Fractional Kelly Strategies for Benchmarked Asset Management

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Abstract
In this paper, we extend the definition of fractional Kelly strategies to the case where the investor’s objective is to outperform an investment benchmark. These benchmarked fractional Kelly strategies are efficient portfolios even when asset returns are not lognormally distributed. We deduce the benchmarked fractional Kelly strategies for various types of benchmarks and explore the interconnection between an investor’s risk-aversion and the appropriateness of their investment benchmarks.

1 Introduction
Classically, a fractional Kelly strategy with fraction $f$ consists in investing a proportion $f$ of one’s wealth in the Kelly criterion, or log utility, optimal portfolio and a proportion $1 - f$ in the risk-free asset. Fractional Kelly strategies play an important role in active investment management in the asset only case, that is when the investor’s objective is to maximize the terminal value of his/her wealth, without any benchmark to track or liability to pay (see for example Thorp (2006) and Ziemba (2003) for discussion and additional references). In this chapter, we analyze the role of fractional Kelly strategies in the asset allocation of a benchmarked investor, that is an investor whose objective is to outperform a given investment benchmarks, such as the S&P 500 or the Salomon Smith Barney World Government Bond Index.

The argument developed in this chapter builds on and applies the stochastic control-based model proposed by Davis and Lleo (2008a). Their methodology is founded on the theory of risk-sensitive stochastic control, rather than on the classical stochastic control theory-based approach proposed by Merton (see for example Merton (1969), Merton (1971) or Merton (1992)).
the asset only case, this choice has the benefit of being consistent with both the Merton model and with the mean-variance analysis while allowing for the explicit inclusion of underlying valuation factors and while admitting a simpler analytical solution than the Merton model. An added advantage of this model is the relative ease with which the asset only problem as formalized by Bielecki and Pliska (1999) can be extended to include a benchmark, as in Davis and Lleo (2008a), or a liability, as in Davis and Lleo (2008b).

In their classical definition, fractional Kelly strategies are not necessarily optimal, with the notable exception of the Merton model where they arise naturally as a consequence of the assumption that asset prices are lognormally distributed. As a result, to be able to interpret the solution to our benchmarked asset allocation problem in terms of fractional Kelly strategies and therefore guarantee their optimality, we will find it necessary to expand slightly their definition.

In Section 2, we introduce risk-sensitive asset management from the perspective of an asset only investor. The interpretation of the resulting optimal asset allocation formula is then formalized in Section 3 both as a Mutual Fund Theorem and in terms of a redefinition of fractional Kelly strategies. Section 4 presents a comparison of the classical Merton model with the risk-sensitive asset management approach from the perspective of fractional Kelly strategies. Then, in Section 5, we present the analytical solution to the risk-sensitive benchmarked asset allocation problem, before interpreting this solution in a Mutual Fund theorem and in terms of benchmarked fractional Kelly strategies in Section 6. Finally, Section 7 contains a number of case studies an application of these ideas to specific types of benchmarks and to Kelly criterion investors.

2 Risk Sensitive Asset Management

2.1 Risk Sensitive Control

Risk-sensitive control is most simply defined as a generalization of classical stochastic control in which the degree of risk aversion or risk tolerance of the optimizing agent is explicitly parameterized in the objective criterion and influences directly the outcome of the optimization. Risk sensitive control was introduced by Jacobson (1973) and has been developed by many authors, notably Whittle (1990) and Bensoussan and Schuppen (1985) before being applied to finance by Lefebvre and Montulet (1994) and to asset management by Bielecki and Pliska (1999).

While in classical stochastic control the objective of the decision maker
is to maximize $E[F]$, the expected value of some performance criterion $F$, in risk-sensitive control the decision maker’s objective is to select a control policy $h(t)$ maximizing the criterion

$$J(t, x; h; \theta) := -\frac{1}{\theta} \ln E \left[ e^{-\theta F(t, x, h)} \right]$$  \hspace{1cm} (1)$$

where

- $t$ and $x$ are the time and the state variable;
- $F$ is a reward function;
- the risk sensitivity $\theta \in (-1, 0) \cup (0, \infty)$ represents the decision maker's degree of risk aversion.

A Taylor expansion around $\theta = 0$ of (1) around 0 evidences the vital role played by the risk sensitivity:

$$J(x, t, h; \theta) = E[F(t, x, h)] - \theta \frac{\theta}{2} \text{Var}[F(t, x, h)] + O(\theta^2)$$  \hspace{1cm} (2)$$

- $\theta \to 0$ corresponds to the “risk-null” case and to classical stochastic control;
- when $\theta < 0$, we have the “risk-seeking” case which is a maximization of the expectation of a convex decreasing function of $F(t, x, h)$;
- finally, $\theta > 0$ is the “risk-averse” case which is to a minimization of the expectation of a convex increasing function of $F(t, x, h)$.

To summarize, risk-sensitive control differs from traditional stochastic control in that it explicitly models the risk-aversion of the decision maker as an integral part of the control framework, rather than importing it in the problem via an externally defined utility function.

### 2.2 Risk Sensitive Asset Management

Bielecki and Pliska (1999) pioneered the application of risk-sensitive control to asset management. They proposed to take the logarithm of the investor’s wealth $V$ as the reward function, i.e.

$$F(t, x, h) = \ln V(t, x, h).$$

The natural interpretation of this choice is that the investor’s objective is to maximize the risk-sensitive (log) return of the his/her portfolio.

With this choice of reward function, the control criterion is

$$J(t, x; h; \theta) := -\frac{1}{\theta} \ln E \left[ e^{-\theta \ln V(t, x, h)} \right]$$  \hspace{1cm} (3)$$
and interpret the expectation
\[ E \left[ e^{-\theta \ln V(t,x,h)} \right] = E \left[ V(t,x,h)^{-\theta} \right] =: U_\theta(V_t) \quad (4) \]
as the expected utility of time t wealth under the power utility (HARA) function. The investor’s objective is to maximize the utility of terminal wealth.

The Taylor expansion becomes
\[ J(t,x;\theta) = E [\ln V(t,x,h)] - \theta \frac{\theta}{2} \text{Var} [\ln V(t,x,h)] + O(\theta^2). \quad (5) \]
Ignoring higher order terms, we recover the mean-variance optimization criterion and the log utility or Kelly criterion portfolio in the limit as \( \theta \to 0 \).

