Valuation, hedging and investment in incomplete financial markets

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1 Introduction

Financial instruments or ‘positions’ are legal contracts between two parties providing for exchange of fixed or contingent payments at various times in the future. For example, in a forward foreign exchange contract entered at time $t_0$, Party A agrees to pay Party B $1.5 at time $t_1 > t_0$ while Party B pays Party A £1. The value of this contract to Party A at $t_1$ is $\$(f(t_1) - 1.5)$ where $f(t_1)$ is the £/$ FX rate at time $t_1$. Since this is unknown at $t_0$, the parties are entering an agreement, with no payment at $t_0$, providing for a net random payment in one direction or the other at time $t_1$. There may also be optionality: if the contract gives Party A the right but not the obligation to effect the above exchange then the value at $t_1$ is $H = \max(f(t_1) - 1.5, 0)$ since Party A will not exercise her right if $f(t_1) < 1.5$. Since $H \geq 0$, Party B will not agree to this arrangement unless there is some compensating payment to her, which could be made at time $t_1$ but more conventionally appears as an “upfront” option premium paid by Party A at $t_0$. Thus Party A is exchanging a fixed payment now for a random payment later. The classic option pricing problem is to determine a ‘fair value’ for the premium.

Banks – or even individuals – will be holding portfolios containing hundreds or thousands of positions. The requirements for mathematical modelling are three:

1. Valuation: determine a value for a portfolio that is consistent with prices of exchange traded assets. This process is known as marking to market. The value of any exchange-traded asset such as an ordinary share is simply its current market price, but a model is needed to value an option on the share or indeed any non-exchange traded position.

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2. *Hedging* is the process of mitigating risk by engaging in offsetting transactions. The writer of an option accepts a premium payment but has to make a – possibly unbounded – random payment later. How should she trade, using the premium payment, so as to be in a position to meet the later payment of at least reduce the risk of loss? This is the hedging problem.

3. *Investment* is the process of trading in the market with a view to increasing the value of one’s assets. Ever since the pioneering work of Markowitz [27] it has been appreciated that there is a trade-off between risk and return: investors can only increase expected return by accepting a higher level of risk (i.e., variance of returns). The portfolio selection problem of Markowitz-style investment theory is to maximize expected returns subject to a bound on the level of risk.

Let $X_t$ be the value of a portfolio obtained by trading in the market starting with initial capital $X_0 = x$ (a more precise description is given in the next section.) An arbitrage opportunity is the availability of ‘something for nothing’, i.e. the existence of a portfolio process $X_t$ and a time $T > 0$ such that:

- $x = 0$
- $X_T \geq 0$ a.s
- $P[X_T > 0] > 0$ (equivalently, $E[X_T] > 0$)

This portfolio requires no initial investment, entails no possibility of loss but a positive probability of gain. Real markets may or may not contain (albeit transitory) arbitrage opportunities, but in mathematical modelling we must insist on models that are arbitrage-free since otherwise there will be strategies leading to infinite riskless profits, an obviously unrealistic conclusion.

Suppose we have an option-like contract in which a premium $p$ paid at time zero is exchanged for a random variable $H$ at time $T > 0$, and suppose there is a portfolio process $X_t$ such that $X_T = H$ almost surely. Then $x = X_0$ is the unique arbitrage-free value for the premium $p$: if $p > x$ then we can simply pocket the difference $p - x$ since $x$ is enough to form the portfolio $X_t$ which perfectly hedges our obligation at time $T$ (this is called a replicating portfolio). If $p < x$ a similar arbitrage accrues to the option purchaser. This argument is sometimes known as the law of one price: if two positions involve identical cash flows in the future then they have the same value now.

It is surprising how far this process of pricing by absence of arbitrage will go. A market model is complete if there is a replicating portfolio for ‘any’ contingent claim $H$ (a contingent claim is one whose exercise value $H$ is a function of traded asset prices) and it is a remarkable fact that the standard market model of Samuelson-Merton-Black-Scholes in which asset prices are geometric Brownian motion (see Section 2 below), is complete. In a complete market, valuation and hedging are one and the same thing: underlying assets are valued at their market price and contingent claims at the capital value required for replication.

At first sight, hedging and investment appear to be very different things since hedging is aimed at mitigating risk while investment is risk-seeking, as we learn from
Markowitz. However on closer inspection the two turn out to be very closely related, the connection being through the duality theory of optimal investment, which is outlined briefly in Section 4 below. This theory gives us very clear hints what to do in the case of incomplete markets, in which there are unhedgeable contingent claims. Generally, markets are incomplete when there are not enough traded assets or when trading restrictions limit the class of portfolios we can construct. In this case hedging and investment become synonymous: writing an option entails real risk, and therefore its value can only be assessed in relation to some investment objective. We will see below in Section 4 that pricing rules can be obtained this way which are direct extensions of the complete market case, i.e. coincide with the no-arbitrage price for all claims that actually are hedgeable. The corresponding investment strategy – replication in the case of hedgeable claims – becomes the utility-maximizing strategy when perfect hedging is impossible. Ideas of this sort have been developed in one form or another by economists for upwards of a century, but it is only recently that the connections with arbitrage pricing theory have been explored.

