A problem of optimal investment with randomly terminating income

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1. Utility maximization. Probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\), normalized asset price semimartingale \(S = (S^1_t, \ldots, S^d_t)\), trading strategy \(\theta_t\), initial endowment \(x\). Portfolio value \(X^\theta_t\) is

\[
X^\theta_t = x + \int_0^t \theta_s dS_s.
\]

General optimal investment problem: calculate

\[
u(x) = \sup_{\theta} E[U(X^\theta_T)].
\]

Complete theory available, based on convex duality [Karatzas et al, 1991, Kramkov & Schachermayer, 1999]

A cumulative random endowment is an \(\mathcal{F}_t\)-adapted process \(\epsilon_t\). (This is income if \(\epsilon_t\) is increasing, liability if \(\epsilon_t\) is decreasing.) New problem: find

\[
u_{\epsilon}(x) = \sup_{\theta} E[U(X^\theta_T + \epsilon_T)].
\]

- Optimal investment.
- Liability management.
- Asset valuation: if \(B\) is the value of a contingent claim exercised at time \(T\), the ‘utility neutral’ value of \(B\) at time 0 is \(p\) satisfying

\[
u(x) = u_B(x - p)
\]

where \(u_B(x) = \sup_{\theta} E[U(X_T + B)]\).


This paper: revisit Merton’s lifetime consumption problem. Certain kinds of income easily allowed for (see Merton). Objective: extend to unhedgeable income streams.

Merton’s Optimal Investment Problem with Secure Income

We maximize the expected discounted utility of consumption

\[
J(x) = \mathbb{E}_x \int_0^\infty e^{-\delta t} U(c_t) dt
\]
for some $\delta > 0$, where $c_t$ is the consumption rate and the money to consume is generated by an investment strategy which generates wealth $X_t$ following

$$dX_t = (r + (\mu - r)\pi_t)X_t dt + \pi_t \sigma dW_t - c_t dt$$

with $X_0 = x > 0$ the initial wealth. We invest a proportion $\pi_t$ of our wealth $X_t$ at time $t$ in the risky asset $S$ and the rest $1 - \pi_t$ in risk-free assets which earn the risk free rate $r < \delta$, where $S$ follows the lognormal price process

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where $\sigma > 0$ and $\mu > r$. $W_t$ is a standard Brownian motion. Comments

The criterion

$$J = \mathbb{E} \left[ \int_0^\infty e^{-\delta t} U(c_t) dt \right] = \int_0^\infty e^{-\delta t} \mathbb{E}[U(c_t)] dt$$

has some shortcomings:

- $J$ depends only on the marginal distribution of $c_t$ for $t \in (0, \infty)$ and doesn’t distinguish different joint distributions.
- There is no reasonable way of approximating consumption at a given rate by discrete consumption.

There are well-known ‘fixes’ for these problems, but they are not related to the issues considered here. The traditional formulation is convenient.

Merton’s problem is solved by dynamic programming. The HJB equation is

$$\sup_{\pi, c} \left( U(c) + \frac{1}{2} \pi^2 x^2 \sigma^2 \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial u_0}{\partial x} [-c + x(r + (\mu - r)\pi)] \right) - \delta u_0 = 0.$$ 

We consider logarithmic utility, $U(c) = \ln(c)$. Performing the maximizations, the HJB equation is

$$u_0' rx - \frac{1}{2} \phi^2 \left( \frac{u_0'}{u_0^2} \right)^2 - 1 - \ln(u_0') - \delta u_0 = 0,$$

with

$$\phi = \frac{\mu - r}{\sigma}.$$ 

It is well known that the value function $u_0(x)$ is

$$u_0(x) = r + \frac{1}{2} \phi^2 - \delta + \delta \ln(\delta) + \frac{1}{\delta} \ln(x),$$

and the optimal strategies are

$$\pi_t = \frac{\phi}{\sigma}, \quad c_t = \delta X_t$$

Under the optimal strategy, the relation between capital ($X_t$) and consumption ($c_t$) is

$$X_t = \mathbb{E}_Q \left[ \int_t^\infty e^{-r(s-t)} c_s ds \bigg| \mathcal{F}_t \right],$$
where $Q$ is the equivalent martingale measure.

