Current Topics in Credit Risk

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Agenda

• The market.
• The data.
• Models and techniques.
• Portfolio credit risk
• Concluding remarks.
1 The Credit Markets

- Corporate and Sovereign bonds
  
  NB: the *credit spread* $s$ for a bond is defined by

  $$\text{Price} = p = \sum_{i=1}^{n} c_i p(0, t_i) e^{-st_i} + p(0, t_n) e^{-st_n}.$$  

- Loans, mortgages, trade receivables ..

- Convertible bonds = corporate bond with right to exchange for equity.

- Asset swaps = exchange of bond coupons for LIBOR + $x$.

- Credit default swaps (CDS) = insurance against default of issuer.

- ‘Structured finance’: CBO (collateralized bond obligation), CLO, CMO, ..

- Standardised portfolio indices (iTraxx).

- Equity Default swaps = far out-of-the-money equity put option.
Protection buyer pays regular premiums $\pi$ until $\min(\tau, T)$ where $T$ is the contract expiry time and $\tau$ the default time of the Reference Bond.

Protection seller pays $(1 - R)1_{(\tau<T)}$ at next coupon date after $\tau$, where $R$ is the recovery rate. If $F$ is the risk-neutral survivor function of $\tau$, the ‘fair premium’ $\pi$ is determined by

$$
\sum_{i=1}^{n} \pi p(0, t_i) F(t_i) = \sum_{i=1}^{n} (F(t_{i-1}) - F(t_i))(1 - R)p(0, t_i).
$$
If we have CDS rates $\pi_k$ for maturities $T_k, k = 1, \ldots, m$ and a family of distributions $\{F_\theta, \theta \in \mathbb{R}^m\}$ then we can determine the ‘implied default distribution’ $F_\hat{\theta}$. Example: $m = 1$ and $F_\theta(t) = e^{-\theta t}$.

Moral: CDS rates determine the risk-neutral marginal default time distribution for the reference issuer.
3 Collateralized Debt Obligations (CDO)

Cash Flow CBO

Investors subscribe $100 to SPV which purchases bond portfolio. SPV issues rated notes to investors. Coupons paid in seniority order.
Here SPV sells credit protection to counterparty as individual-name CDS, buys credit protection on tranches from investors with premiums $x < y < z$.

The *joint* default distribution is the key thing here.

New market product: **iTraxx index** – tranche quotes publicly available on a standardised debt portfolio. Significance: market data directly related to ‘correlation’.
4 Data

- Credit ratings and associated historical data: average credit spreads for each rating, rating transition matrices ($P$-measure, not $Q$-measure!)

- Market bond prices = credit spreads.

- Single-name CDS rates.

- iTraxx quotes.

- Equity prices.
5 Portfolio Credit Risk

Static models

- Moody’s Binomial Expansion Technique: classifies issuers by industry sector, produces a ‘diversity score’. If portfolio size is 60 and diversity score 45, portfolio is deemed equivalent to a portfolio of 45 independent issuers, with notional bond size increased by factor 60/45. Credit rating determined by expected loss criterion.

- Credit metrics: Based on Markov chain model of rating transitions. Fixed credit spread in each rating category. Change of rating simulated by quantiles of $N(0, 1)$ distribution. Correlation determined by assigning a weight vector for industry classification + empirical correlation between industry equity indices.
Copula-based models
Recall the basic relation

\[ F(a, b) = C(F_1(a), F_2(b)). \]

Convenient, because we have information about marginal distributions from CDS market.

Industry standard for basket credit derivatives (small portfolios!) is the \( Li \) model. Individual default times \( \tau_1, \ldots, \tau_n \) have exponential distribution

\[ \mathbb{P}[\tau_i > t] = e^{-\lambda_i t}. \]

Let \( X \) be a normal \( n \)-vector with mean 0 and covariance \( Q = [\rho_{i,j}] \) where \( \rho_{i,i} = 1 \). \( \rho_{i,j} \) is the equity correlation for issuers \( i, j \). We then take

\[ \tau_i = \frac{1}{\lambda_i} \log(N(X_i)) \]

where \( N \) is the \( N(0, 1) \) d.f. The \( \tau_i \) are exponential with gaussian copula.
6 Large Portfolio Models

The requirements in modelling the credit risk of a portfolio are

- Credible modelling of the interaction effects.
- Efficient computational methods
- Ease of calibration

