Collateralized Debt Obligations: Structuring, Pricing and Risk Analysis

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Agenda

Overview
  • Motivation
  • Structure
  • Risk Engineering

Cash Flow
  • Waterfall structure
  • Cash flow vs. Synthetic
  • Funded vs. unfunded

Default risk analysis
  • Moodys diversity scores
  • Bionomial expansion technique
  • Dynamic models
Generic CDO Structure

Swap counterparty

Collateral Portfolio

A (Senior) B (2\textsuperscript{nd} Senior) C (Mezzanine) Equity

SPV
Collateral Pool

- Investment-grade bonds
- High-yield bonds
- Sovereign debt
- Bank loans
- Mortgages
- Tranches of other CDOs
Motivation

• **Balance sheet CDOs:**
  Securitization of bank loans etc.

• **Arbitrage CDOs:**
  Leveraged investment in high yield.
## Typical Term Sheet

### Fixed Collateral

<table>
<thead>
<tr>
<th>Initial Par</th>
<th>225,588,622</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abs Fixed Collateral</td>
<td>75.57%</td>
</tr>
<tr>
<td>Libor</td>
<td>5.20%</td>
</tr>
<tr>
<td>Bond Price</td>
<td>98.00%</td>
</tr>
<tr>
<td>Recovery</td>
<td>50.00%</td>
</tr>
<tr>
<td>Maturity</td>
<td>10</td>
</tr>
<tr>
<td>Frequency</td>
<td>2</td>
</tr>
</tbody>
</table>

### Float Collateral

<table>
<thead>
<tr>
<th>Initial Par</th>
<th>73,202,764</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abs Float Collateral</td>
<td>24.52%</td>
</tr>
<tr>
<td>Libor</td>
<td>5.20%</td>
</tr>
<tr>
<td>Loan Price</td>
<td>98.00%</td>
</tr>
<tr>
<td>Recovery</td>
<td>50.00%</td>
</tr>
<tr>
<td>Maturity</td>
<td>10</td>
</tr>
<tr>
<td>Frequency</td>
<td>2</td>
</tr>
</tbody>
</table>

### Cov Tests

<table>
<thead>
<tr>
<th>Cov Tests</th>
<th>Par</th>
<th>Interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.065</td>
<td>1.050</td>
</tr>
<tr>
<td>B</td>
<td>1.065</td>
<td>1.050</td>
</tr>
<tr>
<td>C</td>
<td>1.010</td>
<td>1.000</td>
</tr>
</tbody>
</table>

### Swap

<table>
<thead>
<tr>
<th>Swap</th>
<th>Years</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>7</td>
<td>5.20%</td>
</tr>
</tbody>
</table>

### Notes

<table>
<thead>
<tr>
<th>Notes</th>
<th>Proceeds</th>
<th>% of Proc</th>
<th>Coupon</th>
<th>Spread bp</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>225,000,000</td>
<td>75.37%</td>
<td>5.850%</td>
<td>65</td>
</tr>
<tr>
<td>B</td>
<td>42,000,000</td>
<td>14.07%</td>
<td>6.000%</td>
<td>80</td>
</tr>
<tr>
<td>C</td>
<td>14,950,000</td>
<td>5.01%</td>
<td>9.200%</td>
<td>400</td>
</tr>
<tr>
<td>Equit</td>
<td>16,576,690</td>
<td>5.55%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>298,526,690</td>
<td>100.00%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Dates

<table>
<thead>
<tr>
<th>Dates</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Today</td>
<td>12-Nov-2001</td>
</tr>
<tr>
<td>Start</td>
<td>12-Dec-2001</td>
</tr>
<tr>
<td>First Coup</td>
<td>12-Mar-2002</td>
</tr>
<tr>
<td>First accrue</td>
<td>0.246575342</td>
</tr>
</tbody>
</table>

### Collateral Portfolio

| Initial | 298,791,386 |
Cash Flow CDO

- Investors subscribe face value of notes
- SPV purchases collateral assets (during ramp-up period)
- Collateral receipts used to pay note coupons to investors in order of seniority.
- Equity investors receive residual receipts.
- Equity investors generally have right to wind up structure after some years.
Synthetic CDO: unfunded