**Solution with constant income at rate $a$**

In this case the wealth equation is

$$dX_t = (r + (\mu - r)\pi)X_t dt + \pi_t \sigma dW_t + (a - c_t) dt.$$ 

HJB equation is

$$u'^a r (x + \frac{a}{r}) - \frac{1}{2} \sigma^2 (u'^a)^2 \frac{u_{aa}}{u^2} - 1 - \ln(u'_a) - \delta u_a = 0,$$

and we see that this is satisfied by

$$u_a(x) = u_0 \left( x + \frac{a}{r} \right).$$

- The optimal strategy is to *borrow the entire capital value $a/r$ of the income stream and add it to the initial capital $x$. The income is used to service the debt.*
- The ‘solvency constraint’ is non-negativity of the investor’s mark-to-market value:

  $$X_t + \frac{a}{r} \geq 0 \quad \text{a.s. for all } t$$

- Similar strategy is optimal for any hedgeable random income stream $a(t)$: value function is

  $$u_a(x) = u_0(x + \theta)$$

where

$$\theta = EQ \int_0^\infty e^{-rt} a(t) dt.$$
2. Randomly Terminating Income. Let $\tau$ be an exponentially distributed r.v. with parameter $\eta \geq 0$, independent of the Brownian motion $W$. The income is now given by

$$a_t = a 1_{\{t \leq \tau\}}$$

Let $u_1$ be the value function with terminating income, with logarithmic utility. Clearly

$$u_0(x) \leq u_1(x) \leq u_0 \left(x + \frac{a}{\tau}\right)$$

We also have $u_1(x) > -\infty$. The following strategy realizes finite utility starting from $x = 0$:

- Up to time $\tau$, consume $c_t = \frac{1}{2}a$ and invest remaining income in riskless account.
- After $\tau$, apply the optimal strategy with no income, starting with capital $x' = a2 \left(e^{r\tau} - 1\right)$

3. Dynamic programming. The time $\tau$ is a ‘killing time’ after which we apply the optimal no-income strategy, with value $u_0(X_{\tau+})$. The HJB equation is therefore

$$0 = \sup_{\pi_1, c_1} \left(U(c_1) + \frac{1}{2}\pi_1^2 x^2 \sigma^2 \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial u_1}{\partial x} [-c_1 + a + x(r + (\mu - r)\pi_1)] \right)$$

$$0 = \sup_{\pi_0, c_0} \left(U(c_0) + \frac{1}{2}\pi_0^2 x^2 \sigma^2 \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial u_0}{\partial x} [-c_0 - x(r + (\mu - r)\pi_0)] \right) - \delta u_0$$

Carrying out the optimization gives the optimal strategies for the amount of money to invest in stock and to consume

$$x\pi_1(x) = \frac{\phi u_1'(x)}{\sigma u_1''(x)} \quad c_1(x) = \frac{1}{u_1'(x)}$$

We assume that the values to invest and consume for zero wealth exist and are positive or zero:

$$\lim_{x \downarrow 0} x\pi_1(x) \geq 0 \quad \lim_{x \downarrow 0} c_1(x) \geq 0$$

The HJB equation for $u_1$ then becomes

$$0 = \eta u_0(x) - \ln(u_1'(x)) - \frac{1}{2}\phi^2 \left(u_1'(x)\right)^2 - 1 + u_1'(x)(a + rx) - (\eta + \delta)u_1(x)$$

or, equivalently

$$0 = \frac{\eta}{\delta^2} (r + \frac{1}{2}\phi^2 - \delta + \delta \ln(\delta x)) + \ln(c_1(x)) - 1 + \frac{a + rx + \frac{1}{2}(\mu - r)x\pi_1(x)}{c_1(x)} - (\eta + \delta)u_1(x)$$

