Asynchronous networks

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[1]Research supported in part by NSF Grant DMS-1265253 & Marie Curie IEF Fellowship (Project 626111);
[2]research supported in part by NSF Grant DMS-1265253 & Marie Curie IIF Fellowship (Project 627590).

1 Asynchronous networks: motivation & characteristics

Asynchronous networks are an approach to network dynamics that takes account of features encountered in networks from engineering and biology, especially neuroscience, and where techniques such as averaging or methods from statistical physics may miss essential structure (see [**3**, §1] for a careful discussion). Asynchronous networks may involve a mix of distributed and decentralized control, adaptivity, event driven dynamics, switching, varying network topology and hybrid dynamics. Network dynamics will generally only be piecewise smooth, nodes may stop and later restart, and there may be no intrinsic global time. Intended applications range from switching problems involving power grids and microgrids [**6**], production and transport networks, and learning mechanisms from neuroscience such as Spike-Timing Dependent Plasticity [**8**]. In these notes we sketch some of the main ideas and refer the reader to [**3, 4**] for more details and references. We summarize below some of the key features of asynchronous networks.

(1) State dependent and/or stochastic variation in connection structure and dependencies between nodes.
(2) Synchronization events associated with stopping or waiting states of nodes.
(3) Order of events may depend on the initialization of the system.
(4) Dynamics is only piecewise smooth.
(5) Aspects involving function, adaptation and control.
(6) Evolution only defined for forward time – systems are not time reversible.

1.1 Reductionism and modularization of dynamics

In nonlinear network dynamics and complex systems generally, there is the question as to how far one can make use of reductionist techniques [**9**, 2.5]. For example, one approach, advanced by Alon and Kastan [**10**] in systems biology, has been the identification and description of *network motifs* (small network configurations that occur frequently in large biological networks). The underlying premise is that a modular, or engineering, approach to network dynamics is feasible: identify building blocks, connect together to form networks and then describe dynamical properties of the resulting network in terms of the dynamics of its components.

> *"Ideally, we would like to understand the dynamics of the entire network based on the dynamics of the individual building blocks." Alon* [**1**, p 27].

Such a reductionist approach works well in linear systems theory, where a superposition principle holds, or in the study of synchronization in weakly coupled nonlinear oscillators (for example, [**11**]), but is usually unrealistic in the study of heterogenous networks modelled by a system of analytic nonlinear differential equations: network dynamics may bear little or no relationship to the intrinsic (uncoupled) dynamics of nodes.

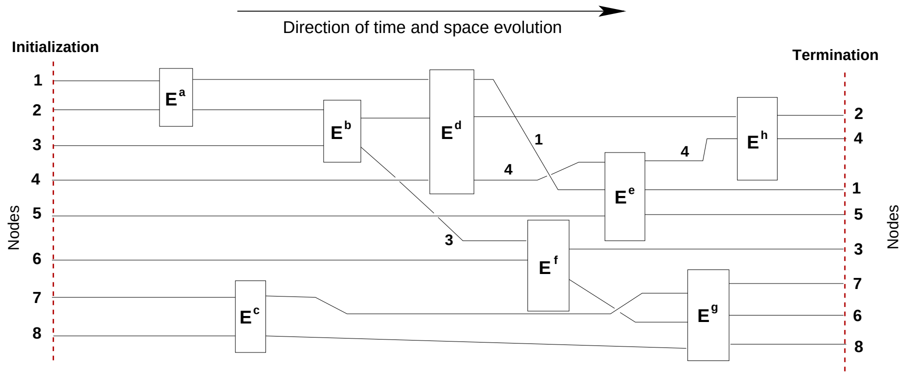

FIGURE 1. A functional feedforward network with 8 nodes

If we emphasize function and allow for intermittent connection structure, then it may be possible to apply reductionist principles. In figure 1 we show schematics of a network with only intermittent connection between eight nodes[1]. Each node N_i will be given an initial state and started at time $T_i \geq 0$. Nodes interact depending on their state. For example, in the evolution depicted in figure 1, nodes N_1, N_2 will first interact during the event indicated by $\mathbf{E}^a$. Observe there is no global time defined for this system but there is a partially ordered temporal structure: event $\mathbf{E}^c$ always occurs after event $\mathbf{E}^a$ but may occur before or after event $\mathbf{E}^b$. This network has also a *function*: reaching the terminal states indicated on the right hand side of the figure. Observe the possibility of a (dynamical) deadlock: network function is not achieved.

