Plan for the day

I Reduced-form models of credit risk.
II Statistical analysis of default-time data.
III Stochastic network models for large portfolios.
IV Optimization of credit portfolios.
I. Reduced-form Models of Credit Risk
Agenda

• Credit Ratings
  – Ratings and rating transitions
  – CDS and CDO
  – Moody’s Binomial Expansion Technique and application to CBOs.
  – Credit Metrics

• Joint Distributions, Hazard Rates and Copulas
  – Definitions of hazard rates and copulas
  – Calibration
  – The Diamond Default model
1 Credit Rating

Rating agencies (Moody’s-KMV, S&P, Fitch) assign credit ratings (AAA, AA, . . .) to firms and transactions on the basis of detailed case-by-case analysis. They also compile statistics of changes of rating and defaults.

Charts show

- Cumulative default probabilities out to 10 years;

- Change of rating matrix

These are obtained by a ‘cohort analysis’: start with (say) all the AA-rated firms on 1 January 1981 ..
### S&P 1-year rating transition matrix

<table>
<thead>
<tr>
<th>Current rating</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>87.74</td>
<td>10.93</td>
<td>0.45</td>
<td>0.63</td>
<td>0.12</td>
<td>0.10</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>AA</td>
<td>0.84</td>
<td>88.23</td>
<td>7.47</td>
<td>2.16</td>
<td>1.11</td>
<td>0.13</td>
<td>0.05</td>
<td>0.02</td>
</tr>
<tr>
<td>A</td>
<td>0.27</td>
<td>1.59</td>
<td>89.05</td>
<td>7.40</td>
<td>1.48</td>
<td>0.13</td>
<td>0.06</td>
<td>0.03</td>
</tr>
<tr>
<td>BBB</td>
<td>1.84</td>
<td>1.89</td>
<td>5.00</td>
<td>84.21</td>
<td>6.51</td>
<td>0.32</td>
<td>0.16</td>
<td>0.07</td>
</tr>
<tr>
<td>BB</td>
<td>0.08</td>
<td>2.91</td>
<td>3.29</td>
<td>5.53</td>
<td>74.68</td>
<td>8.05</td>
<td>4.14</td>
<td>1.32</td>
</tr>
<tr>
<td>B</td>
<td>0.21</td>
<td>0.36</td>
<td>9.25</td>
<td>8.29</td>
<td>2.31</td>
<td>63.89</td>
<td>10.13</td>
<td>5.58</td>
</tr>
<tr>
<td>CCC</td>
<td>0.06</td>
<td>0.25</td>
<td>1.85</td>
<td>2.06</td>
<td>12.34</td>
<td>24.86</td>
<td>39.97</td>
<td>18.60</td>
</tr>
</tbody>
</table>

For example, the probability that a bond rated BBB today will be rated instead AA in one year, is equal to 1.89 %. Note: BBB is the minimum ‘investment grade’.
Is the rating transition process Markovian? Let $M_k$ denote the empirical $k$-year transition matrix. If the process is Markov, we expect to find

$$M_k = (M_1)^k.$$ 

In fact this is not at all accurate. There is a ‘momentum effect’: firms that have been recently downgraded are more likely to be downgraded again than other firms in the same rating category. David Lando suggests a Markov model with additional ‘hyperstates’ A*, BBB* etc. Downgrade probabilities are higher in A* than in A. A company that is downgraded moves first to A* and then, after some time, to A.
1.1 Credit Default Swaps

Protection buyer pays regular premiums $\pi$ until $\min(\tau, T)$ where $T$ is the contract expiry time and $\tau$ the default time of the Reference Bond.

Protection seller pays $(1 - R)1_{(\tau < T)}$ at next coupon date after $\tau$, where $R$ is the recovery rate. If $F$ is the risk-neutral survivor function of $\tau$, the ‘fair premium’ $\pi$ is determined by

$$\sum_{i=1}^{n} \pi p(0, t_i) F(t_i) = \pi \times CV01 = \sum_{i=1}^{n} (F(t_{i-1}) - F(t_i))(1 - R)p(0, t_i).$$
If we have CDS rates $\pi_k$ for maturities $T_k, k = 1, \ldots, m$ and a family of distributions $\{F_\theta, \theta \in \mathbb{R}^m\}$ then we can determine the ‘implied default distribution’ $\hat{F}_\theta$. Example: $m = 1$ and $F_\theta(t) = e^{-\theta t}$.