2.3 The Risk Sensitive Asset Management Model

2.3.1 Asset and Factor Dynamics

Embedding the investor’s risk-sensitivity in the control criterion provides more leeway in the specification of the asset market than would be obtained in the classical stochastic control of the Merton approach. Bielecki and Pliska (1999), in particular, propose a factor model in which the prices of the \( m \) risky assets follow a SDE of the form
\[
\frac{dS_i(t)}{S_i(t)} = (a + AX(t))dt + \sum_{k=1}^{n+m} \sigma_{ik}dW_k(t) \\
S_i(0) = s_i, \quad i = 1, \ldots, m
\]  
where \( W(t) \) is a \( N := n + m \)-dimensional Brownian motion and the market parameters \( a, A, \Sigma := [\sigma_{ij}], \; i = 1, \ldots, m, \; j = 1, \ldots, N \) are matrices of appropriate dimensions. To these \( m \) risky securities, we add a money market asset with dynamics
\[
\frac{dS_0(t)}{S_0(t)} = (a_0 + A'_0X(t)) \, dt, \quad S_0(0) = s_0.
\]  
Finally, the asset prices drift depends on \( n \) valuation factors modelled as affine stochastic processes with constant diffusion
\[
dX(t) = (b + BX(t))dt + \Lambda dW(t), \quad X(0) = x
\]  
where \( X(t) \) is the \( \mathbb{R}^n \)-valued factor process with components \( X_j(t) \) and the parameters \( b, B, \Lambda := [A_{ij}], \; i = 1, \ldots, n, \; j = 1, \ldots, N \) are matrices of
appropriate dimensions. These valuation factors must be specified, but they could include macroeconomic, microeconomic or statistical variables.

To reach our final results, we only need to make the minor assumption that

**Assumption 1.** The matrix $\Sigma\Sigma'$ is positive definite.

The only effect of this assumption is to exclude duplicating assets and clear arbitrage opportunities from our investment universe.

Under these conditions, the logarithm of the investor’s wealth is given by the SDE

$$
\ln V(t) = \ln v + \int_0^t (a_0 + A'_0 X(s)) + h(s)' \left( \hat{a} + \hat{A} X(s) \right) ds
$$

$$
- \frac{1}{2} \int_0^t h(s)' \Sigma \Sigma' h(s) ds + \int_0^t h(s)' \Sigma dW(s),
$$

(9)

where $V(0) = v$, $h$ is the $m$-dimensional vector of portfolio weights and we used the notation $\hat{a} := a - a_0 1$, $A := A - 1 A'_0$ and $1$ is a $n$-element column vector with all entries set to 1.

The equation for $V$ solely depends on the valuation factors (the state process) and is independent from the asset prices. This implies that the effective dimension of the risk-sensitive control problem will be $n$, the number of factors, rather than $m$, the number of risky assets. The limited impact of the number of assets is significant since for practical applications we would typically use only a few factors (possibly 3 to 5) to parametrize a large cohort of assets and asset classes (possibly several dozens). The risk-sensitive asset management model is therefore particularly efficient from a computational perspective.

### 2.3.2 The Associated Linear Exponential-of-Quadratic Gaussian Control Problem

The next step in the analysis is due to Kuroda and Nagai (2002) who ingeniously observed that under an appropriately chosen change of probability measure (via the Girsanov theorem), the risk-sensitive criterion can be expressed as

$$
I(v, x; h; t, T) = \ln v - \frac{1}{\theta} \ln \mathbb{E}_h^\theta \left[ \exp \left\{ \theta \int_t^T g(X_s, h(s); \theta) ds \right\} \right]
$$

(10)
where the expectation $\mathbb{E}_\theta^\theta$ is taken with respect to the newly $\mathbb{P}_\theta^\theta$ measure, the functional $g$ is

$$g(x, h; \theta) = \frac{1}{2} (\theta + 1) h'\Sigma\Sigma'h - a_0 - A_0'x - h'(\hat{a} + \hat{A}x)$$

(11)

and the factor dynamics under the new measure $\mathbb{P}_\theta^\theta$ is

$$dX_s = (b + BX_s - \theta \Lambda\Sigma' h(s)) ds + \Lambda dW_s^\theta$$

(12)

(see Davis and Lleo (2008a) for details).

In this formulation, the problem is a standard Linear Exponential-of-Quadratic Gaussian (LEQG) control problem which can be solved exactly, up to the resolution of a system of Riccati equations. But before solving this control problem and deriving the optimal asset allocation, we will first develop some intuition by considering the simple case in which the security and factor risk are uncorrelated, i.e. when $\Lambda\Sigma' = 0$.

### 2.3.3 Special Case: Uncorrelated Assets and Factors

When $\Lambda\Sigma' = 0$, security risk and factor risk are uncorrelated and the evolution of $X_t$ under the measure $\mathbb{P}_\theta^\theta$ given in equation (12) simplifies to

$$dX_s = (b + BX_s) ds + \Lambda dW_s^\theta.$$

The evolution of the state is therefore independent of the control variable $h$ and, as a result, the control problem can be solved through a pointwise maximisation of the auxiliary criterion function $I(v, x; h, t, T)$.

In this case, the optimal control $h^*$ is simply the maximizer of the function $g(x; h; t, T)$ given by

$$h^* = \frac{1}{\theta + 1} (\Sigma\Sigma')^{-1} \left( \hat{a} + \hat{A}x \right)$$

which represents a position of $\frac{1}{\theta+1}$ in the Kelly criterion portfolio.

Let $\Phi(t, x)$ be the value function corresponding to the auxiliary criterion function $I(v, x; h; t, T)$. Substituting the value of $h^*$ in the equation for $g$ yields

$$\Phi(t, x) = \sup_{h \in \mathcal{A}(T)} I(v, x; h; t, T)$$

$$= -\frac{1}{\theta} \ln \mathbb{E}^\theta \left[ \exp \left\{ \theta \int_0^{T-t} g(x, h^*(s); t, T; \theta) ds \right\} v^{-\theta} \right].$$
The PDE for $\Phi$ can now be obtained directly via an exponential transformation and an application of Feynman-Kac.

2.3.4 The General Case

In the general case, the value function $\Phi$ for the auxiliary criterion function $I(v, x; h; t, T)$, defined as

$$
\Phi(t, x) = \sup_{A(T)} I(v, x; h; t, T)
$$

satisfies the Hamilton-Jacobi-Bellman Partial Differential Equation (HJB PDE)

$$
\frac{\partial \Phi}{\partial t} + \sup_{h \in \mathbb{R}^m} L_h \Phi(X(t)) = 0
$$

where

$$
L_h \Phi(t, x) = \left(b + Bx - \theta \Lambda \Sigma' h(s)\right)' D\Phi + \frac{1}{2} tr \left(\Lambda \Lambda' D^2 \Phi\right) - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi - g(x, h; \theta)
$$

The value function $\Phi$ satisfies the terminal condition $\Phi(T, x) = \ln v$.