Study of incomplete markets is far from being a purely academic matter. In the area of conventional derivatives, ‘stochastic volatility’ models, aiming at a more accurate description than the Black-Scholes model, generally lead to incompleteness (see Davis [11]). In recent years a huge market in credit derivatives has developed (see Schönbucher [33]) and few if any of the contracts traded there are exactly hedgeable other than by trivial back-to-back deals. Another important source of incompleteness is a long time horizon. Generally long-term option contracts are unhedgeable because of uncertainty about future volatility. There is also a huge category of liabilities held by insurance companies involving long-term guarantees on pension payments, annuity rates and the like. In line with our philosophy that “hedging = investment”, we need to study long-term investment to understand valuation and strategies for such contracts. An excellent survey of these problems is given by Campbell and Viceira [4].

This page gives a survey of some of the above topics, and is laid out as follows. Section 2 introduces the basic framework of price processes, investment strategies and arbitrage-free models. Section 3 describes the standard approach to optimal investment based on maximising expected utility and solution techniques based on dynamic programming. Section 4 covers duality theory and associated pricing concepts.

In the long-term problems mentioned above, parameter uncertainty is a crucial feature. Section 5 discusses some ways in which this can be handled. The paper concludes with some final remarks in Section 6.

Throughout the paper, the intention is not to aim at a maximum level of generality but, on the contrary, to concentrate on specific cases and solved problems which give insight into the nature of optimal strategies for hedging and investment.
2 Market model

We start with a vector of continuous semimartingales $S_0(t), \ldots, S_n(t)$ on a filtered probability space $(Ω, \mathcal{F}, (\mathcal{F}_t), P)$, representing the prices of $n + 1$ traded assets. Suppose $S_0(0) = 1$ and $S_0(t) > 0$. A trading strategy is an $n$-vector locally bounded $\mathcal{F}_t$-predictable process $\phi_t$, where $\phi_t(t)$ is the number of units of asset $i$ held at time $t$. All residual value is invested in asset $S_0$. The evolution of portfolio value $X_t$ starting with initial capital $x$ is then

$$dX_t = \phi_t dS(t) + \frac{X_t - \phi_t S(t)}{S_0(t)} dS_0(t), \quad X_0 = x.$$  \hfill (1)

$\phi$ is admissible if $X_t \geq 0$ for all $t$. Define $\tilde{X}_t = X_t/S_0(t)$ etc. Then applying the Ito formula we obtain the following convenient expression for the evolution of portfolio value in normalized units:

$$\tilde{X}_t^\phi = x + \int_0^t \phi_u d\tilde{S}(u).$$  \hfill (2)

The “first fundamental theorem of asset pricing” (Delbaen and Schachermayer, [12]) states, roughly speaking, that there is no arbitrage if and only if there exists an equivalent measure $Q \sim P$ under which the normalised price processes $\tilde{S}(t)$ are local martingales. We say $(S_0, Q)$ is a numéraire pair. Since trading strategies are predictable, the representation (2) shows that $\tilde{X}_t^\phi$ is a local martingale under measure $Q$.

A contingent claim $H$ exercised at time $T$ is an integrable $\mathcal{F}_T$-measurable random variable. The classic example is $H = \max(S_k(T) - K, 0)$ a call option on asset $k$. If $\tilde{H} = \tilde{X}_T^\phi$ a.s. for some $\phi$ then we say $\phi$ ‘replicates’ $H$. If $X^\phi$ is a $Q$-martingale, as opposed to just a local martingale, then by the martingale property

$$x = EQ[\tilde{X}_0^\phi] = EQ[\tilde{X}_T^\phi] = EQ[\tilde{H}]$$

so that

$$x = EQ \left[ \frac{H}{S_0(T)} \right]$$  \hfill (3)

and $x$ is the unique arbitrage-free value for $H$ at time 0. If a replicating strategy exists for ‘any’ contingent claim, the market is complete. The “second fundamental theorem of asset pricing” states that the market is complete if and only if there is a unique martingale measure $Q$. In this case (3) gives the unique arbitrage-free value for arbitrary $H$.

An incomplete market is one in which there are many martingale measures $Q$. In general, replication is not possible in incomplete markets and there is an interval $I$ of arbitrage-free prices given by

$$I = \left( \inf_{Q \in \mathcal{M}} EQ[\tilde{H}], \sup_{Q \in \mathcal{M}} EQ[\tilde{H}] \right)$$

\footnote{$\phi S$ denotes the inner product $\sum^n \phi_i S_i$, so $X_e - \phi_i S(t)$ is the cash available for investment in the 0'th asset}
This was shown by Kramkov [25]. It seems clear that nothing should depend on which asset we choose as the numéraire asset $S_0(t)$ (except that we must have $S_0(t) > 0$ for all $t$), and indeed this is the case: If $(S_0, Q)$ is a numéraire pair and $N$ is another numéraire then $(N, Q_N)$ is a numéraire pair where

$$\frac{dQ_N}{dQ} \bigg|_{F_T} = \frac{N(T)}{S_0(T)}.$$ 

Thus the quantity $\zeta_t = (dQ_N/dP)_t/N(t)$ is numéraire invariant and (3) is expressed as

$$x = E[\zeta_T H].$$

The process $\zeta_t$ is known as a deflator or state price density.