We conclude that

$$\lim_{x \downarrow 0} c_1(x) = 0, \quad \lim_{x \downarrow 0} u_1'(x) = +\infty \quad \lim_{x \downarrow 0} u_1''(x) = -\infty$$
4. Legendre Transformation of Value Functions. Define the Legendre transformations
\[ v(y) = \sup_{x \in \mathbb{R}} (u(x) - xy) = u(I^u(y)) - yI^u(y) \]
\[ u(x) = \inf_{y \in \mathbb{R}} (v(y) + xy) = v(I^v(x)) + xI^v(x) \]
where \( I^u, I^v \) are defined by
\[ \frac{\partial u}{\partial x}(I^u(y)) = y, \frac{\partial v}{\partial y}(I^v(x)) = -x. \]

If we transform our \( x \)-coordinates to \( y \)-coordinates using \( y = I^v(x), x = -(\partial v/\partial y)(y) \), we obtain the HJB equation as
\[
0 = -\delta(v_1(y) - \frac{\partial u_1}{\partial y} y) - 1 - \ln y - \frac{1}{2} \phi^2 y^2 \frac{\partial^2 u_1}{\partial y^2} + ay - \frac{\partial u_0}{\partial y} r y + \phi^2 y^2 \frac{\partial^2 v_1}{\partial y^2} \\
-\eta v_1(y) + \eta \frac{\partial u_0}{\partial y} y + \eta u_0 \left(-\frac{\partial v_1}{\partial y}(y)\right)
\]
or
\[
-\frac{1}{2} \phi^2 y^2 \frac{\partial^2 v_1}{\partial y^2} + (r - \eta - \delta) y \frac{\partial v_1}{\partial y} + (\delta + \eta) v_1 = -1 - \ln y + ay + \eta u_0 \left(-\frac{\partial v_1}{\partial y}(y)\right)
\]

Comments:
- When \( \eta = 0 \) the transformed HJB equation is a linear ODE, with solution \( u_1(x) = u_0(x + a/r) \)
- HJB is nonlinear when \( \eta > 0 \) and has no closed-form solution. Numerical solution is difficult (read: impossible) because of the problem of specifying appropriate boundary conditions.

5. The Dual Problem. In discounted units \( \tilde{X}_t = e^{-rt}X_t \) etc. we can write the wealth equation as
\[
\tilde{X}_t = \theta_t d\tilde{S}_t + (\tilde{a}_t - \tilde{c}_t)dt.
\]

Let
\[ N_t = 1_{t \geq \tau}. \]
The filtration is now \( \mathcal{F}_t = \sigma\{N_s, W_s, \ s \leq t\} \), and ELMMs \( Q \) are in 1-1 correspondence with RN derivatives
\[
\left( \frac{dQ}{dP} \right)_{\mathcal{F}_t} = \exp \left\{ -\phi W_t - \frac{1}{2} \phi^2 t - \int_0^t (\lambda(s) - \eta)(1 - N_s)ds + \int_0^t \ln \frac{\lambda(s)}{\eta} dN_s \right\}
\]
where, as before, \( \phi = (\mu - r)/\sigma \). Under any ELMM \( Q \),
\[
\int_0^T \theta_t d\tilde{S}_t = \tilde{X}_t - x + \int_0^T (\tilde{a}_t - \tilde{c}_t)dt
\]
is a supermartingale with initial value 0, and we obtain the budget constraint
\[
E_Q \int_0^\infty \tilde{c}_t dt \leq x + E_Q \int_0^\infty \tilde{a}_t dt.
\]
Define the deflator

\[ D_t = e^{-rt} \frac{dQ}{dP} \bigg|_{\mathcal{F}_t} . \]

Then the budget constraint is expressed as

\[ E \int_0^\infty D_t c_t \, dt \leq x + E \int_0^\infty D_t a_t \, dt \equiv x + \theta. \quad (5.1) \]

If \( a_t \) is hedgeable then \( \theta \) is the same for every ELLM \( Q \) and the right-hand side is the mark-to-market value of the assets available at time 0. In our case, the market is incomplete and \( \theta \) is \( Q \)-dependent.