Bank

CDS premium

losses

SPV

CDS premium

losses

Investors

No initial investment required
Synthetic CDO: funded

Investors purchase notes with coupons $L+x_1, L+x_2, \ldots$
Waterfall structure

- Fees
- Swap payments
- Class A interest
- Maturing collateral redeems class A notes, then class B...
- Further redemption of class A notes to meet coverage tests
- Class B interest
- ......
- Subordinate fees to collateral manager
- Residual receipts to equity note holders
Coverage tests

N = Collateral face value
A = Face value of class A notes
B = Face value of class B notes
Tests:

\[ \frac{N}{A} \geq a_1 \]
\[ \frac{N}{A + B} \geq a_2 \]

If tests not satisfied at any coupon period, A and B are reduced by redeeming notes. Similar tests for interest payments.
Default and Recovery

If a collateral asset defaults, its recovery value is used to pay down notes.

- Adverse effect on over-collateralization ratios.
- Statistics on recovery values available from Moody’s
Interest rate swap

Swap may be required to hedge the mismatch between fixed and floating coupons on the two sides. Has a significant effect on cash flow analysis

10y Swap rates: USD 4.214%   EUR 4.271%
Forward rate curves
DEFAULT ANALYSIS

- Basic analysis assumes $x\%$ default per year (possibly ‘front loaded’). Demonstrates effect of leverage.
- CDO investment can be in ‘principal protected’ form.

![Equity IRR for Uniform Default](image-url)

- High Leverage
- Low leverage
- Princ Protected
Moody’s Cumulative Default Probabilities

These are empirical estimates based on a ‘cohort’ analysis.
Joint default probabilities: independence

If defaults are independent, distribution of number of defaults in a portfolio of size $n$ is Binomial $B(n,p)$, where $p$ is the individual default probability. Example: $n=60$, $p=0.1$

![Binomial Distribution B(60,0.1)](image)
Moody’s Binomial Expansion Technique

Start with a portfolio of $M$ bonds, each (for simplicity) having the same notional value $X$. Each issuer is classified into one 32 industry classes. The portfolio is deemed equivalent to a portfolio of $M' < M$ independent bonds, each having notion value $XM/M'$. $M'$ is the *diversity score*, determined from the following table:
## Diversity score table

<table>
<thead>
<tr>
<th>No. of firms in same industry</th>
<th>Diversity score</th>
<th>No. of firms in same industry</th>
<th>Diversity score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>6</td>
<td>3.0</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
<td>7</td>
<td>3.2</td>
</tr>
<tr>
<td>3</td>
<td>2.0</td>
<td>8</td>
<td>3.5</td>
</tr>
<tr>
<td>4</td>
<td>2.3</td>
<td>9</td>
<td>3.7</td>
</tr>
<tr>
<td>5</td>
<td>2.6</td>
<td>10</td>
<td>4.0</td>
</tr>
</tbody>
</table>
Diversity Score: Example

Portfolio of 60 bonds

<table>
<thead>
<tr>
<th>No of issuers in sector</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>No of incidences</td>
<td>2</td>
<td>7</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Diversity</td>
<td>2</td>
<td>10.5</td>
<td>18</td>
<td>9.2</td>
<td>5.2</td>
</tr>
</tbody>
</table>

Meaning: 2 cases where issuer is sole representative of industry sector, 7 cases where there are pairs of issuers in same sector,..

Diversity score = 45.
“Loss” in Rated Tranches

Suppose coupon in a rated tranche is $c$ and amounts actually received are $r_1, ..., r_n$. Then the loss is $1 - q$ where

$$q = \frac{r_1}{1+c} + \frac{r_2}{(1+c)^2} + ... + \frac{r_n}{(1+c)^n}$$

Note that loss = 0 when $r_i = c$, $i < n$ and $r_n = 1 + c$. Moody’s rates tranches on a threshold of expected loss.
Example: $n = 60$, $p = 0.1$

With diversity 60, expected number of defaults is 

$\mu = np = 6$, standard derivation $\sigma = \sqrt{np(1-p)} = 2.32$.

In CDO tranches, losses only occur above some threshold level of default. Represented by option-like loss severity function $k$. 