MORAL: CDS rates determine the risk-neutral marginal default time distribution for the reference issuer.

NOTE: Selling credit protection is (nearly) equivalent to buying the reference bond with borrowed funds:

- Borrow $100$ at Libor $L$.
- Buy bond at par for $100$.
- Bond pays coupon $L + x$, so net payment is $(x - \text{losses})$.
- At maturity, sell bond and redeem loan.
1.2 Collateralized Debt Obligations (CDO)

Cash Flow CBO

Investors subscribe $100 to SPV which purchases bond portfolio. SPV issues rated notes to investors. Coupons paid in seniority order.
Here SPV sells credit protection to counterparty as individual-name CDS, buys credit protection on tranches from investors with premiums $x < y < z$.

The *joint* default distribution is the key thing here.

New market product: **iTraxx index** – tranche quotes publicly available on a standardised debt portfolio. Significance: market data directly related to ‘correlation’.
The design process for cash-flow CDOs

Cash Flow CBO

- Set the size of the senior tranche as big as possible while satisfying expected loss constraints needed to secure AAA credit rating (see below).

- Similarly for mezzanine tranches.

- Size of equity tranche is set by risk appetite of investors.

- Pricing (i.e. spreads) on rated tranches are determined by market conditions on launch date.
1.3 The rating process: Moody’s Binomial Expansion Technique

Start with a portfolio of $M$ bonds, each (for simplicity) having the same notional value $X$. Each issuer is classified into one of 32 industry classes. The portfolio is deemed equivalent to a portfolio of $M' \leq M$ independent bonds, each having notional value $X M / M'$. $M'$ is the diversity score, computed from the following table.
Diversity score table:

<table>
<thead>
<tr>
<th>No. of firms</th>
<th>Diversity Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
</tr>
<tr>
<td>3</td>
<td>2.0</td>
</tr>
<tr>
<td>4</td>
<td>2.3</td>
</tr>
<tr>
<td>5</td>
<td>2.6</td>
</tr>
<tr>
<td>6</td>
<td>3.0</td>
</tr>
<tr>
<td>7</td>
<td>3.2</td>
</tr>
<tr>
<td>8</td>
<td>3.5</td>
</tr>
<tr>
<td>9</td>
<td>3.7</td>
</tr>
<tr>
<td>10</td>
<td>4.0</td>
</tr>
</tbody>
</table>
Diversity score example:

\[ M = 60 \text{ bonds.} \]

<table>
<thead>
<tr>
<th>No. of firms in sector</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of incidences</td>
<td>12</td>
<td>12</td>
<td>5</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Diversity</td>
<td>12</td>
<td>18</td>
<td>10</td>
<td>2.3</td>
<td>2.6</td>
</tr>
</tbody>
</table>

Meaning: 12 cases where the firm is the only representative of its industry sector, 12 pairs of firms in the same sector, etc.

**Diversity score = 45.**

The effect of reduced diversity is to push weight out into the tail of the loss distribution, as the chart shows.
Binomial loss distributions

Fraction of portfolio

$d = 60$

$d = 45$

$d = 30$
“Loss” in rated tranches

Suppose the coupon in a rated tranche is $c$ and the amounts actually received are $a_1, \ldots, a_n$. Then the loss is $1 - q$ where

$$q = \frac{a_1}{1 + c} + \frac{a_2}{1 + c} + \cdots + \frac{a_n}{1 + c}.$$  

Note that $q = 1$ (loss zero) when $a_i = c$, $i < n$ and $a_n = 1 + c$.

Moody’s rates tranches on a threshold of expected loss.

Example: $M = 60, p = 0.1$.

Expected no. of defaults is $\mu = np = 6$, standard deviation

$$\sigma = \sqrt{np(1 - p)} = 2.32.$$  

The senior tranche might have a loss threshold of $\mu + 3\sigma = 13$.