### 2.1 Example: the Black-Scholes world

In the market model of Black and Scholes [2] the price processes $S_i(t), \ i = 1, \ldots, n$ satisfy equations

$$dS_i = \mu_i S_i \, dt + \sigma_i S_i \, dw_i^i$$

where $(w_1^1, \ldots, w_n^n)$ is a vector of correlated Brownian motions, while

$$dS_0 = rS_0 \, dt$$

for some constant interest rate $r$. Thus the asset prices $S_1(t) \ldots S_n(t)$ are log-normally distributed, while the numéraire asset is a risk-free savings account. Taking $\gamma = (\mu - r)/\sigma$ we find

$$d\tilde{S} = \sigma \tilde{S}(dw_t + \gamma dt) =: \sigma \tilde{S}d\tilde{w}_t.$$ 

$\tilde{w}_t$ is a Brownian motion under an equivalent measure $Q$ given by the Girsanov theorem. The replication requirement is

$$e^{-rT}H = x + \int_0^T \phi \sigma \tilde{S}d\tilde{w}.$$ 

Existence of $(x, \phi)$ satisfying (4) follows from the martingale representation theorem for BM; this is a complete market. The value of the contingent claim $H$ at time $t < T$ is then given by

$$v_t = E_Q[e^{-r(T-t)}H]$$

so that in particular $v_0 = x$.

### 3 Optimal Investment and Consumption

In this section we see that optimal investment is essentially a stochastic control problem. A special role is played by so-called growth-optimal portfolios, and these can be studied in a very direct way as described below. We then move on to look at a general formulation in terms of maximization of expected utility, and solution of some problems by dynamic programming.
3.1 Growth-optimal portfolios

For a single portfolio trajectory we can define a process $\eta(t)$ by

$$X_t = xe^{\eta(t)t}$$

so that $\eta(t) = \log(X_t/x)/t$ is the growth rate over the time interval $[0,t]$. Thus choosing the investment strategy to maximize $E[\log X_T]$ maximizes the expected growth rate, giving the growth-optimal portfolio. If we take the normalized portfolio process $\tilde{X}_t = X_t/S_0(t)$ then $\log \tilde{X}_t = \log X_t - \log S_0(t)$ so choice of numéraire is irrelevant to the optimization problem, a key property of the logarithmic criterion.

In the semimartingale model, suppose price processes take the form

$$S_i(t) = M_i(t) + \int_0^t \alpha_i(u)d\mu(u)$$

and

$$<M_i, M_j>_t = \int_0^t \beta_{ij}(u)d\mu(u)$$

where the processes $M_i(t)$ are continuous martingales, $\mu$ is a fixed measure $\beta_{ii}(t) > 0$ for all $t$ a.s. and $<M_i, M_j>$ denotes the quadratic co-variation of martingales $M_i, M_j$ (see []). If we write the trading strategy $\phi$ as $\phi_t = \pi_t \tilde{X}_t$ then the portfolio value evolves as

$$d\tilde{X}_t = \pi_t \tilde{X}_t d\tilde{S}_t$$

so that by the Ito formula

$$d(\log \tilde{X}_t) = \left(\pi'\alpha - \frac{1}{2}\pi'\beta\pi\right)d\mu + \pi' dM.$$  \hfill (5)

We maximize the expectation by maximizing the ‘drift’ term, i.e. taking $\pi = \pi^*$ where

$$\pi^* = \beta^{-1}\alpha,$$

giving the maximal value

$$E(\log X_T) = E(\log(xS_0(T))) + E\int_0^T \frac{1}{2}\alpha'\beta^{-1}\alpha d\mu.$$  

The first term is the value obtained by simply investing everything in the numéraire asset.

It turns out that there is a close connection between numéraire pairs and the problem of maximizing logarithmic utility [1]. Indeed, suppose $(Y,Q)$ is a numéraire pair; then using the inequality $\log x \leq x - 1$ we have for an arbitrary portfolio process $X_t$

$$E_Q \log X_T - E_Q \log Y_T = E_Q[\log(X_T/Y_T)] \leq E_Q[X_T/Y_T] - 1 = 0.$$  

Thus $Y_T$ maximises logarithmic utility under $Q$.

The converse is also true: if $Y_T$ maximises logarithmic utility under a certain measure $Q$ then $(Y,Q)$ is a numéraire pair. In particular if we use the log-optimal
portfolio, calculated as above, as numéraire, the ‘physical’ measure $P$ is a martingale measure. This is intriguing because it shows we do not need to change the measure to do pricing: we can change the numéraire instead.

To show this, assume $S_0$ is the log-optimal portfolio process under $P$. Then

$$E \log \tilde{X}_T = E \log X_T - E \log S_0(T) \leq 0$$

for any portfolio process $X$ corresponding to trading strategy $\phi$, and using (5) and (6) we see that for all $\pi$

$$-E \log \tilde{X}_T = \frac{1}{2} \int_0^T (\pi' \beta \pi - 2\pi' \alpha) d\mu$$

$$= \frac{1}{2} \int_0^T [(\pi - \beta^{-1} \alpha)' \beta (\pi - \beta^{-1} \alpha) - \alpha' \beta^{-1} \alpha] d\mu \geq 0.$$  

Thus $\alpha_i(t) = 0$ a.s. $d\mu$, showing that $\tilde{S}_i(t) = M_i(t)$, a local martingale.