Solving the optimization problem gives the optimal investment policy $h^*(t)$

$$
h^*(t) = \frac{1}{\theta + 1} \left(\Sigma \Sigma'\right)^{-1} \left[\hat{a} + \hat{A}X(t) - \theta \Sigma \Lambda' D\Phi(t, X(t))\right].
$$

Moreover, the solution of the PDE is of the form

$$
\Phi(t, x) = x' Q(t) x + x' q(t) + k(t)
$$

where $Q(t)$ solves a $n$-dimensional matrix Riccati equation and $q(t)$ solves a $n$-dimensional linear ordinary differential equation depending on $Q$ (see Kuroda and Nagai (2002) for details).
3 Fractional Kelly Strategies in the Risk-Sensitive Asset Management Model

3.1 A Mutual Fund Theorem

Theorem 2 (Mutual Fund Theorem (Davis and Lleo, 2008a)). Any portfolio can be expressed as a linear combination of investments into two “mutual funds” with respective risky asset allocations:

\[
\begin{align*}
    h^K(t) &= (\Sigma^\prime \Sigma)^{-1} (\hat{a} + \hat{A}X(t)) \\
    h^C(t) &= -(\Sigma^\prime \Sigma)^{-1} \Sigma \Lambda^\prime (q(t) + Q(t)X(t))
\end{align*}
\]  

(15)

and respective allocation to the money market account given by

\[
\begin{align*}
    h^K_0(t) &= 1 - 1'(\Sigma^\prime \Sigma)^{-1} (\hat{a} + \hat{A}X(t)) \\
    h^C_0(t) &= 1 + 1'(\Sigma^\prime \Sigma)^{-1} \Sigma \Lambda^\prime (q(t) + Q(t)X(t))
\end{align*}
\]

Moreover, if an investor has a risk sensitivity \( \theta \), then the respective weights of each mutual fund in the investor’s portfolio equal \( \frac{1}{\theta+1} \) and \( \frac{\theta}{\theta+1} \), respectively.

The main implication of this theorem is that the allocation between the two funds is a sole function of the investor’s risk sensitivity \( \theta \). As \( \theta \to 0 \), the investor’s wealth gets invested in the Kelly criterion portfolio (portfolio \( K \)). On the other hand, as \( \theta \to \infty \), the investor’s wealth gets invested in the correction portfolio \( C \). The investment strategy of this portfolio can be interpreted as a large position in the short-term rate and a set of positions trading on the comovement of assets and valuation factors. In financial economics, portfolio \( C \) is referred to as the ‘intertemporal hedging term’.

When we assume that there are no underlying valuation factors, the risky securities follow geometric Brownian motions with drift vector \( \mu \) and the money market account becomes the risk-free asset (i.e. \( a_0 = r \) and \( A_0 = 0 \)). In this case \( \Sigma \Lambda' = 0 \) and we can then easily see that fund \( C \) is fully invested in the risk-free asset. As a result, we recover Merton’s Mutual Fund Theorem for \( m \) risky assets and a risk-free asset (see, for example, Merton (1992)).

3.2 Fractional Kelly Strategies in the Risk-Sensitive Asset Management Model

Fractional Kelly strategies arise naturally in the Merton investment model, since Merton’s Mutual Fund Theorems guarantees that the optimal investment strategy can be split in an allocation to the risk-free asset and an allocation to a mutual fund investing in the Kelly criterion portfolio. However, the optimality of fractional Kelly strategies is the exception rather than
the rule: in the Merton model, fractional Kelly strategies are optimal as a result of the assumption that asset prices are lognormally distributed. In the factor-based risk-sensitive asset management model, we cannot expect either that the ‘classical’ definition of fractional Kelly strategies to yield optimal or near optimal asset allocations as soon as $\theta \neq 0$.

To address this difficulty, Davis and Lleo (2008a) proposed a generalization of the concept of fractional Kelly strategy based on the findings expressed in the Mutual Fund Theorem 2. Rather than regarding the fractional Kelly strategy as a split between the Kelly portfolio and the short-term rate, Davis and Lleo propose to define it as a split between the Kelly portfolio and the portfolio $C$, as defined in the Mutual Fund Theorem 2. In this case, the Kelly fraction, which represents the proportion of wealth invested in the Kelly portfolio, is inversely proportional to the investor’s risk sensitivity and is equal to $\frac{1}{\sigma+1}$.

This redefinition of fractional Kelly strategies has two important consequences:

- the fractional Kelly portfolios are always optimal portfolios;
- in the lognormal case (i.e. when $n = 0$), the generalized definition of fractional Kelly strategies reverts to the ‘classical’ definition. This can be verified from the fact that in the lognormal case, the Mutual Fund Theorem 2 simplifies into Merton’s Mutual Fund Theorem for $m$ risky assets and a risk-free asset.

4 Fractional Kelly Strategies, Risk-Sensitive Asset Management and the Merton Model with Power Utility

4.1 Objective

In this section, we go further in our analysis by drawing a comparison between risk-sensitive asset management and the Merton model from the perspective of fractional Kelly strategies.

For comparison purpose, we consider:

- a fractional Kelly strategy with fraction $f$;
- the Merton model maximizing utility of terminal wealth, with the utility function chosen to be the homogeneous power utility or hyperbolic absolute risk aversion (HARA) function; and
• the diffusion risk-sensitive asset management model without any underlying valuation factors.

In order to perform a comparison between the risk-sensitive asset allocation model and the original Merton model, we assume that the market is comprised of \( m \) risky assets and that the investor’s objective is to find an optimal \( m \)-dimensional portfolio allocation vector \( h(t) \), where \( h_i(t) \) represents the proportion of the portfolio invested in risky security \( i \). We also assume that there are no valuation factors and hence \( n = 0 \). We will show that the treatment of fractional Kelly strategies is comparable in both cases up to a sign difference in the risk-aversion / risk-sensitive coefficient.

4.2 A Brief Review of The Merton Model

In the Merton model, the objective of an investor is to maximize the utility of terminal wealth represented by the criterion:

\[
I(t, s, h; \theta) := E[U(V(t, s, h))] \tag{16}
\]

where

• \( V(t, x, h) \) is the investor’s wealth at time \( t \) in response to a securities market with price vector \( s \) and an investment policy \( h \);

• \( U \) is the homogeneous power utility or hyperbolic absolute risk aversion (HARA) function, defined as

\[
U(z) = \frac{z^\gamma}{\gamma}
\]

where \( \gamma \in (−\infty, 0) \cup (0, 1) \) is the risk-aversion coefficient.

The dynamics of the prices of the \( m \) risky assets follows a geometric Brownian motion of the form

\[
\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sum_{k=1}^{m} \sigma_{ik} dW_k(t), \quad S_i(0) = s_i, \quad i = 1, \ldots, m
\]

where \( W(t) \) is a \( m \)-dimensional Brownian motion. The price of the money market asset satisfies

\[
\frac{dS_0(t)}{S_0(t)} = r dt, \quad S_0(0) = s_0.
\]
In this setting, the optimal asset allocation is given by

\[ h^*(t) = \frac{1}{1 - \gamma} (\Sigma \Sigma')^{-1} \hat{\mu} \]

with \( \hat{\mu} := \mu - r \mathbf{1} \) where \( \mathbf{1} \) is the \( m \)-element unit vector and \( \Sigma \) is the diffusion matrix, defined as \( \Sigma = [\sigma_{ij}] \).