Our main result is a *modularization of dynamics theorem*. Specifically, we give general conditions that enable us to describe the function of a large class of functional asynchronous networks in terms of the function of constituent subnetworks (as in figure 1; details are in [**4**]). The theorem allows for events depending on local times and for variation in the number of nodes (a dynamics version of a *Petri Net* [**5**]). Nonsmoothness is a crucial ingredient needed for this result. In networks modelled by smooth dynamical systems, all nodes are effectively coupled to each other at all times and so information propagates instantly across the entire network. A spatiotemporal decomposition, such as is given by the modularization of dynamics theorem, is only possible if the network dynamics is nonsmooth and (subsets of) nodes are allowed to evolve independently of each other for periods of time. This allows us to construct discrete dynamical units (for example, Alon's motifs), each with its own function, that together make up the dynamics of the network. The result highlights what can get lost when averaging over a network: the averaging out of the functional units and their temporal relations that yield network

[1]For example, view figure 1 as being part of a threaded computer program and $\mathbf{E}^a, \ldots, \mathbf{E}^h$ as being synchronization events – evolution of associated threads is stopped until each thread has finished its computation. Variables are then synchronized across the threads.

function. A consequence is that rather than asking how network dynamics can be understood in terms of the dynamics of constituents, one has to ask how network function can be understood in term of the function of constituents.

2 Abstraction

We give the formal setup for asynchronous networks in the simplest case, omitting most technical details (for these see [**3**]), and conclude with an example of a transport network.

For $k \in \mathbb{N}$, define $\mathbf{k} = \{1, \ldots, k\}$, $\mathbf{k}^\bullet = \mathbf{k} \cup \{0\}$. Assume a network with k nodes, $N_1, \ldots, N_k$. Let M_i denote the phase space of N_i, $i \in \mathbf{k}$, and set $\mathbf{M} = \prod_{i\in\mathbf{k}} M_i$ – the network phase space. A vector field $\mathbf{f}$ on $\mathbf{M}$ is a *network vector field*.

Stopping, waiting, and synchronization are characteristic features of asynchronous networks. If the node of a network is stopped or partially stopped, then node dynamics will be constrained to a subset of the node phase space (a single point if the node is stopped). We codify this situation by introducing a *constraining node* N_0 that, when connected to N_i, implies that dynamics on N_i is constrained. Set $\mathcal{N} = \{N_0, \ldots, N_k\}$. We often abuse notation and refer to the *network* $\mathcal{N}$.

2.1 Connection structures & admissible vector fields

We represent interactions between distinct nodes in $\mathcal{N}$ by the network graph. Connections $N_j \to N_i$ encode *dependencies*, if $i, j \in \mathbf{k}$, and *constraints* if $j = 0$, $i \in \mathbf{k}$.

A *connection structure* α is a directed network graph on the nodes $\mathcal{N}$ such that for all $i \in \mathbf{k}$, $j \in \mathbf{k}^\bullet$, $i \neq j$, there is at most one directed connection $N_j \to N_i$. An α-admissible network vector field $\mathbf{f}$ has dependencies given by α (if $N_j \to N_i \notin \alpha$, $\mathbf{f}_i^\alpha$ will not depend on x_j and conversely). A *generalized connection structure* $\mathcal{A}$ is a set of connection structures on $\mathcal{N}$. An $\mathcal{A}$*-structure* $\mathcal{F}$ is a set $\mathcal{F} = \{\mathbf{f}^\alpha \mid \alpha \in \mathcal{A}\}$ of admissible vector fields.

2.2 The event map & asynchronous networks

Suppose given a generalized connection structure $\mathcal{A}$ and an $\mathcal{A}$-structure $\mathcal{F}$. Interactions between nodes in asynchronous networks may be state or time dependent. We consider state dependence and handle interactions and constraints using an *event map* $\mathcal{E} : \mathbf{M} \to \mathcal{A}$.