### 3.2 Utility functions

A utility function is a smooth, concave function $U : R^+ \to R$ satisfying

$$U'(0) = +\infty, \quad U'(\infty) = 0.$$  

The function

$$I(y) = (U')^{-1}(y)$$

is then decreasing and convex. A utility function $U$ defines a preference ordering $\preceq$ on the set of probability measures on $R^+$, given by

$$\mu_1 \preceq \mu_2 \iff \int_{R^+} U(x) \mu_1(dx) \leq \int_{R^+} U(x) \mu_2(dx),$$

and the objective of maximizing expected utility with a concave utility function corresponds to ‘rational behaviour’ of a risk-averse agent; see for example Ferguson [14].

We consider various problems of maximizing expected utility of the form

$$E \left\{ \int_0^T U_1(c_t) dt + U_2(X_T) \right\},$$

where $c_t$ is a consumption rate. This objective trades off personal gratification (the first term) against leaving something for the next generation (the second term). General methods for solving such problems are

- Dynamic programming [15]: this provides a very direct computational technique but applies to Markovian systems only

- Convex duality [24]: this can be applied to very general price process models but the computational side is less obvious except in very special cases.
3.3 Infinite-horizon dynamic programming

The problem described here was originally studied by Merton [28] and is remarkable in being one of the few nonlinear stochastic control problems that can be solved explicitly, as well as having a solution with intuitively satisfying content. See [9] for a clean treatment.

The objective is to maximize

$$E \int_0^\infty e^{-\delta t} U(c_t) dt$$

for a scalar price model

$$dS_t = \mu S_t dt + \sigma S_t dw_t.$$

The wealth equation is

$$dX_t = X_t (r + (\mu - r) \pi) dt - c_t dt + X_t \pi_t \sigma dw_t$$

where \( \pi \) is the fraction of wealth invested in the risky asset. The Bellman equation of dynamic programming is

$$\max_{\pi, c} \left\{ \frac{\partial v_0}{\partial x} (x(r + (\mu - r) \pi) - c) + \frac{1}{2} \pi^2 \sigma^2 x^2 \frac{\partial^2 v_0}{\partial x^2} + U(c) - \delta v_0 \right\} = 0. \tag{8}$$

Performing the maximization this becomes (with \( v'_0 = \frac{\partial v_0}{\partial x} \) etc.)

$$v'_0 r x - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 \left( \frac{v'_0}{v_0} \right)^2 + U(I(v'_0)) - I(v'_0) v'_0 - \delta v_0 = 0. \tag{9}$$

If \( v_0 \) satisfies (9) then \( v_0(x) \) is the maximum achievable expected utility starting with capital \( x \), and the optimal investment strategy is given by the maximizing values of \( c, \pi \) in (8). For example, when \( U(c) = c^\gamma \) then

$$\pi^* = \frac{1}{1 - \gamma} \frac{\mu - r}{\sigma^2}, \quad c^*_t = C\gamma X_t.$$

The optimal strategy is to invest a constant fraction of total wealth in the risky asset and consume at a rate proportional to total wealth. The same form of strategy is optimal for logarithmic utility, with \( \pi^* = (\mu - r)/\sigma^2 \).

3.4 Optimal investment with income

In the previous section we are simply concerned with investing an initial ‘endowment’ \( x \). It is perhaps more natural to consider the case in which the investor also has an income, and this was already studied by Merton in the following simple case. Suppose we start with capital \( x \) and receive an income at a constant rate \( a \) per unit time. The wealth equation becomes

$$dX_t = X_t (r + (\mu - r) \pi) dt + (a - c_t) dt + X_t \pi_t \sigma dw_t \tag{10}$$
and the Bellman equation is
\[
\begin{align*}
v' r \left( x + \frac{a}{r} \right) - & \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 \frac{(v')^2}{v''} + U(I(v')) - I(v')v' - \delta v = 0. 
\end{align*}
\]
(11)
The following is easily checked.

**Proposition 1.** If \( v_0 \) satisfies (9) then \( v_a(x) := v(x + a/r) \) satisfies (11).

The intuition here is clear: the value at time 0 of the income stream is
\[
a = \int_0^\infty a e^{-rt} dt.
\]
The optimal strategy is to borrow this amount and apply the the optimal strategy with initial capital \( x + a/r \) and no income. The income stream finances the debt. We will see later that this is a special case of a much more general result.

### 3.5 Optimal hedging with annuity liability

Suppose we start with capital \( x \) and have the obligation to pay a perpetual annuity at rate \( a \). This is now a hedging problem: we have to manipulate our funds so as to be in a position to meet our obligation to pay the annuity, if it is possible to do so. If \( x \geq a/r \) the solution is simple: the liability is perfectly hedged by placing \( a/r \) in the riskless account. The remaining funds can be invested optimally as before, achieving expected utility \( v(x - a/r) \).