For large portfolio models we can’t estimate the covariance matrix $Q$ in the Li model and it makes sense to go to ‘factor’ or ‘latent variable’ models.
Large portfolio models

- Vasicek – homogeneous large portfolio model (see below)
- Hull and White – extension to non-homogeneous portfolios.
- CreditRisk+ – saddle point approximations (Wilde, Gordy)
- Giesecke and Weber – gaussian approximations in a voter model.
- Frey and Backhaus – similar model to the one presented here using an arbitrary continuous space state latent process and ‘mean field’ interaction. Strong convergence results are derived.
7 Vasicek Large Portfolio Model

Obligor $i$ defaults if $X_i < K_i$ where $X_i \sim N(0, 1)$ so $K_i = N^{-1}(p_i)$ where $p_i$ is the marginal default probability. Represent $X_i$ as

$$X_i = \rho X + \sqrt{1 - \rho^2} \epsilon_i,$$

where $X, \epsilon_1, \epsilon_2, \ldots$ are independent $N(0, 1)$. In homogeneous case $p_i = p_1$ for all $i$ and

$$P[\text{Obligor } i \text{ defaults}|X] = N \left( \frac{K_1 - \rho X}{\sqrt{1 - \rho^2}} \right) \equiv p(X).$$

Conditional distribution of proportion $\pi$ of obligors defaulting is then binomial with mean $p(X)$ and standard deviation $\sqrt{p(X)(1 - p(X))/n}$ where $n$ is the portfolio size.
For large $n$ the standard deviation is small and we have approximately

$$(\pi > \alpha) \Leftrightarrow p(X) > \alpha$$

giving the *unconditional* distribution

$$P[\pi > \alpha] \sim N \left( \frac{K_1 - \sqrt{1 - \rho^2 N^{-1}(\alpha)}}{\rho} \right).$$

If we apply this model to the iTraxx portfolio, we get a ‘correlation smile’: the model has only one correlation parameter $\rho$, and different values $\rho_1, \rho_2 \ldots$ are needed to fit different tranches of iTraxx. The situation is analogous to ‘volatility smiles’ in Black-Scholes.
8 Dynamic Large Portfolio models.

- Empirical evidence (Crowder, Giampieri & Davis 2003) suggests that the pattern of realized defaults is well represented by a latent variable model where the latent process $X_t$ is a 2-state (good times/bad times) economic variable.

- Obligors move around rating categories at a faster time scale than the economic cycle.

- These facts suggest a model in which obligors move around the rating categories at rates depending on the latent process and occasionally default.

- There is an obvious analogy with queueing networks in which ‘jobs’ move around ‘service stations’ for processing.

- Recent work by Choudhury, Mandelbaum et al. studies fluid and diffusion limits for queueing networks under random environments.
9 Rating transitions, no latent variable

For a portfolio with \( n \) obligors and \( K \) possible ratings \( k = 1, \ldots, K \) we define a vector process \( Q^n(t) \) taking values in \( \mathbb{N}^k \), with each component \( Q^n_k(t) \) containing the number of elements in each rating category at time \( t \). Then, \( \sum Q^n_k(t) \) is the number of non defaulted obligors at time \( t \). When \( n = 5, K = 2 \), the state space of \( Q^n(t) \) is as shown above. The figure shows all the possible movements given the current credit ratings: transitions (move along the diagonal) and defaults (move to the next diagonal).
Leaky bucket analysis

Here $\pi$ is the proportion of obligors in rating category A. Assuming $\alpha, \beta \ll \mu, \lambda$ we have the mass balance equation

$$\mu(1 - \pi) = \lambda\pi,$$

giving $\pi = \mu/(\mu + \lambda)$ and a default rate

$$d = \alpha\pi + \beta(1 - \pi) = \frac{\alpha\mu + \beta\lambda}{\mu + \lambda}.$$
Since the obligors are independent in this model, the standard deviation of cumulative defaults in $[0, t]$ is just

$$\sqrt{\frac{d_t}{n}}.$$  

This simple analysis is surprisingly successful in predicting the mean and variance of the exact default distribution obtained by solving the forward equation for the finite-state Markov process, if an adjustment for mean defaults is made.