Taking \( U_1(t, c) = e^{-\delta t} U(c) \), the dual function is

\[ V_1(t, y) = \max_c U_1(t, c) - cy = \max_c e^{-\delta t} (U(c) - e^{\delta t} cy) = e^{-\delta t} V(e^{\delta t} y). \]

For any \( c', y' \) we have \( U_1(t, c') \leq V_1(t, y') + c'y' \). Applying this with \( c' = c_t, y' = yD_t \) and using (5.1) gives the basic duality relationship

\[ E \int_0^\infty e^{-\delta t} U(c_t) \, dt \leq E \int_0^\infty e^{-\delta t} V(e^{\delta t} yD_t) \, dt + xy + yE \int_0^\infty D_t a_t \, dt \]

\[ \equiv \tilde{J}(y) + xy \]

- Duality relationship shows easily that \( u(x) = u_0(x + \theta) \) for hedgeable income.
- For \( U(c) = \ln(c) \) we have \( V(y) = -1 - \ln(y) \)

**Theorem** \( \tilde{J}(y) = v_0(y) + J(y) \), where

\[ v_0(y) = r - 2\delta + \frac{1}{2}\phi^2 + -\ln y \]

is the dual function for the no-income case, while

\[ J(y) = ay \int_0^\infty e^{-rt - \int_0^t \lambda(s)ds} \, dt + \frac{y}{2} \int_0^\infty e^{-(\delta + \eta)t} \left( \frac{\lambda(t) - \eta}{\eta} - \ln \frac{\lambda(t)}{\eta} \right) \, dt \]

(The function \( J(y) \) equals zero if \( a = 0 \) since \( \lambda(t) = \eta \) for all \( t \) is optimal in that case. This recovers the dual of the value function when there is no income.)

6. The Deterministic Optimal Control Problem. Consider the system

\[ \frac{dx(t)}{dt} = -x(t) (r - \alpha + \lambda(t)), \quad x(0) = 1 \]

We aim to minimize the value of the functional

\[ J(y) = \frac{\eta}{\delta} \int_0^\infty e^{-\alpha t} [g(\lambda(t)) + Kx(t)] \, dt \]

where

\[ K = \frac{ay\delta}{\eta}, \quad \alpha = \eta + \delta, \]
are positive constants and where the function

\[ g(\lambda) = \frac{\lambda - \eta}{\eta} - \ln \frac{\lambda}{\eta} \]

has a unique global minimum zero in \( \lambda = \eta \). In the sequel we will use the notation

\[ \Lambda(t) = \int_0^t \lambda(s)ds \]

**Theorem** The optimal control function \( \lambda_y(t) \) which minimizes \( J(y) \) satisfies

\[ \frac{1}{\eta} \frac{1}{\lambda_y(t)} = \frac{ay\delta}{\eta} e^{\alpha t} \int_t^\infty e^{-rs-\int_s^t \lambda_y(u)du}ds \]

and the optimal initial conditions \( L(y) = \lambda_y(0) \) satisfy the differential equation

\[ \frac{dL(y)}{dy} = \begin{cases} \frac{L(y)}{L(y)-\eta}\left(\frac{a\delta}{\eta} L(y) - \frac{a}{\eta^2} (L(y) - \eta)\right) & y \neq y^* \\ \frac{a\delta(\alpha-r)}{2(\delta-r)} \left( \sqrt{1 + 4(\delta-r)(\alpha-r)\eta} - 1 \right) & y = y^* = \frac{\alpha(\delta-r)}{a\delta(\alpha-r)} \end{cases} \]

with the midpoint condition that

\[ L(y^*) = \alpha - r. \]

The minimal value of the functional \( J(y) \) then equals

\[ J^*(y) = \frac{\alpha}{\delta a} \left[ \frac{ag\delta}{\eta} + g(L(y)) + (\alpha - r - L(y))g'(L(y)) \right] \]