If \( 0 < x < a/r \) we cannot guarantee to pay the annuity for ever. Let \( X_t \) be the wealth process and define stopping times
\[
\begin{align*}
\tau_0 &= \inf \{ t : X_t = 0 \} \\
\tau_1 &= \inf \{ t : X_t = a/r \} \\
\tau &= \begin{cases} \\
\tau_0 \text{ if } \tau_0 < \tau_1 \\
\infty \text{ if } \tau_1 < \tau_0 
\end{cases}
\end{align*}
\]
Then \( \tau \) is the time for which we can pay the annuity and it is reasonable to maximize the expected NPV
\[
E \int_0^{\tau} a e^{-rt} dt = \frac{a}{r} \left( 1 - E e^{-\tau r} \right).
\]
This problem has the following explicit solution, due to Sid Browne [3]: define
\[
\begin{align*}
\theta &= \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 \\
d &= \theta^2 + 4r \theta \\
\eta^+ &= \frac{1}{2r} \left( 2r + \theta + \sqrt{d} \right).
\end{align*}
\]
Then the optimal strategy is $c_t \equiv 0$ and $\pi_t = \pi^*(X_t)$ where

$$\pi^*(x) = \frac{\mu - r}{\sigma^2(\eta^+ - 1)} \left( \frac{a}{r} - x \right)$$

The value function is

$$E^\pi x \exp(-rt) = (a - rx)^{\eta^+}.$$ Again, this solution is intuitively reasonable: the proportion of wealth invested in stock is proportional to the distance from the 'safe' level $a/r$. When wealth is low one has to gamble to have any chance of survival, whereas near the safe level volatility should be reduced. (In this solution the upper barrier $x = a/r$ is never actually hit.)

4 Duality and Utility-Based Pricing

4.1 Duality

In this section we take the numéraire asset as $S_0(t) \equiv 1$ for ease of exposition. Equivalently, we consider everything in discounted units. Let $X(x)$ be the set of wealth processes corresponding to admissible trading strategies $\phi$ with initial capital $x$ and define

$$u(x) = \sup_{X \in X(x)} E[U(X_T)].$$

(12)

Here $U$ is a utility function as before. This is the problem studied by Kramkov and Schachermayer [26] with semimartingale price processes $S_t(\cdot)$, following earlier work by several authors based on a diffusion process price model (see Karatzas and Shreve [24] and the references there).

The utility function $U$ is said to have reasonable asymptotic elasticity if

$$\lim_{x \to \infty} \frac{xU'(x)}{U(x)} < 1 \quad (13)$$

Define the dual function $V$ by

$$V(y) = \max_x (U(x) - xy).$$

This is a convex decreasing function, and the maximum is achieved at $x = I(y) = (U')^{-1}(y)$. The dual optimization problem is to calculate

$$v(y) = \inf_{Y \in \mathcal{Y}} E[V(Y_T)]$$

(14)

where $\mathcal{Y}$ is the set of non-negative processes such that $Y_0 = 1$ and $XY$ is a supermartingale for all $X \in X(1)$. Note that $L \in \mathcal{Y}$ if

$$L_t = E \left[ \frac{dQ}{dP} | \mathcal{F}_t \right]$$

for some equivalent martingale measure $Q$. 
The main general result of [26] is that if $U$ satisfies (13) then optimal elements $\hat{X}(x), \hat{Y}(y)$ exist for (12),(14) and the functions $u, v$ are continuously differentiable. Taking $y = u'(x)$ we have

$$\hat{X}_T(x) = I(y\hat{Y}_T(y)), \quad E[\hat{Y}_T(x)\hat{X}_T(x)] = x. \quad (15)$$

It may not however be the case that $\hat{Y}$ is the density of a martingale measure.

If there is cumulative ‘random endowment’ $C_t$ then formally the dual problem is modified to

$$v_C(y) = \inf_{Y \in \mathcal{Y}} E \left[ V(yY_T) + y \int_0^T Y_t dC_t \right] \quad (16)$$

though the exact duality relationship is complicated in this case. See Cuoco [5], Cvitanic, Schachermayer and Wang [6] and Hugonnier and Kramkov [20].

4.2 Example: Optimal investment with randomly terminating income

In the problem of Section 3.4 income at rate $a$ is guaranteed for all time, so that the capital value of future income is $a/r$ at any time. Thus investor’s ‘solvency constraint’ is $X_t \geq -a/r$ for all $t$, a.s. The investor is able to borrow against future income because there is no doubt that this income will be received. The same idea applies much more generally. Suppose the investor has cumulative random income $c_t$ which is hedgeable, i.e. can be replicated by trading in the market. Then the value of the income stream at time 0 is

$$p = E_Q \int_0^\infty e^{-rt} dC_t$$

where $Q$ is any equivalent martingale measure\(^2\). We assume $p < \infty$. We can then borrow $p$ and form a replicating portfolio $Z_t$ with initial value $Z_0 = -p$, so that

$$\int_0^t e^{r(t-s)} dC_s + Z_t + E_Q \left[ \int_t^\infty e^{-r(s-t)} dC_s | \mathcal{F}_t \right] = 0 \quad \text{a.s.}$$

