MARTINGALES OF WIENER AND POISSON PROCESSES

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ABSTRACT

Using the Hilbert space theory of square integrable martingales it is shown that each such martingale with respect to the family of σ-fields generated by independent Wiener and Poisson processes is a sum of stochastic integrals of the generating processes.

The fact that all martingales of a Brownian motion are stochastic integrals has received several proofs, a spectacularly short one, which also works for the Poisson case, being due to Dellacherie [1]. There is, however, a small defect in Dellacherie’s proof, and in circumventing this we arrive at an argument which can be applied to the combined case of σ-fields generated by independent Wiener and Poisson processes. This gives a very quick proof of a special case of Elliott’s “double martingales” [3].

It should be pointed out that since this note was written the succeeding volume of the Strasbourg Séminaire de Probabilités [2] has appeared, in which Dellacherie gives, in a correction, very much the argument of the proof of Theorem 2 below. But the Poisson case has, as will be seen, some additional features.

The situation is most easily described in the framework of Kunita and Watanabe [4]. Let (Ω, ℱ, P) be a probability space carrying a standard Brownian motion b, (continuous paths, b₀ = 0) and an independent standard Poisson process p, (right-continuous paths, p₀ = 0, Epᵩ = t). Let (ℱᵦ) and (ℱₚ) be the generated σ-fields of (bᵦ) and (pᵦ) respectively, completed with all subsets of null sets of ℱ, and let ℱᵦ = ℱᵦ ∨ ℱᵦ. For convenience, assume that ℱᵦ = ℱ. Let ℳ be the set of square-integrable martingales of ℱᵦ, i.e. M ∈ ℳ if and only if M is a martingale, M₀ = 0 and sup₀<∞EMᵦ² < ∞. It is no restriction to assume that each M ∈ ℳ is right-continuous.

ℳ is a Hilbert space under the inner product (M, N) = EMᵦNᵦ. A subspace ℒ of ℳ is stable if it is closed under the formation of stochastic integrals, i.e. M ∈ ℒ ⇒ ∫₀¹ φ dM ∈ ℒ where φ is any ℱᵦ-predictable process such that

\[ E ∫₀¹ φᵦ² d⟨M⟩ᵦ < ∞. \]

In particular, it is closed under stopping. The orthogonal complement ℒ¹ of a stable subspace is itself stable. Let ℒ(b, p) denote the stable subspace generated by bᵦ and pᵦ, i.e.

\[ ℒ(b, p) = \left\{ ∫₀¹ ϕ dp + ∫ ψ(dpᵦ−dt) : φ and ψ predictable, E ∫₀¹ (φᵦ² + ψᵦ²)ds < ∞ \right\}. \]

Then the martingale representation result can be stated simply as follows:

THEOREM 1. ℒ¹(b, p) = {0}.

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To prove this we have first to consider \((b_t)\) and \((p_t)\) separately. Let \(\mathcal{M}_b\) be the set of square integrable martingales of \(\mathcal{B}_t\), and \(\mathcal{L}_b\) the stable subspace of \(\mathcal{M}_b\) generated by \((b_t)\).

**Theorem 2.** \(\mathcal{L}_b^\perp = \{0\}\).

**Proof.** First, all stopping times of \(\mathcal{B}_t\) are predictable, as the following argument shows. Let \(T\) be any stopping time of \(\mathcal{B}_t\) and define

\[ x_t = I_{t \geq T}. \]

\(x_t\) is a submartingale with the Meyer decomposition

\[ x_t = y_t + q_t, \]

where \(q_t\) is a martingale and \(y_t\) a predictable integrable increasing process whose jumps are bounded by 1. Now let \(q_t^n = q_{t \wedge \tau_n}\), where

\[ \tau_n = \inf\{t : y_t \geq n\} \]

and define

\[ L = 1 - \frac{1}{2} q_\infty^n. \]

Then \(L \geq \frac{1}{2}\) and \(EL = 1\) so that a probability measure \(Q\) on \((\Omega, \mathcal{B}_\infty)\) is defined by the formula

\[ dQ = L \, dP \]

and \(Q\) is mutually absolutely continuous with respect to \(P\). Now \(q^n \in \mathcal{M}_b\) and in fact \(q^n \in \mathcal{L}_b^\perp\) since it has no continuous component. Thus \(q^n \cdot b_t\) and \(q^n \cdot (b_t^2 - t)\) are martingales, which implies that \(b_t\) and \(b_t^2 - t\) are martingales of \((\Omega, \mathcal{B}_t, \mathcal{Q})\). According to the Lévy-Doob characterization of Brownian motion, this implies that \(b_t\) is Brownian under measure \(Q\) as well as under measure \(P\), so that \(Q\) and \(P\) coincide on \(\mathcal{B}_\infty\) and \(q_\infty^n = 0\) a.s. It follows that \(x_t = y_t\) a.s., i.e. \(x_t\) is a predictable process and \(T\) is a predictable stopping time.