**Proof.** To find the optimal control function, define for a fixed \( y > 0 \) the Hamiltonian

\[ H = (g(\lambda) + Kx)e^{-\alpha t} - px(r - \alpha + \lambda) \]

The Hamiltonian equation gives

\[ \dot{p} = -\frac{\partial H}{\partial x} = -Ke^{-\alpha t} + p(r - \alpha + \lambda) \]

with boundary condition \( \lim_{t \to \infty} p(t) = 0 \) which implies that

\[ \frac{d}{dt}(xp) = xp \dot{p} + px = -Ke^{-\alpha t} = -Ke^{-\alpha t - \Lambda(t)} \]

so

\[ p(t)x(t) = K \int_t^\infty e^{-rs-\Lambda(s)}ds \]

The optimal control function now satisfies

\[ 0 = \frac{\partial H}{\partial \lambda} = g'(\lambda)e^{-\alpha t} - xp \]
so we have that
\[
\frac{1}{\eta} - \frac{1}{\lambda_y(t)} = Ke^{\alpha t} \int_t^\infty e^{-rs - \Lambda_y(s)} ds \tag{6.1}
\]
From this it turns out that the value function can be expressed as
\[
J(y) = \frac{\eta}{\delta} [K + g(\lambda(0)) + (\alpha - r - \lambda(0))g'(\lambda(0))]
\]
as claimed. To analyze the optimal strategy \(\lambda\) we differentiate (6.1) to find
\[
\frac{\dot{\lambda}_y(t)}{\lambda_y(t)} = \alpha \frac{\lambda_y(t)}{\eta} - 1 - K\lambda_y(t)e^{(\alpha - r)t - \Lambda_y(t)}.
\]
Define
\[
M(t) = -\ln K + \int_0^t (\lambda_y(s) - (\alpha - r)) ds
\]
and
\[
m(t) = \lambda_y(t) - (\alpha - r).
\]
Then the dynamics for \(M\) and \(m\) do not depend explicitly on \(t, y\).
\[
\dot{M}(t) = m(t)
\]
\[
\dot{m}(t) = (m(t) + \alpha - r) \left( \frac{m(t) + \alpha - r - \eta}{\eta} - (m(t) + \alpha - r)e^{-M(t)} \right)
\]
The initial conditions are \((M(0), m(0)) = (-\ln K, \lambda_y(0) - \alpha + r)\). This dynamical system has only one equilibrium point
\[
M^* = -\ln \frac{\alpha (\delta - r)}{\eta (\alpha - r)}, \quad m^* = 0
\]
and the linearized system matrix in the equilibrium point equals
\[
\begin{bmatrix}
0 & 1 \\
\frac{\alpha (\delta - r)(\alpha - r)}{\eta} & \alpha
\end{bmatrix}
\]
which has eigenvectors \((1, f_+ + f_-)\) and \((1, f_+ - f_-)\) for the eigenvalues
\[
f_\pm = \frac{1}{2} \alpha \pm \frac{1}{2} \sqrt{\alpha^2 + 4\alpha (\delta - r)(\alpha - r)/\eta}
\]
Since \(f_- < 0\) and \(f_+ > 0\) the equilibrium point is unstable; for almost all starting points \((M(0), m(0))\), \(\lambda_y(t)\) converges to zero or infinity. Since this is never optimal the trajectories \((M(t), m(t))\) must be one of the only three stable trajectories which converge to \((M^*, m^*)\): the equilibrium point itself, a trajectory for \(M(0) < M^*\) and one for \(M(0) > M^*\). The last two trajectories are characterized by
\[
\frac{dm}{dM} = \frac{\dot{m}}{M} = \frac{m + \alpha - r}{m} \left( \frac{\eta}{2} (m + \alpha - r - \eta) - (m + \alpha - r)e^{-M} \right).
\]
In particular the initial conditions
\[
(M(0), m(0)) = (-\ln \frac{\alpha \delta}{\eta}, L(y) - \alpha + r)
\]
are on these trajectories for all possible \( y \). Transforming back to our original coordinates we find that

\[-y \frac{dL}{dy} = \frac{L}{L - \alpha + r} \left( \frac{\eta}{\eta} (L - \eta) - \frac{ay \delta}{\eta} L \right)\]