Given a generalized connection structure $\mathcal{A}$, $\mathcal{A}$-structure $\mathcal{F}$ and event map $\mathcal{E}$, the quadruple $(\mathcal{N}, \mathcal{A}, \mathcal{F}, \mathcal{E})$ defines an *asynchronous network*. Dynamics on $\mathcal{N}$ is given by the state dependent network vector field $\mathbf{F}$ defined by

$$\mathbf{F}(\mathbf{X}) = \mathbf{f}^{\mathcal{E}(\mathbf{X})}(\mathbf{X}),\ \mathbf{X} \in \mathbf{M}. \tag{1}$$

Subject to quite simple regularity conditions [**3**], the network vector field (1) will have a uniquely defined semi-flow (continuous in time but not necessarily in the initial state).

Although the integral curves of (1) are not always the same as those obtained using standard approaches to nonsmooth systems, there are relations between asynchronous networks and Filippov systems [**7**]. This is explored further in [**3, 2**].

2.3 Example: two passing loops on single track lines

In figure 2, we show three trains that have to traverse two passing loops. Train $\mathfrak{T}_1$ starts at station A; train $\mathfrak{T}_2$ at B; train $\mathfrak{T}_3$ at C. Once train $\mathfrak{T}_2$ has traversed the first passing loop it will continue on the branch line towards the second passing loop and station B.

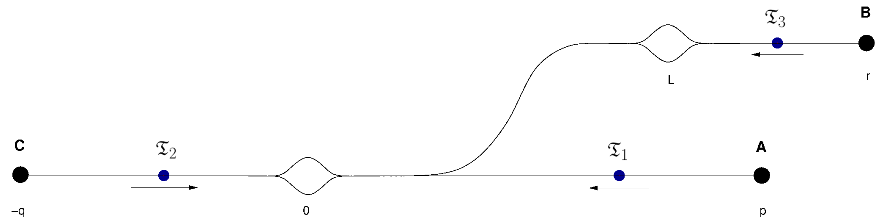

FIGURE 2. Three trains going through passing loops at 0 and L.

We model the transport network using our abstraction of an asynchronous network (see also[**4**]). Take $M_i = \mathbb{R}$, $\alpha_i = N_0 \to N_i$, $i \in \mathbf{3}$ (α_i corresponds to $\mathfrak{T}_i$ stopped). Take generalized connection structure $\mathcal{A} = \{\boldsymbol{\emptyset}, \alpha_1, \alpha_2, \alpha_3, \alpha_1 \vee \alpha_3, \alpha_2 \vee \alpha_3\}$, where $\boldsymbol{\emptyset}$ denotes the empty connection structure. Let $v_2 > 0 > v_1, v_3$ and define the $\mathcal{A}$-structure $\mathcal{F}$ by

$$\mathbf{f}^{\boldsymbol{\emptyset}} = (v_1, v_2, v_3), \qquad \mathbf{f}^{\alpha_1} = (0, v_2, v_3), \qquad \mathbf{f}^{\alpha_2} = (v_1, 0, v_3),$$
$$\mathbf{f}^{\alpha_3} = (v_1, v_2, 0), \qquad \mathbf{f}^{\alpha_1 \vee \alpha_3} = (0, v_2, 0), \qquad \mathbf{f}^{\alpha_2 \vee \alpha_3} = (v_1, 0, 0).$$

Define the event map $\mathcal{E} : \mathbf{M} \to \mathcal{A}$ by

$$\mathcal{E}(x_1, x_2, x_3) = \begin{cases} \alpha_1 & \text{if } x_1 = 0, x_2 < 0, \\ \alpha_2 & \text{if } x_1 > 0, x_2 = 0 \text{ or } x_2 = L, x_3 > L, \\ \alpha_3 & \text{if } x_2 < L, x_3 = L, \\ \alpha_1 \vee \alpha_3 & \text{if } x_1 = 0, x_2 < 0, x_3 = L, \\ \alpha_2 \vee \alpha_3 & \text{if } x_1 < 0, x_2 = 0, x_3 = L, \\ \boldsymbol{\emptyset} & \text{otherwise.} \end{cases}$$

This defines the asynchronous network $\mathfrak{N} = (\mathcal{N}, \mathcal{A}, \mathcal{F}, \mathcal{E})$ and gives the correct train dynamics. The modularization of dynamics theorem applies to $\mathfrak{N}$; see [**4**] for details.

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Abstract: Asynchronous networks form a natural framework for many classes of dynamical networks encountered in technology, engineering and biology. Typically, nodes can evolve independently, be constrained, stop, and later restart, and interactions between components of the network may depend on time, state, and stochastic effects. We outline some of the main ideas, motivations and a basic result.