4.3 Observations and Conclusions

From a fractional Kelly perspective, the two models are strikingly similar. Taking the case of an investor allocating a fraction \( f \) of his/her wealth to the Kelly portfolio, we see that

- in the Merton model, such strategy would be optimal for an investor with a level of risk aversion \( \gamma \) equal to
  \[ \gamma = 1 - \frac{1}{f}. \]

- in the risk-sensitive model, such strategy would be optimal for an investor with a level of risk-sensitivity \( \theta \) equal to
  \[ \theta = 1 - \frac{f}{f}. \]

Looking solely at the Kelly component of the optimal investment policy, this would imply that

\[ \theta = -\gamma. \]

This similarity is in not surprising. Indeed, when we restrict the risk-sensitive approach to a 0 factor model, we get the same optimal asset allocation as in the Merton model with homogeneous power utility, but for the fact that \( \gamma \) has been replaced by \( -\theta \), i.e.

\[ h^*(t) = \frac{1}{\theta + 1} (\Sigma \Sigma')^{-1} \hat{\mu}. \]

This observation is confirmed by both the range of \( \theta \) and \( \gamma \) and by the functional form of the utility function associated with the risk-sensitive approach.

These findings are summarized in Table 1.

**Key Points:**
Risk-Sensitive Asset Management

<table>
<thead>
<tr>
<th>Risk-sensitive parameter / Risk aversion coefficient</th>
<th>( \theta \in (-1, 0) \cup (0, \infty) )</th>
<th>( \gamma \in (-\infty, 0) \cup (0, 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range of the risk-sensitive parameter / risk aversion coefficient for risk-averse investors</td>
<td>((0, \infty))</td>
<td>((-\infty, 0))</td>
</tr>
<tr>
<td>Form of Utility Function</td>
<td>( U(z) = z^{-\gamma} )</td>
<td>( U(z) = \frac{z^\gamma}{\gamma} )</td>
</tr>
<tr>
<td>Optimal asset allocation</td>
<td>( h^*(t) = \frac{1}{\theta+1} (\Sigma \Sigma')^{-1} (\hat{\mu}(X(t)) + \Sigma \Lambda D \Phi) )</td>
<td>( h^*(t) = \frac{1}{1-\gamma} (\Sigma \Sigma')^{-1} \hat{\mu} )</td>
</tr>
<tr>
<td>Value of the risk aversion coefficient / risk-sensitive parameter corresponding to a Kelly fraction ( f )</td>
<td>( 1/f - 1 )</td>
<td>( 1 - 1/f )</td>
</tr>
<tr>
<td>Range of Kelly fractions for risk-averse investors</td>
<td>((0, 1))</td>
<td>((0, 1))</td>
</tr>
</tbody>
</table>

Table 1: Comparison of the Merton Model and of the Risk-Sensitive Asset Management Model from a Kelly Perspective

- The situation between the Merton model with homogeneous power utility and the risk-sensitive asset management model is parallel, and we can use the guideline correspondence \( \theta = -\gamma = 1/f - 1 \) to link them with the fractional Kelly approach;

- The risk-sensitive approach is therefore consistent with the Merton model with homogeneous power utility;

- In the case when there are no factors (i.e. \( n = 0 \)), the optimal asset allocation obtained in the risk-sensitive approach reverts to that associated with the Merton model;

- In the general case \( (n > 0) \), the definition of fractional Kelly strategies in the risk-sensitive asset management model can be extended as proposed above while remaining consistent with the “classical” definition of fractional Kelly strategies as a split between the Kelly portfolio and the risk-free asset.
5 Adding an Investment Benchmark: Risk Sensitive Benchmarked Asset Management Model

So far, we have introduced risk sensitive asset management in the classical asset only context of an investor attempting to maximize the terminal utility of his/her wealth. We will now examine the related benchmarked asset management problem in which the investor selects an asset allocation to outperform a given investment benchmark.

In the benchmarked case, Davis and Lleo (2008a) propose that the reward function $F(t,x,h)$ be defined as the (log) excess return of the investor’s portfolio over the return of the benchmark, i.e.

$$F(t,x,h) := \ln \frac{V(t,x,h)}{L(t,x,h)}$$

where $L$ is the level of the benchmark.

Furthermore, the dynamics of the benchmark is modelled by the SDE:

$$\frac{dL(t)}{L(t)} = (c + C'X(t))dt + \varsigma'dW(t), \quad L(0) = l$$ (17)

where $C$ is a scalar constant, $C$ is a $n$-element column vector, and $\varsigma$ is a $N$-element column vector.

This formulation is wide enough to encompass a multitude of situations such as:

- **the single benchmark case**, where the benchmark is, for example, an equity index such as the S&P500 or the FTSE 100.

- **the single benchmark plus alpha**, where, for example, a hedge fund has for benchmark a target based on a short-term interest rate plus alpha.

- **the composite benchmark case**. for example a benchmark constituted of 5% cash, 35% Citigroup World Government Bond Index, 25% S&P 500 and 35% MSCI EAFE.

- **the composite benchmark plus alpha** which is a combination of the previous two cases.

By Itô’s lemma, the log of the excess return in response to a strategy $h$
is

\[
F(t, x; h) = \ln \frac{v}{t} + \int_0^t d \ln V(s) - \int_0^t d \ln L(s)
\]

\[
= \ln \frac{v}{t} + \int_0^t \left( a_0 + A'_0 X(s) + h(s)' \left( \dot{a} + \dot{A} X(s) \right) \right) ds
\]

\[
\quad - \frac{1}{2} \int_0^t h(s)' \Sigma \Sigma' h(s) ds + \int_0^t h(s)' \Sigma dW(s)
\]

\[
\quad - \int_0^t (c + C' X(s)) ds + \frac{1}{2} \int_0^t \dot{\varsigma}' \varsigma ds
\]

\[
\quad - \int_0^t \dot{\varsigma}' dW(s)
\]

\[
F(0, x; h) = f_0 := \ln \frac{v}{t}.
\]

Following an appropriate change of measure, the criterion function can be expressed as

\[
I(f_0, x; h; t, T) = \ln f_0 - \frac{1}{\theta} \ln E^\theta \left[ \exp \left\{ \theta \int_0^T g(X_s, h(s); \theta) ds \right\} \right]
\]

where

\[
g(x, h; \theta) = \frac{1}{2} (\theta + 1) h' \Sigma \Sigma' h - a_0 - A'_0 x - h'(\dot{a} + \dot{A} x)
\]

\[
\quad - \theta h' \Sigma \varsigma + (c + C' x) + \frac{1}{2} (\theta - 1) \dot{\varsigma}' \varsigma.
\]

Once again, the control problem simplifies into a LEQG problem.