There is a singularity in the equilibrium point which corresponds to

\[ L(y^*) = \alpha - r, \quad y^* = \frac{\alpha (\delta - r)}{(\alpha - r) \delta a} \]

but we know that the derivative there equals

\[
\left. \frac{dL}{dy} \right|_{y=y^*} = \frac{1}{-y} \frac{dm}{dM} \bigg|_{M=M^*} = \frac{f_{-y^*}}{\alpha - \frac{1}{2} \sqrt{\alpha^2 + 4 \alpha (\delta - r) (\alpha - r) / \eta}} \frac{-\alpha (\delta - r) / (\alpha - r) \delta a}{\eta^2} \]

This proves the result. \( \blacksquare \)

7. Value Function and Optimal Strategies. Recall that

\[ v_1(y) = v_0(y) + J(y). \]

We now substitute the optimized function

\[
J(y) = \frac{\eta}{\delta \alpha} \left( \frac{ay \delta}{\eta} + \frac{\eta}{\eta} L(y) - \eta - \ln L(y) + \ln \eta + \alpha - r - L(y) \left( \frac{1}{\eta} - \frac{1}{L(y)} \right) \right) \]

\[ = \frac{ay}{\alpha} + \frac{\eta}{\delta \alpha} \left( -\ln L(y) - \frac{\alpha - r}{L(y)} + \ln \eta + \frac{\alpha - r}{\eta} \right) \]

and transform back to our primal coordinates to find

\[ u_1(x) = \inf \left( xy + v_1(y) \right) \]

\[ = \inf \left[ xy + \frac{r - 2 \delta + \frac{1}{2} \delta^2 + \frac{1}{\delta} + \frac{ay}{\alpha} + \frac{\eta}{\delta \alpha} \left( -\ln L(y) - \frac{\alpha - r}{L(y)} + \ln \eta + \frac{\alpha - r}{\eta} \right) }{\eta} \right] \]

We find that

\[ \frac{d}{dy} v_1(y) = -\frac{\eta}{\delta y L(y)} \]

and hence \( u_1(x) = xy + v_1(y) \) when

\[ x = \frac{\eta}{\delta y L(y)} \]

so

\[
u_1 \left( \frac{\eta}{\delta y L(y)} \right) = \frac{\eta}{\delta L(y)} + v_0(y) + \frac{ay}{\alpha} + \frac{\eta}{\delta \alpha} \left( -\ln L(y) - \frac{\alpha - r}{L(y)} + \ln \eta + \frac{\alpha - r}{\eta} \right) \]

\[ = v_0(y) + \frac{ay}{\alpha} - \frac{\eta}{\delta \alpha} \ln \frac{L(y)}{\eta} + \eta (\alpha - r) \left( \frac{1}{\eta} - \frac{1}{L(y)} \right) \]

(7.3)
This gives the value function \( u_1 \) in terms of the function \( L \) which satisfies a 1st-order ODE with known boundary conditions.

**Small-\( x \) behaviour of \( u_1 \)**

To analyze the behaviour of \( u_1 \) for \( x \downarrow 0 \) we need to study the dynamics of \( v_1 \) for \( y \to \infty \). To do so, we first characterize the growth of \( L(y) \) for large \( y \).

**Lemma** There exists a strictly positive constant \( \chi(\delta, r, \eta, a) \) such that

\[
\lim_{y \to \infty} \frac{L(y)}{y^{\alpha/\eta} e^{\alpha y/\eta}} = \chi
\]

Using this result gives the finite value \( \lim_{x \downarrow 0} u_1(x) \) as

\[
u_1(0) = \lim_{y \to \infty} v_1(y) = \frac{r - \delta + \frac{1}{2} \phi^2}{\delta^2} + \frac{\phi}{\delta \eta} \ln \eta - \frac{\xi}{\eta} - \ln \chi
\]

but we do not have an explicit expression for \( \chi \).