Thus the mark-to-market value of the portfolio plus income is always zero, and the investor can use his enhanced initial capital $x + p$ for investment, achieving a maximum expected utility $v_0(x + p)$ where $v_0(x)$ is the maximum utility starting with capital $x$ and no income. This argument applies equally well to the general incomplete-market finite time horizon models of section 4.1 above. When the income stream is not hedgeable, the investor will not be able to borrow against it in the same way, and the problem becomes more interesting (and realistic). The papers [5], [6], [20] mentioned above give existence results but few hints as to what optimal strategies are like. To get some insight, Davis and Vellekoop [10] consider Merton’s problem, as described above, with constant income at rate $a$, but supposing that this income terminates at a random time $\tau$, exponentially distributed

\(^2\)More precisely, $Q$ is equivalent to $P$ on the $\sigma$-field $\mathcal{F}_T$ for any $T > 0$
with parameter $\eta$ and independent of $(w_t)$. The market is now incomplete since the random cancellation is an unhedgeable risk.

For this problem, the portfolio value should be thought of as a jump-diffusion process $X_t = (X_t, Y_t)$ on the state space $\mathbb{R}^+ \times \{0, 1\}$ where $X_t$ is the cash value of the portfolio as before and $Y_t = 1_{(t<\tau)}$. The process $X_t$ starts at $(x, 1)$ and jumps from $(X_t, 1)$ to $(X_t, 0)$ at $t = \tau$. Writing the value function of dynamic programming as $v(x, i) = v_i(x)$, $x \in \mathbb{R}^+$, $i = 0, 1$, we see that $v_0(x)$ is the value function as derived previously with no income, while $v_1$ satisfies the Bellman equation

$$v'_1 r(x + a) - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 \frac{(v'_1)^2}{v'^2_1} + U(I(v'_1)) - I(v'_1) v_0 + (\delta + \eta) v_1 = 0. \quad (17)$$

There is no simple closed form solution to (17) for, say, logarithmic or power utility.

Turning to duality, the densities of martingale measures $Q_\lambda$ take the form

$$\frac{dQ_\lambda}{dP} \bigg|_{F_t} = \exp \left\{ \int_0^t \left[ r - \nu \right] dw - \frac{1}{2} \left( \frac{r - \nu}{\sigma} \right)^2 dt - \int_0^t (\lambda \eta - \eta)(1 - N_s) ds + \int_0^t \log(\lambda \eta) dN_s \right\}.$$

Here $N_t = 1_{t \geq \tau} = 1 - Y_t$. Under measure $Q_\lambda$, $N_t$ has hazard rate $\lambda_t$, replacing the hazard rate $\eta$ under measure $P$. We find that the dual optimal hazard rate $\hat{\lambda}(\cdot)$ is the solution to the following deterministic optimal control problem: minimize

$$J(\lambda) = \int_0^\infty e^{-\alpha t} (g(\lambda(t)) + ayx(t)) dt \quad (18)$$

over pairs $(\lambda(\cdot), x(\cdot))$ satisfying

$$\frac{dx}{dt} = -x(t)(r - \alpha + \lambda(t)) \quad x(0) = 1, \quad (19)$$

where $\alpha = \delta + \eta$ and

$$g(\lambda) = \frac{\eta}{\delta} \left( \frac{\lambda - \eta}{\eta} - \log \frac{\lambda}{\eta} \right).$$

(Note that $g(\lambda)$ has a global minimum of 0, achieved at $\lambda = \eta$) Analysis of the optimal control problem (18), (19) proceeds by application of the Pontryagin maximum principle and is surprisingly delicate: the Hamiltonian and control functions have to lie on a certain manifold to secure stable solutions. Having realized this, the problem can be solved and the value function $v_1$ computed. (Naive attempts to solve the Bellman equation (17) numerically are doomed to failure). Figure 1 shows the solution for some specific parameter values with logarithmic utility. It is notable that $v_1(0) > -\infty$ but $v'_1(0) = \infty$.

### 4.3 A general pricing formula

In incomplete markets most contingent claims involve unhedgeable risks. Their ‘value’ must be related to their use in constructing portfolios that are attractive
from an investment point of view. Considerations of this sort go back to, the 'principle of equi-marginal utility' formulated by Jevons [23] in the 19th century and have been extensively developed in the economics literature; see Foldes [17] for an exposition. To formalize these ideas in the present framework, suppose an investor’s objective is to maximize expected utility. As in Section denote

\[ u(x) = \sup_{X \in \mathcal{X}(x)} E[U(X_T)]. \]

In an option contract, we exchange a sure payment \( p \) at time zero for the random exercise value \( H \) of the option at time \( T \). An ‘indifference price’ for the option is the number \( \hat{p} \) such that

\[ u(x) = u_H(x - \hat{p}) \]

where

\[ u_H(x) = \sup_{X \in \mathcal{X}(x)} E[U(X_T + H)]. \tag{20} \]