Chart shows expected loss as function of diversity score. Expected loss increases by a factor of 10 as diversity score is reduced from 60 to 30.
1.4 CreditMetrics

CreditMetrics is aimed at producing the Value-at-Risk over a 1-year time horizon for – say – a bond portfolio. Assumptions:

- There is a fixed credit spread for each credit rating
- Change in value is due only to change in credit rating
- Change in credit rating follows the Moody’s 1-year transition matrix.

What about correlation?

- There are $N$ industry sectors and each obligor $i$ has a weight $N$-vector $w_i$ such that $w_{i,j}$ represents the participation of obligor $i$ in sector $j$.
- CreditMetrics estimates the equity return correlation for sector indices $I_1, \ldots, I_N$, giving a correlation matrix $Q$. 
• The return for obligor $i$ is

$$r_i = w_{i,0}r_{i,0} + \sum_{j=1}^{N} w_{i,j}R_j$$

where $R_j$ is the normalized return for sector index $I_j$ and $r_{i,0}$ is an idiosyncratic factor.

• The obligor correlations are

$$\rho_{ik} = \text{corr}(r_i, r_k) = \frac{w_i^T Qw_k}{\sqrt{(w_i^T Qw_i + w_{i,0}^2)(w_k^T Qw_i + w_{k,0}^2)}}$$

• Generate a normal $M$-vector $X$ with mean 0 and covariance matrix $A$ with diagonal elements 1 and off-diagonal elements $\rho_{ik}$.

• Choose quantiles in each coordinates direction so that $X_i$ gives the correct transition probabilities for obligor $i$ (see picture)
This procedure gives the joint transition probabilities for all obligors. (In the figure, Obligor 1 starts at BBB, obligor 2 at A.)
2 Joint distributions, Hazard Rates and Copulas

Let $\tau \geq 0$ be a random variable with density $f(t)$. The survivor function $G$ and distribution function $F$ are

$$P[\tau > t] = G(t) = 1 - F(t) = \int_t^\infty f(u)du.$$  

The hazard rate is

$$h(t)dt = \frac{f(t)}{G(t)}dt \approx P[\tau \in ]t, t + dt[|\tau > t],$$

and there is a 1-1 relation between $h$ and $G$ in that

$$G(t) = e^{-\int_0^t h(u)du}.$$  

Thus specifying $h$ is equivalent to specifying $f$.  

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For random variables $\tau_1, \tau_2 \geq 0$ with joint density $f(t_1, t_2)$ the marginal and joint distributions are

\[
F_1(t) = \int_0^t \int_0^{\infty} f(u, v) \, dv \, du
\]
\[
F_2(t) = \int_0^{\infty} \int_0^t f(u, v) \, dv \, du
\]
\[
F(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} f(u, v) \, dv \, du,
\]

while the survivor function is

\[
G(t_1, t_2) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} f(u, v) \, dv \, du
\]

The joint distribution $F$ is related to the marginals by the \textit{copula function} $C$ given by

\[
F(t_1, t_2) = C(F_1(t_1), F_2(t_2)).
\]

Given marginal distributions $F_1, F_2$, formula (1) defines a \textit{bone fide} joint distribution for \textit{any} choice of copula function $C$. 
Let $\tau_{\text{min}} = \min(\tau_1, \tau_2)$, $\tau_{\text{max}} = \max(\tau_1, \tau_2)$. Then

$$P[\tau_{\text{min}} > t] = G(t, t).$$

The initial hazard rate is therefore (see picture)

$$h_0(t) = \frac{1}{G(t, t)} \left( \int_t^\infty f(u, t) du + \int_t^\infty f(t, v) dv \right)$$
Suppose \( \tau_1 \) occurs first. The conditional density of \( \tau_2 \) is then
\[
\frac{f(\tau_1, t)}{\int_{\tau_1}^{\infty} f(\tau_1, v) dv}
\]
for \( t \geq \tau_1 \), so that the new hazard rate is
\[
h_2(t) = \frac{f(\tau_1, t)}{\int_{t}^{\infty} f(\tau_1, v) dv},
\]
while if \( \tau_2 \) occurs first the hazard rate switches to
\[
h_1(t) = \frac{f(t, \tau_2)}{\int_{t}^{\infty} f(u, \tau_2) du}.
\]

The hazard rate process is therefore
\[
h(t) = h_0(t)1_{(t<\tau_{min})} + (h_2(t)1_{(\tau_{min}=\tau_1)} + h_1(t)1_{(\tau_{min}=\tau_2)})1_{(\tau_{min} \leq t < \tau_{max})}
\]
2.1 Copula-based calibration

Let $\tau_1, \tau_2, \ldots$ default times of issuers $A, B, \ldots$. If $F$ is the joint distribution and $F_i$ is the marginal distribution of $\tau_i$ then

$$F(t_1, \ldots, t_n) = C(F_1(t_1), \ldots, F_n(t_n))$$

for some copula function $C$ (= a multivariate DF with uniform marginals).