The value function \( \Phi \) for the auxiliary criterion function \( I(f_0, x; h; t, T) \). Then \( \Phi \) is defined as

\[
\Phi(t, x) = \sup_{A(T)} I(f_0, x; h; t, T)
\]

and it satisfies the HJB PDE

\[
\frac{\partial \Phi}{\partial t} + \sup_{h \in \mathbb{R}^m} L_t^h \Phi = 0
\]

where

\[
L_t^h \Phi = (b + B x - \theta \Lambda (\Sigma' h - \varsigma))' D \Phi + \frac{1}{2} \text{tr} (\Lambda \Lambda' D^2 \Phi)
\]

\[
- \theta (D \Phi)' \Lambda \Lambda' D \Phi - g(x, h; \theta).
\]
Solving the optimization problem gives the optimal investment policy \( h^*(t) \)

\[
h^* = \frac{1}{\theta + 1} (\Sigma \Sigma')^{-1} \left( \hat{a} + \hat{A} x - \theta \Sigma \Lambda' \Phi + \theta \Sigma \varsigma \right).
\] (21)

The solution of the PDE is still of the form

\[
\Phi(t, x) = x'Q(t)x + x'q(t) + k(t)
\]

where \( Q(t) \) solves a \( n \)-dimensional matrix Riccati equation and \( q(t) \) solves a \( n \)-dimensional linear ordinary differential equation.

6 What About the Kelly Criterion?

6.1 A Mutual Fund Theorem

In the benchmarked case, we will follow the same logic as in the asset only and rewrite the optimal asset allocation as an allocation between two funds. We will then use this result to define benchmarked optimal fractional Kelly strategies, that is fractional Kelly strategies which are also optimal investment policies in the benchmarked asset management model. All we will need to do after that is to verify that this new definition is consistent with the definition we gave earlier in the asset only case, and thus with the ‘classical’ definition of fractional Kelly strategies in the limit as our model converges to the Merton model.

The following benchmarked mutual fund theorem is due to Davis and Lleo (2008a):

**Theorem 3** (Benchmarked Mutual Fund Theorem). Given a time \( t \) and a state vector \( X(t) \), any portfolio can be expressed as a linear combination of investments into two “mutual funds” with respective risky asset allocations

\[
h^K(t) = (\Sigma \Sigma')^{-1} \left( \hat{a} + \hat{A} X(t) \right)
\]

\[
h^C(t) = (\Sigma \Sigma')^{-1} \left[ \Sigma \varsigma - \Sigma \Lambda' \left( q(t) + Q(t) X(t) \right) \right]
\] (22)

and respective allocation to the money market account given by

\[
h^K_0(t) = 1 - \left( \Sigma \Sigma' \right)^{-1} \left( \hat{a} + \hat{A} X(t) \right)
\]

\[
h^C_0(t) = 1 - \left( \Sigma \Sigma' \right)^{-1} \left[ \Sigma \varsigma - \Sigma \Lambda' \left( q(t) + Q(t) X(t) \right) \right]
\]

Moreover, if an investor has a risk sensitivity \( \theta \), then the respective weights of each mutual fund in the investor’s portfolio equal \( \frac{1}{\theta + 1} \) and \( \frac{\theta}{\theta + 1} \), respectively.
There are two main differences between Theorems 2 and 3: the definition of portfolio $C$ and the role played by the risk sensitivity $\theta$.

In the asset only case of Theorem 2, portfolio $C$ is comprised of the money market asset and of a strategy trading the comovement of assets and valuation factors. In the benchmark case of Theorem 3, portfolio $C$ still includes an allocation to the money market asset and the asset-factor comovement strategy, but it also contains an allocation to a strategy designed to replicate the risk profile of the benchmark. Indeed, the term $\hat{u} := (\Sigma^\prime \Sigma^\prime)^{-1} \Sigma \varsigma$ represents an unbiased estimator of a linear relationship between asset risks and benchmark risk $\varsigma = \Sigma^\prime u$. In financial economics, the interpretation of $\hat{u}$ is as the vector of asset systematic exposure, or “betas”, computed with respect to the benchmark.

Hence, when $\theta$ is low, the investor will take more active risk by investing larger amounts into the log-utility or Kelly portfolio. On the other hand, when $\theta$ is high, the investor will divert most of his/her wealth to the correction fund, which is dominated by the term $(\Sigma^\prime \Sigma^\prime)^{-1} \Sigma \varsigma$ and designed to track the index. In short, while an investor with $\theta = 0$ is a Kelly criterion investor, an investor with extremely high $\theta$ is a passive investor.

This first observation leads us to reconsider the role and definition of the risk sensitive parameter $\theta$. Indeed, while in the asset only case $\theta$ represents the sensitivity of an investor to total risk, in the benchmark case, $\theta$ corresponds to the investor’s sensitivity to active risk. To some extent, in the benchmark case the investor already takes the benchmark risk as granted. The main unknown is therefore how much additional risk the investor is willing to take in order to outperform the benchmark. This amount of risk is directly quantified by the risk sensitive parameter $\theta$.

### 6.2 Fractional Kelly Strategies Revisited

In the benchmark case, as in the asset only case, we can expand the classical definition of fractional Kelly strategies by defining them as a split between the Kelly portfolio $K$ and the benchmark-tracking portfolio $C$, as per the Mutual Fund Theorem 3 and which is includes a benchmark replicating strategy.

We can quickly check that the extended definition in the benchmarked case is consistent with the definition given earlier in the asset only case by setting the benchmark to 0, which de facto implies the absence of a benchmark. We see then that the definitions of fund $C$ given in Theorems 3 and 2 are identical: the two theorems now coincide. As expected, the asset only
problem can be viewed as a special case of the benchmarked allocation.

Thus, if we set \( n = 0 \) so that no valuation factor is considered and set the benchmark to 0, the risk-sensitive benchmarked asset management model is the Merton model and the definition of optimal benchmarked fractional Kelly strategies reverts to the classical definition of fractional Kelly strategies.

7 Case Studies

We will now apply the model to study some specific benchmark structures and assess the appropriateness of benchmarks for Kelly criterion investors.

7.1 Benchmark as a Portfolio of Traded Assets

First, we revisit two examples considered by Davis and Lleo (2008a) in which the benchmark is a portfolio of traded assets respectively with or without the money market asset.

7.1.1 Benchmark as a Portfolio of Traded Assets Only

We assume that the benchmark follows a constant proportion strategy invested in a combination of traded assets. Its dynamics is given by the equation

\[
\frac{dL_t}{L_t} = \nu'(a + AX(t))dt + \nu'\Sigma dW_t
\]

where \( \nu \) is a \( m \)-element allocation vector satisfying the budget equation

\[
1'\nu = 1.
\]

In this setting, the benchmarked Mutual Fund Theorem 3 can be expressed as:

**Corollary 4.** *(Fund Separation Theorem with a Constant Proportion Benchmark (I)).* Given a time \( t \) and a state vector \( X(t) \), any portfolio can be expressed as a linear combination of investments into a “mutual fund”, an
index fund and a “long-short hedge fund” with respective risky asset allocations

\[ h^K(t) = (\Sigma\Sigma')^{-1}(\hat{a} + \hat{AX}(t)) \]
\[ h^I(t) = \nu \]
\[ h^H(t) = -(\Sigma\Sigma')^{-1}\Sigma\Lambda'(q(t) + Q(t)X(t)) \] (23)

and respective allocation to the money market account given by

\[ h^K_0(t) = 1 - 1'(\Sigma\Sigma')^{-1}(\hat{a} + \hat{AX}(t)) \]
\[ h^I_0(t) = 0 \]
\[ h^H_0(t) = 1'(\Sigma\Sigma')^{-1}\Sigma\Lambda'(q(t) + Q(t)X(t)) \] (24)

Moreover, if an investor has a risk sensitivity \( \theta \), then the respective weights of each mutual fund in the investor’s portfolio equal \( \frac{1}{\theta+1}, \frac{\theta}{\theta+1} \) and \( \frac{\theta}{\theta+1} \), respectively.