Thus indifference pricing is a special case of the ‘random endowment’ problem (16). The optimal ‘hedge’ is the maximizing strategy for the problem (20). Note that \( \hat{p} \) is a nonlinear pricing function: \( 2\hat{p} \) is not the indifference price for \( 2H \). To get a linear pricing rule, consider buying \( \epsilon \) units at price \( p \) per unit and define

\[ v(x, \epsilon, p) = \sup_{X \in \mathcal{X}(x - \epsilon p)} E[U(X_T + \epsilon H)]. \tag{21} \]
\( \hat{p} \) is the marginal utility price if
\[
\left. \frac{\partial}{\partial \epsilon} v(x, \epsilon, \hat{p}) \right|_{\epsilon=0} = 0.
\]

Formally differentiating (21) we obtain
\[
\left. \frac{\partial}{\partial \epsilon} v(x, \epsilon, \hat{p}) \right|_{\epsilon=0} = -pu'(x) + E[U'(\hat{X}_T)H],
\]
where \( \hat{X}_T \) is the optimal terminal wealth, giving the following formula for the marginal utility price:
\[
\hat{p} = E \left[ \frac{U'(\hat{X}_T)}{u'(x)} \right].
\] 
(22)

Now recall the characterization of the optimal investment portfolio \( \hat{X}_T \) given at (15) above, namely \( \hat{X}_T = I(\hat{y}\hat{Y}_T) \) where \( \hat{Y}_T \) is the solution of the dual minimization problem. Since \( I = (U')^{-1} \) and \( \hat{y} = u'(x) \)
\[
\hat{Y}_T = \frac{1}{u'(x)} U'(\hat{X}_T). 
\] 
(23)

Now suppose that \( \hat{Y}_T \) happens to be a martingale (rather than just a supermartingale). Then the measure \( \hat{Q} \) defined by
\[
\frac{d\hat{Q}}{dP} = \hat{Y}_T
\] 
(24)
is an equivalent martingale measure, and the price \( \hat{p} \) of (22) is expressed as\(^3 \)
\[
\hat{p} = E_{\hat{Q}}[H]. 
\] 
(25)

As was mentioned earlier, all arbitrage-free valuations lie in the interval 
\( (\inf_{\hat{Q} \in \mathcal{M}} E_{\hat{Q}}[H], \sup_{\hat{Q} \in \mathcal{M}} E_{\hat{Q}}[H]) \), so we see that the marginal utility price picks a specific value from this interval, justified on economic grounds and depending on the investor’s preferences.

If the optimal dual supermartingale \( \hat{Y}_T \) is not a martingale, then the marginal utility price is not uniquely defined; see Kramkov and Schachermayer [26].

5 Parameter Uncertainty

Recall the Black-Scholes price model
\[
dS_t = \mu S_t dt + \sigma S_t dw_t.
\]
The Black-Scholes value at time \( t < T \) for an option with exercise value \( h(S_T) \) is a function \( C_h(S, r, \sigma, T-t) \) where \( S \) is the price at time \( t \) and \( r \) is the riskless interest...

\(^3\)Recall we have taken \( S_0(t) \equiv 1 \), so no ‘discount factor’ appears.
rate. It is a key point that $C_h$ does not depend on the growth rate $\mu$, and this is fortunate because $\mu$ is essentially impossible to estimate. Indeed, if $\sigma$ is known, the minimum variance unbiased estimate of $\mu$ is

$$\hat{\mu}_t = \frac{1}{t} \log \frac{S_t}{S_0} + \frac{1}{2} \sigma^2 = \mu + \frac{\sigma w_t}{t}$$

with variance $\text{var}(\hat{\mu}_t) = \sigma^2 / t$. A typical volatility might be $\sigma = 20\%$, so to achieve 95% confidence that $|\mu - \hat{\mu}| < 1\%$ we need $1.96\sigma / \sqrt{t} < 0.01$, i.e. $t > 1521$ years. For shorter periods, the estimation error is enormous; for example, at $t = 3$ years the standard deviation is 11.5%. Estimating the volatility $\sigma$ is easier, not surprisingly since in continuous time $\sigma^2 \varepsilon$ is exactly observable as the quadratic variation of the sample path log$S_t$ over any time interval of length $\varepsilon$. See Rogers [32] for properties of estimates based on sampled data.

A further source of estimation error is that it is surely not credible to assume that the parameters are constant over lengthy periods such as 20 or 30 years. Financial data is thus subject to a kind of “uncertainty principle”. Parameters can never be estimated to arbitrarily high precision: if the sampling interval is short then estimation error variance is high, while if the sampling interval is long then there will be bias errors due to non-stationarity. Figure 2 shows sample estimates based on simulated data from the log-normal model with $\mu = 5\%$, $\sigma = 20\%$ using an increasing number of years of data back from the present, while similar estimates for the growth rate of the FTSE100 index, based on a log-normal model with the sample volatility, are shown in Figure 4. The index itself is shown in Figure 3. One can see how useless these estimates are, even in the artificial case with simulated data. The lesson is that parameter uncertainty must be explicitly allowed for in the model.

Optimization under parameter uncertainty uses one of two approaches

- Stochastic control with random parameters
- Stochastic programming using a small number of random scenarios.

We very briefly describe these approaches in the next sections.