![Diagram showing loss function with thresholds at $\mu + 3\sigma$ and $\mu + 4\sigma$.]
Cumulative Loss distributions with $d=60, 45, 30$

Std Dev: 0 1 2 3 4

$d=60$
$d=45$
$d=30$
Tail of Cumulative Distribution

Std dev: 3 4 5

Data points for different standard deviations:
- d=60
- d=45
- d=30
Expected Loss

Chart shows expected loss with option-like loss function, for diversity scores 60, 45, 30. (Arbitrary scale). Shows massive increase in ‘tail risk’ as diversity is reduced.
CDO Pricing

- Rated notes pay fixed-rate coupon, quoted as Treasury + x, or floating rate coupon Libor + x.
- Credit rating is fixed by expected loss.
- Spread x is the current market spread for that credit rating.
- Specification of notes determined by investor demand.
- Equity tranche has no guaranteed coupon or credit rating. It is a leveraged investment.
- Design problem is to obtain leverage required by equity investor while maintaining low expected losses on rated tranches as required by rating agency.
Dynamic Default Models

Model single default by exponentially distributed time $T$

$$P[T > t] = e^{-\lambda.t}$$

Then $\lambda$ is the hazard rate

$$P[t < T \leq t + dt \mid T > t] \approx \lambda dt$$

Calibrate to 10-year default probability $p = 0.1$ by

$$0.1 = P[T \leq 10] = 1 - e^{-10\lambda}$$

giving $\lambda = 0.01054$
Multiple independent defaults

For $n$ independent issuers with default times $T_1, \ldots, T_n$, the hazard rate up to the first default in time $S_1$ is $n\lambda$, then $(n-1)\lambda$ until the second time $S_2$ etc.

Distribution of number of defaults in 10 years is Binomial $B(n,0.1)$, as in Moodys with diversity score $n$.

Easy to simulate since for example

$$S_1 = -\frac{1}{n\lambda} \log U$$

where $U$ is a uniform random number in $[0,1]$. 
‘Enhanced Risk’ Model

Consider a macroeconomic “shock” model in which

- Initially, all bonds have hazard rate $\lambda$.
- When one bond defaults, remaining bonds become more risky: hazard rate increases to $a\lambda$, where $a > 1$ is the ‘risk enhancement’ parameter.
- After an exponentially-distributed time (parameter $\mu$), hazard rates return to normal level.

This models a situation where default occurs in ‘bursts’, triggered off by an actual default or some other external event.
Enhanced risk model is Markov process on state space $E = \{(i,j): i=0,1; j=0,1,...,n\}$. Index $i=0$ represents ‘normal risk’ and $i=1$ represents ‘enhanced risk’. Transition rates between states are shown on next slide.

- Simulation is simple because sojourn times in each state are exponential – same relation with uniform random numbers as before.
- Computation of distributions and expectations only involves solving ordinary differential equations.
Enhanced Risk Model: state space

Enhanced Normal

Number ‘alive’
Calibration

As enhancement factor $a$ increases expected number of defaults, and hence marginal defaults probability, increase. To maintain, say, a 10-year default probability 0.1 per Issuer, recalibrate the hazard rate $\lambda$. Example ($\mu=0.5$)

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\lambda$ (bp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>105.4</td>
</tr>
<tr>
<td>2</td>
<td>74.8</td>
</tr>
<tr>
<td>3</td>
<td>61.2</td>
</tr>
<tr>
<td>5</td>
<td>47.6</td>
</tr>
</tbody>
</table>
Cumulative probabilities in the enhanced risk model
Tail of distribution

Cumulative probabilities, enhanced risk

Number defaulting

- $a=1$
- $a=2$
- $a=3$
- $a=5$
Default time samples

\[ \lambda = 0.015 \]
\[ a = 1 \]

\[ \lambda = 0.01 \]
\[ a = 3 \]
Enhanced risk model provides

• Dynamic analysis of losses in rated tranches.

• Monte Carlo evaluation of risk-return characteristics of equity tranche.

• Easy ‘stress testing’ of collateral ‘concentration risk.’
Summary

- CDOs provide useful ‘risk engineering’ for balance sheet and arbitrage purposes
- Returns of equity tranche have low correlation with other equity investments
- Performance largely depends on success of collateral manager in avoiding realized defaults
- ‘Concentration risk’ is the key factor in analysing CDO risk and performance
References