From the perspective of optimal fractional Kelly strategies, the implication of this corollary is that the second component of the strategy, the correction fund \( C \), can now be explicitly split into two sub portfolios:

1. a portfolio \( I \) with asset allocation given by \( \nu \) which is designed to strictly replicate the risk exposure of the index. Observe that the linear estimator \( \hat{u} := (\Sigma\Sigma')^{-1}\Sigma\upsilon \) has vanished and been replaced by the actual asset allocation \( \nu \): since the benchmark is now comprised of traded assets it can be replicated by a direct investment in the appropriate asset allocation and does not require any further estimation:

2. a “long-short hedge fund” \( H \) with risky asset allocation given by

\[-(\Sigma\Sigma')^{-1}\Sigma\Lambda'(q(t) + Q(t)X(t)) \]

and whose sole purpose is to trade the comovement of assets and factors.

The “long-short hedge fund” \( H \) is particularly interesting because it has zero net weight in the sense that the sum of all the long positions in the portfolio is exactly matched by the sum of short positions. As a result, fund \( H \) satisfies the budget equation

\[ 1'h^H + h^H_0 = 0. \]

This “long-short hedge fund” can therefore be viewed as a macro-oriented overlay strategy within the asset allocation.
7.1.2 Benchmark as a Portfolio of Traded Assets and the Money Market Asset

Developing the ideas from the previous paragraph, we consider a benchmark whose dynamics is given by

\[
\frac{dL_t}{L_t} = \left( a_0 + A'_0 X(t) \right) + \nu'(t) (\hat{a} + \hat{A} X(t)) dt + \nu'(t) \Sigma dW_t \\
= \left[ (1 - \nu'1) a_0 + \nu' a + ((1 - \nu'1) A'_0 + \nu' A) X(t) \right] dt \\
+ \nu'(t) \Sigma dW_t
\]  

where \( \nu \) is an \( m \)-element allocation vector satisfying the budget equation

\[ 1' \nu = 1 - h^K_0 \]

and \( h^K_0 \) is the allocation left in the money market account. The process \( L(t) \) represents the (log) return of a constant proportion portfolio with risky allocation vector \( \nu \) and the remainder (i.e. \( 1 - 1' \nu \)) invested in the money market account.

**Corollary 5. (Fund Separation Theorem with a Constant Proportion Benchmark (II)).** Given a time \( t \) and a state vector \( X(t) \), any portfolio can be expressed as a linear combination of investments into a “mutual fund”, an index fund and a “long-short hedge fund” with respective risky asset allocations

\[
h^K(t) = (\Sigma \Sigma')^{-1} \left( \hat{a} + \hat{A} X(t) \right) \\
h^I(t) = \nu \\
h^H(t) = -(\Sigma \Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t) X(t))
\]  

and respective allocation to the money market account given by

\[
h^K_0(t) = 1 - 1'(\Sigma \Sigma')^{-1} \left( \hat{a} + \hat{A} X(t) \right) \\
h^I_0(t) = 1 - 1' \nu \\
h^H_0(t) = 0
\]

Moreover, if an investor has a risk sensitivity \( \theta \), then the respective weights of each fund in the investor’s portfolio equal \( \frac{1}{\theta + 1} \), \( \frac{\theta}{\theta + 1} \) and \( \frac{\theta}{\theta + 1} \), respectively.

We can interpret this Corollary in terms of optimal fractional Kelly strategies in a similar way as we did for Corollary 4. The second component of the strategy, the correction fund \( C \), can again be split into:

1. a portfolio \( I \) with risky asset allocation given by \( \nu \) which is designed to strictly replicate the risk exposure of the index;
2. A “long-short hedge fund” $H$ with risky asset allocation given by $-(\Sigma \Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t)X(t))$ and whose sole purpose is to trade the comovement of assets and factors.

Here again, we could show that $H$ is a zero net weight strategy, i.e. that

$$I' h^H = 0.$$ 

### 7.2 Benchmark With Alpha Target

We now consider two new examples involving a benchmark based on an interest rate plus some measure of risk-adjusted excess return known as alpha.

#### 7.2.1 Money Market Rate Plus Alpha Benchmark

Money market rate plus some predetermined alpha has been adopted as a benchmark by a large number of hedge funds. What are the implications of this choice in terms of the optimal fractional Kelly strategies?

The dynamics of such a money market rate plus alpha benchmark can be expressed as

$$\frac{dL_t}{L_t} = (a_0 + A_0 X(t))dt + \alpha dt$$

where $\alpha$ represents the instantaneous level of risk-adjusted excess return, or alpha, required of the fund manager.

The optimal asset allocation in this case is

$$h^* = \frac{1}{\theta + 1} \left( (\tilde{a} + (\tilde{A} - \theta \Sigma \Lambda' Q)x - \theta \Sigma \Lambda' q \right)$$

which is simply the asset only asset allocation as given in (14). As a result, the conclusion of the Mutual Fund Theorem 2 hold and optimal fractional Kelly strategies can be defined accordingly.

So, what has happened to the benchmark? In our risk-sensitive framework, the benchmark is tracked through its risk profile rather than its return profile. Since a money market plus alpha benchmarks does not have any direct risk, there cannot be any comovement between the assets and the benchmark. Thus, the benchmark risk profile cannot be replicated, and as a result, the asset allocation is established without any regard to the benchmark.

The money market plus alpha benchmark will have an impact on the value function $\Phi$. Indeed, in the benchmarked case, the auxiliary criterion function $I(f_0, x; h; t, T)$ reflects excess return over the benchmark, rather
than total return as in the asset only case. The value function for the benchmarked problem is therefore consistent with excess return rather than total return, resulting in different value functions for the benchmark problem and for the asset only case.

The main conclusion from this case study is that from the perspective of a rational risk-sensitive investor with no investment constraints, money market plus alpha benchmarks results in the same investment strategy as not having any benchmark at all. To some extent, in a risk-sensitive setting, money market plus alpha is a benchmark-free benchmark. In particular, and somehow counter-intuitively, the choice a higher value for alpha does not result in a the design of a riskier asset allocation. In fact, the value of alpha has no impact at all!