### 5.1 Stochastic control with random parameters

Here asset prices are modelled as

$$dS_t = \mu(S_t, Z_t)S_t dt + \sigma(S_t, Z_t)S_t dw_t.$$  

$w_t$ is vector Brownian motion and $Z_t$ is a ‘factor process’ representing economic conditions. It could be another diffusion or, more tractably, a finite-state process $Z_t \in \{1, 2, \ldots, \kappa\}$ with transition rates

$$P[Z(t + dt) = j | \mathcal{F}_t] 1_{Z(t) = i} = a_{ij}(S_t, Z_t)dt + o(dt), \quad j \neq i$$

We define $a_{ii} = - \sum_{j \neq i} a_{ij}$. Take the scalar case (1 risky asset) and suppose that as above the objective is to maximize

$$E \int_0^\infty e^{-\delta t} U(c_t)dt.$$
If the factor process is observed, the wealth equation is unchanged:

\[ dX_t = X_t(r + (\mu - r)\pi)dt - c_t dt + X_t\pi_t\sigma dw_t \]

except that \( \mu, \sigma \) now depend on \( Z_t \). Write the value function as \( v_z(x) = v(x, z) \); then the Bellman equation is

\[
\max_{u,c} \left\{ \frac{\partial v_i}{\partial x}(x(r + (\mu - r)\pi) - c) + \frac{1}{2}u^2\sigma^2x^2 \frac{\partial^2 v_i}{\partial x^2} + U(c) + \sum_j a_{ij}v_j - \delta v_i \right\} = 0.
\]

This is a coupled system of equations for the functions \( v_1(x), \ldots, v_\kappa(x) \).

### 5.2 Stochastic programming

Stochastic programming is a discrete-time approach using a ‘scenario tree’ with a small number \( T \) of time periods. This is probably the best available method for analysing long-term asset and liability management for large insurance and pensions companies; see Mulvey and Ziemba [30] for a comprehensive treatment. A scenario is any path from left to right through the tree; in the example shown in Figure 2. Sample estimates of \( \mu \), simulated data.

![Growth rate estimate, simulated data](image)

Figure 2. Sample estimates of \( \mu \), simulated data.
4 there are $N = 6$ scenarios. In realistic applications, for example the pension planning model described in [35], a typical number is $N = 10000$.

Asset classes available for investment are labelled $a \in \{1, \ldots, M\}$. For each scenario $s$, returns

$$\{R_{s,i}^a, a \in \{1, \ldots, M\}, i \in \{1, \ldots, T\}\}$$

are specified (by sampling and quantizing historical return distributions), so that $\$1$ invested in asset class $a$ at time $t_{i-1}$ becomes $\$R_{s,i}^a$ at time $t_i$. This scenario occurs with probability $p_s$. Investment decisions are made at each node at times $t_i < t_T$. Tree structure enforces ‘non-anticipativity’: if scenarios $s, s'$ coincide up to time $t_i$ then the investment decisions for $s$ and $s'$ at $t_i$ are the same. The objective is to maximize

$$EU(X) = \sum_s [p_s U(X(s))]$$

where $X(s)$ is the wealth at the terminal node of scenario $s$ and $U$ is a concave utility function.

Various penalties and constraints can be included, for example investment constraints (no shortselling etc.), penalties for not meeting growth targets, penalties for failing to pay liabilities, or transaction costs.
This problem can be formulated and solved as a linear programming problem. (The sub-problem of evaluating piecewise-linear approximations to the concave utility function $U$ is also a linear program.) The art of this approach is to include a sufficiently rich set of scenarios to cover a wide range of favourable and adverse economic conditions. Then solutions are ‘robust’ in that they guard against adverse conditions. It turns out that choice of the probabilities $p_s$ associated with the scenarios is less important.

6 Concluding Remarks

In a word, the message of this paper is as follows. “Traditional” mathematical finance, centred around the Black-Scholes model and its extensions, is a complete-market theory based on pricing by replication. In this setting only volatilities are important, and in applications most of the modelling effort is devoted to deriving appropriate volatility structure for particular products such as fixed-income derivatives.

In recent years there has been a huge increase in the trading of products involving unhedgeable risks. Examples include credit derivatives, counter party de-
fault risk, long-term equity risk and contracts on non-standard underlying variables such as the weather or insurance indices. In all these cases perfect hedging is not possible and therefore the investor’s attitude to risk is an essential ingredient of the problem.

Our contention is that unhedgeable contracts should be thought of as investment opportunities, and this approach leads both to pricing methods based on utility indifference and ‘hedging’ in the form of optimal investment strategies to maximize utility and/or minimize the probability of failing to honour liabilities. The chief technical tool is then stochastic control theory, either in continuous time or using discrete-time scenario-based methods. Stylized problems, such as the annuity payment problem of section $x$ or the terminating income problem of section $y$ are useful in giving guidance as to the form of optimal strategies for specific purposes, and in the author’s opinion the repertoire of solved problems of this sort is, at the present time, insufficiently large.

If one seeks to build a model for a specific application then an extra difficulty appears, namely parameter uncertainty. As we have pointed out, there is no way of estimating relevant parameters with any degree of precision. How to operate in this world of gross uncertainty is still far from settled. The stochastic programming method described in section $z$ provides a systematic approach for long-term problems, but for shorter term problems such as hedging credit risk some fresh thinking is required to provide solutions that are both implementable and soundly based. This is the challenge for the future.

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