\mathcal{W}_2$	$\dot{x}_1 = f(x_1; x_2, x_4, x_3)$ $\dot{x}_2 = f(x_2; x_1, x_4, x_3)$ $\dot{x}_3 = f(x_3; x_4, x_2, x_1)$ $\dot{x}_4 = f(x_4; x_3, x_2, x_1)$	(12), (34), (13‖24)	Yes: (1234)	Yes: (12‖34)	Yes

FIGURE 16. Schematic for 3-face heteroclinic network Σ_N supported in $\mathbf{\Delta}_5$.

The index of the equilibria $\mathbf{v}_{135}$ and $\mathbf{v}_{124}$ is 2, all other equilibria have index 1.

The network $\mathcal{P}_6$ has lattice of synchrony subspaces $\mathcal{S}$ generated by $\{(1j) \mid j = 2, \ldots, 6\}$. We have an associated synchronization transform $(T, \mathcal{B}, \mathcal{S})$ associated to the network $\mathcal{P}_6$ (given by theorem 4.5). The synchronization map T maps the hyperplanes in $\mathcal{B} = \{H_j \mid j = 2, \ldots, 6\}$ according to $H_j \longrightarrow (1j)$, $j = 2, \ldots, 6$.

Apply the synchronization map T to the the heteroclinic network Σ_N to obtain the predicted heteroclinic network Σ for the architecture $\mathcal{P}_6$. We show the result in figure 17 (nodes and connections are labelled by synchrony type). After some lengthy, but straightforward,
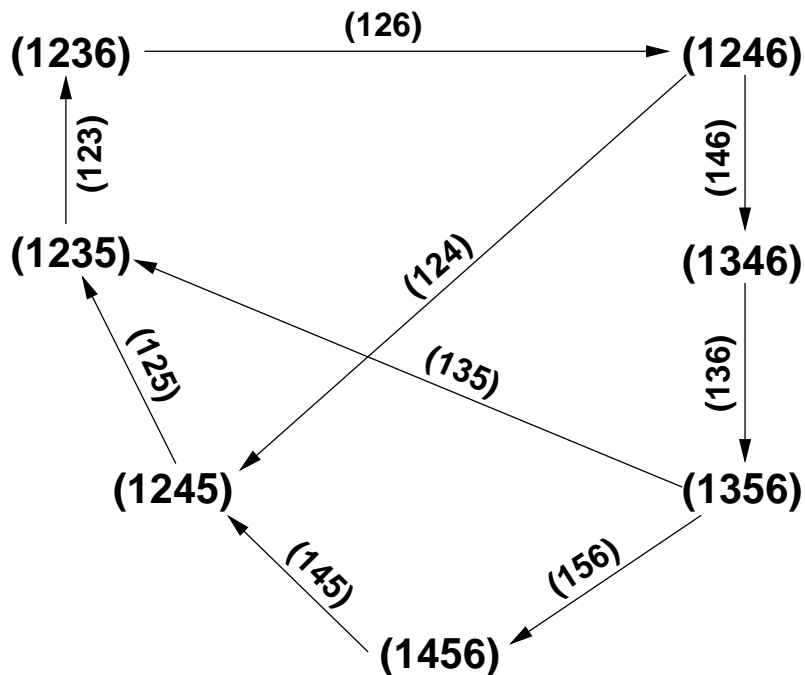
FIGURE 17. The robust heteroclinic network Σ realized in $\mathcal{P}_6$

computations it may be shown that a network vector field can be chosen for which Σ is a robust heteroclinic network in the architecture $\mathcal{P}_6$ with 1-dimensional cell dynamics. In particular, all equilibria will be hyperbolic saddles with the correct index, and connections will lie in the synchrony subspaces indicated in figure 17.

Finally, we remark that the heteroclinic network has 7 equilibria and 9 connections and that no two connections lie in the same synchrony subspace. The realization is more efficient than that given in [26, Theorem 1.1] which realizes the heteroclinic network in $\mathcal{P}_{10}$.   ※

6. Outstanding Questions and Concluding Comments

6.1. The Realization Conjectures.

(1) Part I of the conjectures proposes that if there is a synchronizations transform $(T, \mathcal{B}, \mathcal{S})$, then there is a coupled identical cell network $\mathcal{N}$ with $\mathcal{S}(\mathcal{N}) = \mathcal{S}$. Is it possible to find an algorithm that gives $\mathcal{N}$ (that is, the architecture) in terms of $(T, \mathcal{B}, \mathcal{S})$?

(2) Is it possible to find strong sufficient conditions on a coupled identical cell network $\mathcal{N}$ that imply that linearization of network vector fields have no restrictions on index or multiplicity? (Conditions (A,B) of section 4.)

Regarding (2), symmetries of $\mathcal{N}$ and $\mathcal{S}(\mathcal{N})$ seem to play a role. However, conditions such as requiring symmetries of $\mathcal{N}$ to act trivially on $\mathcal{S}$ or all symmetries of $\mathcal{S}$ to act trivially on $\mathcal{S}$ are either false (see $\mathcal{W}_2$) or too weak.

6.2. **Transition from SLF to coupled identical cell networks.** It is generally straightforward to construct explicit vector fields that realize heteroclinic cycles and networks in an SLF system [21, 11, 12]. Often (perhaps always) cubic vector fields suffice. On the other hand it seems difficult to construct explicit vector fields that realize heteroclinic cycles and networks in coupled identical cell systems. Can these difficulties be overcome by working with heterogeneous networks with two cell types or by allowing the node (or coupling) dynamics to be 2-dimensional? We refer to the recent work of Ashwin & Postlethwaite [12] for the use of two cell types and to the comments on scalar signalling in [3, 26] for restrictions on coupling. We are inclined to the view that there should be a natural way of transforming from a coupled identical cell system of the type we construct to systems with two or more distinct cell types for which it is easy to construct explicit and natural vector fields realizing heteroclinic cycles and networks[1]. In short, we would like to think of the identical cell network, with one dimensional node dynamics, as a *minimal model* for coupled cell networks supporting robust heteroclinic phenomena (just as the Kuramoto phase oscillator model can be viewed as a minimal model for describing the phase dynamics of a weakly coupled system of nonlinear oscillators). The question then is how to unfold the minimal model to obtain more physically realistic models with, for example, additive input structure [26, §2, §5] (a crucial assumption in the reduction of weakly coupled networks of nonlinear oscillators to phase oscillator models [43, 38]).

If a heteroclinic network is clean, can we always require the same of the network when realized in an coupled identical cell system? In particular, how straightforward is it to construct asymptotically stable heteroclinic attractors in coupled cell systems?

6.3. **Symmetric inputs and sparseness of coupling.** Suppose that an r-face heteroclinic network is realized in a coupled identical cell system using one of the architectures given by theorem 4.5(2). Under what conditions can we 'symmetrize' some of the inputs to the cells without breaking the robust structure – for example, by introducing

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[1]However, this might require thresholds and vector fields may only be piecewise smooth or pulse coupled [47].

multiple eigenvalues. Is it the case that the larger r is the more inputs we can symmetrize and can this be quantified?

In biological and technological networks, coupling is typically sparse and far from "all-to-all". In examples 3.15(2) we gave a simple six identical cell example, with two symmetric inputs, which supported a robust heteroclinic cycle. Is it possible to find families of identical cell networks which have relatively few inputs, compared to the total number of cells, such that (a) the networks support robust heteroclinic networks, and (b) robustness persists if we allow for (approximately) symmetric inputs? We would expect that a positive answer to this question would yield potentially realistic networks for which there were robust heteroclinic networks connecting relatively small clusters of synchronized cells.

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