7.2.2 Bill Rate or Bond Yield Plus Alpha Benchmark

The situation would however be slightly different if the benchmark was for example 3 month Treasury bill rate plus alpha or a 5 year Treasury note yield plus alpha. To generalize slightly, the benchmark dynamics could be expressed as an extension of the model considered above in the “Benchmark as a Portfolio of Traded Assets Only” case (in Section 7.1.1) \(^1\):

\[
\frac{dL_t}{L_t} = \nu'(a + AX(t))dt + \alpha dt + \nu'\Sigma dW_t
\]

where \(\nu\) is a \(m\)-element allocation vector satisfying the budget equation

\[1'\nu = 1.\]

As expected, the level of alpha does not influence the optimal asset allocation and the conclusions stemming from Corollary 4 are still valid in this case.

7.2.3 Solving the Alpha Puzzle

Why is the alpha not producing any impact on the asset allocation? The reason is that the risk-sensitive benchmarked asset management model does

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\(^1\)This slight generalization is required in the event the benchmark is based on a bond. Indeed, in the risk-sensitive model, zero coupon bonds are the representatives of the fixed income asset class in the investment universe. Any coupon bond must therefore be “recreated” as a linear combination of zero coupon bonds. In the event the benchmark is a Treasury bill, then we have the degenerate case in which \(\nu\) is a vector with the element corresponding to the Treasury bill set to 1 and all other elements equal to 0.
not penalize for a non-achievement of the benchmark return. As we have seen above, risk sensitivity implies that the risk of the asset portfolio relative to the benchmark matters, but not the expected return. Since an alpha target is in essence a “pure return” target, it does not change the behaviour of the risk sensitive investor.

Should the investor be penalized for a non-achievement of the benchmark return? Not in our opinion. For an investor who decides of their own asset allocation, the benchmark is a measure of acceptable risk rather than an minimum return objective. Indeed, the portfolio optimization process will always select, for a given risk sensitivity, the asset allocation which maximizes the relative return of the asset portfolio with respect to its benchmark. The issue of an alpha target is irrelevant in this problem.

The situation might however be different if the investor hires a manager to perform the asset allocation. The alpha imperative may in this case be understood as a control imposed by the investor to force the asset manager to select an optimal (or near optimal) asset allocation. The investor should still not be penalized for a non achievement of the benchmark, but it is conceivable that the investor may want to penalize the manager for non achievement of the benchmark return. Here, the penalization would occur at the level of the fee earned by the manager and not at the level of the investor’s terminal wealth. This would result in a completely different control problem in which we would need to take the perspective of the manager whose objective is to allocate the investor’s assets in order to maximize the management fee received and subject to a non-achievement penalty.

### 7.3 Alpha-Omega Targets

An alternative to the pure alpha target is an alpha-omega target, where omega represents the variability of alpha (see Grinold and Kahn (1999) for a view of the role of alpha and omega in active management). This approach implicitly recognises that alpha generation is not only uncertain but also risky. The introduction of the omega terms changes only slightly the asset allocation problem. The dynamics of a benchmark with alpha-omega targets can be modelled as

\[
\frac{dL(t)}{L(t)} = [(c + C'X(t))dt + \varsigma' dW(t)] + [\alpha dt + \omega' dW(t)]
\]

\[
= (c + \alpha + C'X(t))dt + (\varsigma + \omega')dW(t), \quad L(0) = l \quad (29)
\]

where the scalar constant \(c\), the \(n\)-element column vector \(C\), and \(\varsigma\) is a \(N\)-element column vector are related to the benchmark itself and the scalar \(\alpha\) and vector \(\omega\) refer to the excess return demanded by the investor. The last
element of \( \varsigma \) is set to 0 while the first \( n+m \) element of the vector \( \omega \) are set to 0. This condition ensures that the variability of alpha is uncorrelated with the factors and assets, as active asset management theory suggests. We will still make Assumption 1.

### 7.3.1 The Optimal Investment Policy

Extending the reasoning in Davis and Lleo (2008a), we would find that the optimal investment policy \( h^* \) is

\[
h^* = \frac{1}{\theta + 1} (\Sigma \Sigma')^{-1} \left( \dot{\alpha} + \dot{\Lambda} x - \theta \Sigma \Lambda' D \Phi + \theta \Lambda' (\varsigma + \omega) \right).
\]  

(30)

The solution of the PDE is still of the form

\[
\Phi(t, x) = x' Q(t) x + x' q(t) + k(t)
\]

where \( Q(t) \) solves a \( n \)-dimensional matrix Riccati equation and \( q(t) \) solves a \( n \)-dimensional linear ordinary differential equation. Specifically,

\[
\dot{Q}(t) - Q(t) K_0 Q(t) + K'_1 Q(t) + Q(t) K_1 + \frac{1}{\theta + 1} \dot{\Lambda}' (\Sigma \Sigma')^{-1} \dot{\Lambda} = 0
\]  

(31)

for \( t \in [0, T] \), with terminal condition \( Q(T) = 0 \) and with

\[
K_0 = \theta \left[ \Lambda \left( I - \frac{\theta}{\theta + 1} \Sigma' (\Sigma \Sigma')^{-1} \Sigma \right) \Lambda' \right]
\]  

(32)

\[
K_1 = B - \frac{\theta}{\theta + 1} \Lambda \Sigma' (\Sigma \Sigma')^{-1} \dot{\Lambda}
\]

\[
\dot{q}(t) + (K'_1 - Q(t) K_0) q(t) + Q(t) b + \theta \dot{Q}'(t) \Lambda (\varsigma + \omega) + A_0 - C + \frac{1}{\theta + 1} \left( 2 \dot{\Lambda}' - \theta \dot{Q}'(t) \Lambda \Sigma' \right) (\Sigma \Sigma')^{-1} (\dot{\alpha} + \theta \Sigma (\varsigma + \omega)) = 0
\]  

(33)

with terminal condition \( q(T) = 0 \).

\[
k(s) = f_0 + \int_s^T l(t) dt
\]  

(34)
for $0 \leq s \leq T$ and where

$$l(t) = \frac{1}{2} \text{tr} \left( \Lambda \Lambda' Q(t) \right) - \frac{\theta}{2} q'(t) \Lambda \Lambda' q(t) + b q(t)$$

$$+ \frac{1}{\theta + 1} \hat{a}'(\Sigma \Sigma')^{-1} \hat{a} + \frac{1}{\theta + 1} \theta^2 q'(t) \Lambda \Sigma' (\Sigma \Sigma')^{-1} \Sigma \Lambda' q(t)$$

$$- \frac{\theta}{\theta + 1} q'(t) \Lambda \Sigma' (\Sigma \Sigma')^{-1} \hat{a} - \frac{2 \theta^2}{\theta + 1} q'(t) \Lambda \Sigma' (\Sigma \Sigma')^{-1} \Sigma \gamma$$

$$+ \theta (\varsigma + \omega)' \Lambda q(t) - \frac{1}{2} (\theta - 1) (\varsigma + \omega)' (\varsigma + \omega) + \frac{\theta}{\theta + 1} \hat{a}'(\Sigma \Sigma')^{-1} \Sigma (\varsigma + \omega)$$

$$+ \frac{1}{\theta + 1} \theta^2 \gamma' \Sigma (\Sigma \Sigma')^{-1} \Sigma (\varsigma + \omega) + a_0 - (c + a)$$