However, the leaky bucket analysis doesn’t depend on the initial distribution of the obligors’ ratings and therefore doesn’t capture the short-term default behaviour.
11 Fluid and Diffusion Limits

The finite state environment process defines different ‘layers’ in which the transition parameters are different.
Conditional in the realisation of the random environment, the process $Q^n(t)$ may be approximated by two processes

$$Q^n(t) \simeq nQ^{(0)}(t) + \sqrt{n}Q^{(1)}(t) \quad (1)$$

where $Q^{(0)}(t)$ is a deterministic process called the \textit{fluid limit} and $Q^{(1)}(t)$ is a diffusion called the \textit{diffusion limit} of the sequence $Q^n(t)$. This is, conditional on the random environment, the distribution of the process $Q(t)$ may be approximated by a normal distribution.
The random environment process $X(t)$ is a finite state process in continuous time, having at most a finite number of jumps in any bounded interval of $[0, \infty)$.

To construct the process $Q(t)$ we consider a collection of mutually independent Poisson processes $\{A_i\}_{i \in I = \{1, \ldots, n\}}$ and a collection of vectors $\{v_i\}$ in $\mathbb{R}^K$, $K \in \mathbb{N}$, and a collection of non-negative functions of the form $\alpha_i(\cdot, \cdot, x) : [0, \infty) \times \mathbb{R}^K \to [0, \infty)$ for all $i \in I$ and $x \in \mathcal{X}$. We assume each $\alpha_i(t, \cdot, x)$ is Lipschitz bounded with respect to the second argument, this is, exist a locally integrable function $\beta_t : [0, \infty) \to [0, \infty)$ such that

$$\|\alpha_i(t, \cdot, x)\| \leq \beta_t$$

where $\|\cdot\|$ is the Lipschitz norm defined as

$$\|f\| = \sup_{x,y \in D_1, x \neq y} \frac{|f(x) - f(y)|_{D_2}}{|x - y|_{D_1}} \lor |f(0)|_{D_2}$$
We define a mapping $Q$ from $\Omega$ into $D([0, \infty), \mathbb{R}^K)$ by $(\omega_1, \omega_2) \mapsto Q$, where $Q$ is the process solution to the equation

$$Q(t) = Q(0) + \sum_{i=1}^{n} A_i \left( \int_0^t \alpha_i(s, Q(s), X(s)) ds \right) v_i$$

(3)

The law of $Q(t)$ is $P_{Q}(\cdot) = P(Q^{-1}(\cdot))$. The probability conditional on the environment process $X(t)$ denoted by $P_{Q}^{\omega_1}$ is then defined by the conditional probability under the inverse mapping.

We are concerned with the convergence of sequences $Q^n : (\omega^1, \omega^2) \rightarrow \mathbb{R}^K$ of the form

$$\frac{Q^n(t)}{\eta} = \frac{Q^n(0)}{\eta} + \frac{1}{\eta} \sum_{i=1}^{n} A_i \left( \eta \int_0^t \alpha_i(s, Q^n(s), X(s)) ds \right) v_i$$

for $\eta > 0$. 
13 Strong Approximations

We define a sequence of network processes \( \{(X(t), Q^n(t)/\eta); \eta > 0\} \) associated to \((X(t), Q(t))\) as the set of network processes where \(Q^n(t)\) is the solution to the system

\[
Q^n(t) = Q^n(0) + \sum_{i=1}^{N} A_i \left( \int_{0}^{t} \alpha^i_n(s, Q^n(s), X(s)) \, ds \right) v_i
\]  

(4)

where \(\{\alpha^i_n(s, \cdot, x)\}\), with \(x \in \mathcal{X}\) and \(i \in I\), is a collection of functions satisfying

\[
\|\alpha^i_n(t, \cdot, x)\| \leq \eta \beta_t
\]  

(5)

with \(\beta_t\) a locally integrable function.