Since all stopping times, and in particular all accessible stopping times, are predictable, we have from [5; III, T51] that the family \((\mathcal{B}_t)\) is quasi-left-continuous: \(\mathcal{B}_{T'} = \mathcal{B}_T\) for any predictable time. But according to [5; V, T10], for any \(M \in \mathcal{M}_b\) and predictable time \(T\)

\[ M_{T'} = E[M_T | \mathcal{B}_{T'}] = E[M_T | \mathcal{B}_T] = M_T. \]

Since all stopping times are predictable this shows that each \(M \in \mathcal{M}_b\) has continuous paths. Suppose \(M \in \mathcal{L}_b^\perp\) and define

\[ \sigma_n = \inf\{t: |M_t| \geq n\} \]

and

\[ M_t^n = \frac{1}{2n} M_{t \wedge \sigma_n}; \]

then \(|M_t^n| \leq \frac{1}{n}\) and \(M^n \in \mathcal{L}_b^\perp\) since \(\mathcal{L}_b^\perp\) is closed under stopping. Defining \(L = 1 + M_\infty^n\), we can apply an exactly similar argument to the above to show that \(M_\infty^n = 0\) a.s. This completes the proof.

Similar arguments apply also to the Poisson component. Let \((T_i)_{i=1,2,...}\) be the jump times of the Poisson process and denote by \([T]\) the graph of a stopping time \(T\).
Suppose $T$ is an accessible stopping time of $\mathcal{P}$. Then since each $T_i$ is totally inaccessible, $T$ must have the property that $[T] \cap (\bigcup_i [T_i])$ is evanescent (a $(t, \omega)$-set $A$ is evanescent if the process $z_t(\omega) = I_A(t, \omega)$ has almost all sample functions zero); see [5; III D39]. Hence $x_t = I_{t \geq T}$ has no discontinuities in common with $p_t$, and neither has its compensating process since this is predictable. The same argument as above now shows that $T$ is predictable, so that $\mathcal{P}$ is quasi-left-continuous. If $Z \in \mathcal{L}_p^\perp$ and $T$ is a $\mathcal{P}$-stopping time such that $P[Z_T \neq Z_T-] > 0$ then $T$ must have the property mentioned above. But this means that $Z$ must be sample-continuous, and the remaining argument is exactly as before.

**Proof of Theorem 1.** Let $T$ be a stopping time of $\mathcal{F}$ such that $[T] \cap (\bigcup_i [T_i])$ is evanescent and let $x_t = I_{t \geq T}$ as before, with the Meyer decomposition

$$x_t = y_t + q_t$$

but note that this is now relative to the $\sigma$-fields $\mathcal{F}_t$. Now $q^n_t \in \mathcal{L}_p^\perp(b, p)$ since $q^n$ and $p$ have no common discontinuities. Carrying through the argument of Theorem 2, one concludes that $b_t$ and $p_t$ are still Wiener and Poisson processes respectively under the new measure $Q$. We want to show they are still independent.

Fix $r > 0$ and let $X_r, Y_r$ be bounded zero-mean $\mathcal{B}_r$- and $\mathcal{P}_r$-measurable random variables respectively. For $t \in \mathbb{R}^+$ let $X_t = E(X_r | \mathcal{B}_t)$ and $Y_t = E(Y_r | \mathcal{P}_t)$. In view of Theorem 2, $X = (X_t) \in \mathcal{L}_p$ and $Y = (Y_t) \in \mathcal{L}_p$. Since $\mathcal{B}_\infty$ and $\mathcal{P}_\infty$ are independent, $X$ and $Y$ are also $\mathcal{F}_r$-martingales, i.e. $X, Y \in \mathcal{M}$ and $X \perp Y$. Since $X, Y \in \mathcal{L}(p, b)$, and $q^n \in \mathcal{L}_p^\perp(p, b)$, we have, using the change of variables formula [6],

$$q^n_t X_r = \int_0^r q^n_s \, dX_s + \int_0^r X_s- \, dq^n_s.$$

Because of the boundedness, each term on the right is a martingale on $[0, r]$, so that $E q^n_t X_r = 0$. Denoting expectation with respect to measure $Q$ by $E_Q$, this means that $E_Q X_r = 0$. Similarly $E_Q Y_r = 0$, and since $X, Y \in \mathcal{L}(p, b)$ by orthogonality, $E_Q X_r Y_r = 0$ too. Taking $X_r = I_{B-PB}, Y_r = I_{A-PA}$ where $B \in \mathcal{B}$, and $A \in \mathcal{P}$, this shows that $Q(A \cap B) = QA \cdot QB$. Hence $\mathcal{B}_\infty$ and $\mathcal{P}_\infty$ are independent under measure $Q$. Thus $P$ and $Q$ coincide on $\mathcal{F}$, so that $q^n_t = 0$ a.s. and $T$ is predictable. The proof is completed using once more the argument of Theorem 2.

References


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