(i) Back out marginal default distributions $F_1(t), F_2(t), \ldots$ from credit spreads or CDS rates.

(ii) Choose your favourite copula function – say, Gaussian copula.

(iii) Take correlation parameter $\rho = \text{correlation of equity returns}$.

(iv) Define joint distribution $F$ as above.
In more detail:

(i) In a CDS contract, protection buyer pays regular premiums $\pi$ until $\min(\tau, T)$ where $T$ is the contract expiry time and $\tau$ the default time of the Reference Bond. Protection seller pays $(1 - R)1_{(\tau<T)}$ at next coupon date after $\tau$, where $R$ is the recovery rate. If $F$ is the risk-neutral survivor function of $\tau$, the ‘fair premium’ $\pi$ is determined by

$$
\sum_{i=1}^{n} \pi p(0, t_i) F(t_i) = \pi \times CV01 = \sum_{i=1}^{n} (F(t_{i-1}) - F(t_i))(1 - R)p(0, t_i).
$$

If we have CDS rates $\pi_k$ for maturities $T_k, k = 1, \ldots, m$ and a family of distributions $\{F_\theta, \theta \in \mathbb{R}^m\}$ then we can determine the ‘implied default distribution’ $F_{\hat{\theta}}$. 
A standard procedure is to take

\[ F_{\theta}(t) = \exp \left( - \int_0^t h(s) ds \right) \]

where

\[ h(s) = \sum_{i=1}^{m} \theta_i 1_{[T_{i-1}, T_i]}(s). \]

Then \( \theta_1, \theta_2, \ldots \) are determined recursively using \( \pi_1, \pi_2, \ldots \).

Note: in a multivariate setting there is a different parametrization \( F_{j,\theta_j} \) for each issuer.

(ii), (iii) To generate a random vector \( U \) with uniform marginals and a gaussian copula, take \( U_i = N^{-1}(X_i) \) where \( X \) is a normal random vector with \( \mathbb{E}X_i = 0, \mathbb{E}X_i^2 = 1, \mathbb{E}X_iX_j = \rho_{ij} \).
NB: Care must be taken that the covariance matrix

\[ \Sigma = \begin{bmatrix} 1 & \rho_{11} & \rho_{12} & \cdots \\ \rho_{21} & 1 & \rho_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \]

is actually non-negative definite. If the \( \rho_{ij} \) are obtained from sample estimates, it may not be!

To generate \( X \), take \( X = AZ \) where \( A \) is the Cholesky factorization of \( \Sigma \) (i.e. \( \Sigma = AA^T \)) and \( Z \) are independent \( N(0,1) \). In the 2-dimensional case we can simply take \( X_1 = Z_1, X_2 = \rho_{12}Z_1 + \sqrt{1 - \rho_{12}^2}Z_2 \).

(iv) Default times \( \tau_i \) are given by

\[ \tau_i = F_{i,\theta_i}^{-1}(U_i). \]

(There is an obvious algorithm for computing this when \( F_{i,\theta} \) is as above.)
2.2 Simultaneous calibration: The Diamond Model

An obvious drawback of copula methods is: how do you choose the copula? An alternative is to think in terms of ‘infection’.

![Diagram of the Diamond Model]

- **Node 0**: A non-def, B non-def
- **Node 1**: A def, B non-def
- **Node 2**: A non-def, B def
- **Node 3**: A def, B def

- **Edges**:
  - From 0 to 1: $h_1$
  - From 0 to 2: $h_2$
  - From 1 to 3: $a h_2$
  - From 2 to 3: $a h_1$
• Hazard rate of remaining issuer increases by a factor $a$ after first default.