(35)

7.3.2 Mutual Fund Theorem

We can now restate the optimal asset allocation in terms of the following Mutual Fund Theorem:

**Theorem 6** (Alpha-Omega Benchmarked Mutual Fund Theorem). Given a time $t$ and a state vector $X(t)$, any portfolio can be expressed as a linear combination of investments into two “mutual funds” with respective risky asset allocations

$$h^K(t) = (\Sigma \Sigma')^{-1} \left( \hat{a} + \hat{A} X(t) \right)$$

$$h^C(t) = (\Sigma \Sigma')^{-1} \left[ \Sigma (\varsigma + \omega) - \Sigma \Lambda' (q(t) + Q(t) X(t)) \right]$$

(36)

and respective allocation to the money market account given by

$$h^K_0(t) = 1 - 1'(\Sigma \Sigma')^{-1} \left( \hat{a} + \hat{A} X(t) \right)$$

$$h^C_0(t) = 1 - 1'(\Sigma \Sigma')^{-1} \left[ \Sigma (\varsigma + \omega) - \Sigma \Lambda' (q(t) + Q(t) X(t)) \right]$$

Moreover, if an investor has a risk sensitivity $\theta$, then the respective weights of each mutual fund in the investor’s portfolio equal $\frac{1}{\theta + 1}$ and $\frac{\theta}{\theta + 1}$, respectively.

**Proof.** This Corollary can be proved in a similar fashion to Theorem 3 in Davis and Lleo (2008a).

7.3.3 Kelly Strategies

In terms of Kelly strategies, this Mutual Fund Theorem represents a split between a fraction $\frac{1}{\theta + 1}$ invested in the Kelly portfolio $K$, and a fraction $\frac{\theta}{\theta + 1}$
invested in the correction fund $C$. As in Theorem 3, portfolio $C$ includes an allocation to the money market asset and the asset-factor comovement strategy and an allocation to a strategy designed to replicate the risk profile of the benchmark. In addition, the allocation to portfolio $C$ features a new term related to the variability of alpha, defined as $\hat{u} := (\Sigma\Sigma')^{-1}\Sigma\omega$ and which represents an unbiased estimator of a linear relationship between asset risks and alpha risk $\omega = \Sigma'\omega$.

When $\theta$ is high, the investor will divert most of the wealth to the correction fund in order to replicate the risk profile of the benchmark and the appropriate level of active risk given by $\omega$. On the other hand, when $\theta$ is low, the investor will take more active risk by investing larger amounts into the Kelly portfolio and will not attempt to stay within the bounds of active risk imposed by $\omega$.

### 7.4 Benchmarks and Kelly Investors: No Benchmark for Buffett!

Our last application of the idea of benchmarked fractional Kelly strategies takes the form of an anecdote concerning the appropriateness of benchmarks for Kelly investors. A few years ago, a controversy shook the North American investment industry: what would be a proper investment benchmark to assess the performance of legendary investor Warren Buffett’s Berkshire Hathaway?

Although this question may appear, at first glance, trivial, it is in fact the reflection of a wider concern shared by asset managers and investors alike on what constitutes an appropriate investment benchmark. This question has received an increasing deal of attention in the past two decades and it is considered so important in the investment management industry that the Research Foundation of the CFA Institute (then called AIMR), a leading professional association in the investment management industry, has recently devoted a monograph to the question (see Siegel (2003) for more details).

Although we do not intend on entering the benchmark design and specification debate\textsuperscript{2}, the results we derived reveal a new dimension to the problem: the impact of the investment benchmark on a fund manager’s investment strategy depends to a significant extend on the risk-aversion of the manager or investor. Indeed, we have shown that in the risk-sensitive benchmarked

\textsuperscript{2}We have, after all, conveniently assumed throughout that an appropriate care had been taken beforehand to select the benchmark.
asset management model, the importance of the benchmark in the investment decision increases as the risk aversion of the investor increases. In fact, the risk-sensitivity $\theta$ is a direct indication of the amount of active risk that an investor is willing to take. Thus, the exercise consisting in setting an investment benchmark becomes increasingly irrelevant as risk-aversion gets nearer to 0: Kelly criterion investors invest in the log-utility portfolio and have no regards for a benchmark.

Before setting a benchmark, it is imperative to know the investor’s risk preferences as this will influence his/her investment style. A very rough correspondence would be to identify extremely high risk aversion investors to passive portfolio managers whose mandate is to track an index, low risk aversion investors to purely active managers, whose mandate is to generate the best possible risk-adjusted return in a specific market, and medium risk-aversion investors to so called “core plus” or “index plus” managers who track closely and index while trying to generate some incremental extra return or “alpha”. The only time these three broad categories of investors will coincide is when their benchmark is the Kelly portfolio.

Going back to our original question, Warren Buffet focuses on long-term growth maximization rather than the avoidance of short-term losses (see Ziemba (2005) for a discussion of Berkshire Hathaway’s risk-adjusted performance and Thorp (2006) for a Kelly criterion perspective). In this sense, Mr Buffet behaves similarly to a Kelly criterion bettor. The RSBAM model then demonstrates that benchmarks are irrelevant to Mr. Buffet’s investment strategy.

As a concluding note, since at least 1995, the Berkshire Hathaway annual report features a table presenting a proxy for the performance of the Berkshire Hathaway share against the return of the S&P500 going back to 1965. Warren Buffet, however, does not recognize the S&P500 as his benchmark. He made it clear that this comparative table had been included in the annual reports at the request of investors but that he personally sees little value in it.

8 Conclusion

Historically, Kelly and fractional Kelly strategies have played an important part in asset-only investment management. But the essential role of fractional Kelly strategies does not stop at this level: it also extends to benchmarks and even to asset and liability management (see Davis and Lleo (2008b)). In benchmarked asset allocation problems, fractional Kelly strate-
gies highlight the fundamental split between a purely active growth maximizing strategy (the Kelly criterion portfolio) and pure benchmark replication. Moreover, the Kelly fraction, which represent the fraction of one’s wealth invested in the Kelly portfolio, is a sole function of the investor’s degree of risk-sensitivity.

This has profound implications. A Kelly investor, with 0 risk-sensitivity, will invest solely in the purely active portfolio and as a result, benchmarks are irrelevant to Kelly investors. On the other end of the spectrum, investors with extremely large risk-sensitivity will tend to opt for full replication of the benchmark: they are passive investors. Somewhere in the middle, we find investors with an average risk-sensitivity, who adopt core-plus strategies: a basic replication of the benchmark with some departure in order to generate excess returns. The only time these three broad categories of investors will coincide is when their benchmark is the Kelly portfolio.

References