**8. Calculation of optimal strategies** \( c_1(x) \) and \( \pi_1(x) \).

Recall that \( x \) and \( y \) are related by \( x = \eta y L(y) \) and that

\[
u_1'(x) = y, \\
u''_1(x) = -1/\delta^2 y L(y) = 1/\delta^2 = -\frac{1}{\delta^2} \left( \frac{1}{\delta^2} \right)
\]

so

\[
\frac{c_1(x)}{x} = \frac{1}{x u'_1(x)} = \frac{\delta L(y)}{\eta}, \\
\pi_1(x) = -\frac{\phi}{\sigma} \frac{u'_1(x)}{x u''_1(x)} = \frac{\phi \delta y L(y)}{\sigma \eta} \frac{y L(y) + y L'(y)}{\delta y^2 L^2(y)} = \frac{\phi}{\sigma} \left( 1 + \frac{y L'(y)}{L(y)} \right)
\]

To obtain a more explicit expression for the investment strategy \( \pi_1 \) we can substitute the differential equation for \( L(y) \) and we find that

\[
\pi_1(x) = \frac{\phi}{\sigma} \frac{L(y) - \alpha + r - \left( \frac{\alpha}{\eta} L(y) - \alpha - \frac{a \delta}{\eta} L(y) \right)}{L(y) - \alpha + r} = \frac{\phi}{\sigma} \frac{r \eta + \delta L(y)(a y - 1)}{\eta \sigma (L(y) - \alpha + r)}
\]

**Comments**

- The case \( a = 0 \) corresponds to \( L(y) = \eta \), which gives us the old (proportional) strategies for optimal investment without income. Since \( \lim_{y \downarrow 0} L(y) = \eta \) we have for \( x \to \infty \) the same proportional strategies.
• For small values of our current wealth $x$ there is a marked difference. Indeed for large $y$ we have

$$\ln \frac{1}{x} \approx \ln \cfrac{\phi \delta}{\eta} + \phi \delta (ay - \ln y)$$

so, since $L(y) \to \infty$ with a speed indicated in the lemma, for $x$ close to zero

$$\pi_1(x) = \phi \eta \frac{\delta L(y)(ay - 1)}{\eta \sigma(L(y) - \alpha + r)} \approx \phi \delta a \frac{\eta}{\eta \sigma} y \approx \frac{\phi \ln \frac{1}{x}}{\sigma}$$

$$c_1(x) = \frac{\delta L(y)}{\eta} \approx \frac{1}{y} \approx \frac{a \delta}{\eta \ln \frac{1}{x}}$$

which shows that the invested wealth $c_1(x)$ and $x \pi_1(x)$ converge to zero for $x$ to zero, while the invested percentages of wealth $c_1(x)/x$ and $\pi_1(x)$ go to infinity.

The figure gives the results for the value function $u_1(x)$, with parameter values $\mu = 0.05, \sigma = 0.2, \eta = 0.06, \delta = 0.06, r = 0.03, a = 0.09$. 

Fig. 8.1. Value functions $u_0, u_1, u_a$
Fig. 8.2. The relative invested and consumed amounts \( \pi_1(x) \) and \( c_1(x)/x \) (highest one for high values is \( \pi_1(x) \)).

Fig. 8.3. The absolute invested amounts \( x\pi_1(x) \) and \( c_1(x) \): (highest one for high values is \( x\pi_1(x) \)).
Fig. 8.4. Ratios $c(x)/c_{a}(x)$ (blue) and $\pi(x)/\pi_{a}(x)$ (red) of consumption and investment $c, \pi$ to 'sure income' values $c_{a}, \pi_{a}$. Parameters $r = .03, a = r + \eta, \eta = 0.06, 0.12, 0.18$. Other parameters as before.

Fig. 8.5. Consumption (blue) and investment (red) ratios: zoom scale.