• If functions $h_1, h_2$ are time-dependent and we replace $ah_1, ah_2$ by general $h_3, h_4$ then we can represent any joint distribution this way.
Marginal distributions

\[ F_1(t) = 1 - e^{-(h_1+h_2)t} - \frac{h_2 e^{-ah_1 t}}{h_1 + h_2 - ah_1} \left( 1 - e^{-(h_1+h_2-ah_1)t} \right) \]

\[ F_2(t) = 1 - e^{-(h_1+h_2)t} - \frac{h_1 e^{-ah_2 t}}{h_1 + h_2 - ah_2} \left( 1 - e^{-(h_1+h_2-ah_2)t} \right) \]

Double Default

\[ F_{DD}(t) = 1 - e^{-(h_1+h_2)t} - \frac{h_2 e^{-ah_1 t}}{h_1 + h_2 - ah_1} \left( 1 - e^{-(h_1+h_2-ah_1)t} \right) \]

\[ - \frac{h_1 e^{-ah_2 t}}{h_1 + h_2 - ah_2} \left( 1 - e^{-(h_1+h_2-ah_2)t} \right) \]

Calibration

*Joint* calibration to credit spreads/CDS rates for issuers A and B, for given ‘enhancement’ parameter \( a \). (Time-varying \( h_1, h_2 \) required for term structure of credit spreads)
Continuous-time CDS premium $\pi_i$ on asset $i$ is determined by

$$
\pi_i \int_0^T e^{-rt}(1 - F_i(t))dt = (1 - R) \int_0^T e^{-rt} f_i(t)dt,
$$

where $r$ is the riskless rate and $f_i(t) = dF_i(t)/dt$. From the model, we find

$$
\pi_1 = (1 - R) \frac{I_2}{I_1}
$$

where (with $m(\alpha, T) = \frac{1}{\alpha}(1 - e^{-\alpha T})$)

$$
I_1 = h_1(1 - a)m(r + h_1 + h_2, T) + h_2m(r + ah_1, T),
$$

$$
I_2 = (h_1 + h_2)h_1(1 - a)m(r + h_1 + h_2, T) + ah_1h_2m(r + ah_1, T).
$$

The first default time $\tau_{\text{min}} = \tau_1 \land \tau_2$ is exponential with rate $(h_1 + h_2)$. Hence the FTD premium is

$$
\pi_{\text{FTD}} = (1 - R)(h_1 + h_2).
$$

Chart shows calibrated $h_1, h_2$ when CDS rates are $\pi_1 = 75bp$, $\pi_2 = 200bp$. 
Calibrated parameters $h_1$, $h_2$, and First-to-Default premium

Enhancement factor $a$

$0$ $0.005$ $0.01$ $0.015$ $0.02$ $0.025$ $0.03$ $0.035$

$1$ $2$ $3$ $4$ $5$ $6$ $7$ $8$

$h_1$ $h_2$ FTD

enhancement factor $a$
Generators and Backward Equations

The *generator* of a Markov process $x_t$ is an operator $\mathcal{A}$ acting on functions $\mathcal{D}(\mathcal{A})$ such that

$$M_t^f = f(x_t) - f(x_0) - \int_0^t \mathcal{A}f(x_s)ds$$

is a martingale for $f \in \mathcal{D}(\mathcal{A})$. The corresponding *backward equation* is

$$\frac{\partial v}{\partial t} + \mathcal{A}v - \beta v = 0$$

$$v(T, x) = h(x)$$

The solution of the backward equation is

$$v(t, x) = E_{t,x} \left[ e^{-\int_t^T \beta(x_s)ds} h(x_T) \right]$$

(as in Black-Scholes).
For the diamond model, \( x_t \in \{0, 1, 2, 3\} \) and the generator can be expressed in matrix form as

\[
A = \begin{bmatrix}
-(h_A + h_B) & h_A & h_B & 0 \\
0 & -\beta h_B & 0 & \beta h_B \\
0 & 0 & -\alpha h_A & \alpha h_A \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Solving the backward equation amounts to computing the matrix exponential \( e^{At} \). We can also solve the Forward equation

\[
\frac{d}{dt}p(t) = p(t)A
\]

for the probability distribution of the process at time \( t \) (expressed as a row vector)

Other, more complex, models are also easily computable.