The interpretation of the pair \((X(t), \frac{1}{\eta}Q^n(t))\) for some \(\eta > 0\) is a process with the same characteristics of \(Q(t)\) under the same environment \(X(t)\) but where the number of servers and rates of arrivals have increased \(\eta\) times.
Theorem 1 If \( \{\alpha_i^\eta | i \in I, \eta > 0\} \) are Lipschitz bounded and
\[
\lim_{\eta \to \infty} \sum_{i=0}^{N} \int_0^t \left\| \frac{\alpha_i^\eta(s, \cdot, x)}{\eta} - \alpha_i^{(0)}(s, \cdot, x) \right\| ds = 0 \tag{6}
\]
for \( \omega_1 \in \Omega^1 \) where the process \( Q^\eta(t) \) is such that \( \lim_{\eta \to \infty} Q^\eta(0)/\eta = Q^{(0)}(0) \) then
\[
\frac{Q^\eta(t)}{\eta} \to Q^{(0)}(t) \text{ as } \eta \to \infty \tag{7}
\]
a.s. in \( P_{\omega_1}^Q \), where \( Q^{(0)}(t) \) defined in \( \Omega^2 \) is the solution to the equation
\[
Q^{(0)}(t) = Q^{(0)}(0) + \int_0^t \alpha^{(0)}(s, Q^{(0)}(s), X(s))ds \tag{8}
\]
where
\[
\alpha^{(0)}(t, \cdot, x) = \sum_{i=1}^{N} \alpha_i^{(0)}(t, \cdot, x)v_i \tag{9}
\]
and the integral equation is deterministic conditional on the random environment process \( X(t) \).

The process \( Q^{(0)}(t) \) is referred as the fluid limit of the sequence \( \{Q^\eta(t)\}_{\eta \geq 0} \).
14 Weak convergence

It is possible to derive a quenched functional version of the central limit theorem for $(X(t), (1/\eta)Q^n(t))$ conditional on $\mathcal{F}_{t^1}$.

Some weak convergence results in queuing theory assume a heavy traffic condition, this is, the arrival and total service rates are nearly the same. Here we have no arrivals and eventually $Q^n(t) = 0$, so no stationarity may be assumed.

**Theorem 2** Assume

\[
\sum_{i \in I} \int_0^t \lim_{\eta \to \infty} \sqrt{\eta} \left\| \frac{\alpha^n_i(s, \cdot, x)}{\eta} - \alpha_i^{(0)}(s, \cdot, x) \right\| ds < \infty
\]  

(10)

and

\[
\lim_{\eta \to \infty} \sum_{i \in I} \int_0^t \left\| \sqrt{\eta} \left[ \frac{\alpha^n_i(s, \cdot, x)}{\eta} - \alpha_i^{(0)}(s, \cdot, x) \right] - \alpha_i^{(1)}(s, \cdot, x) \right\| ds = 0
\]

(11)

and for all $x \in \mathcal{X}$ and $i \in I$. Assume the function $\alpha^{(0)}(s, \cdot, x)$ is continuously
differentiable for any values $X(t)$ and $Q(0)(t)$. For $P^1$ a.e. $\omega_1 \in \Omega^1$, if

$$\lim_{\eta \to \infty} \sqrt{\eta} \left[ \frac{Q^\eta(0)}{\eta} - Q(0)(0) \right] =^d Q^{(1)}(0)$$

(12)

w.r.t $P_{\omega_1}^Q$, then

$$\lim_{\eta \to \infty} \sqrt{\eta} \left[ \frac{Q^\eta(t)}{\eta} - Q(0)(t) \right] =^d Q^{(1)}(t)$$

(13)

w.r.t $P_{\omega_1}^Q$, where the process $Q^{(1)}(t)$ takes values in $\Omega^2$ and it is the solution to the stochastic integral equation

$$Q^{(1)}(t) = Q^{(1)}(0) + \int_0^t D\alpha_i^0(s, X_s, Q^0; Q^{(1)}(s)) + \alpha^{(1)}(s, Q^0(s), X_s)ds$$

$$+ \sum_{i \in I} \mathbf{B}_i \left( \int_0^t \alpha_i^0(s, Q^0(s), X_s)ds \right) \mathbf{v}_i$$

(14)

where \{\mathbf{B}_i(t)\} is a collection of independent standard Brownian motions in $(\Omega^2, \mathcal{F}^2, P_{\omega_1})$. 

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15 Application to Correlated Defaults

We propose a model for the default/rating process of a set of obligors. There is a finite-state random environment process \( \{X(t) | t \geq 0\} \) representing some macroeconomic (or sector associated) process that influences the default/transition rates of the obligors. The obligor credit events are independent conditional on the realisation of the environment process and follow a Markov chain with rates being function of the environment process.

Assume a portfolio with \( n \) obligors and \( K \) possible ratings \( 1, \ldots, K \). The initial rating composition of the portfolio is represented by the rating distribution vector \( Q^n(0) \in \mathbb{R}^k \). \( Q^n(t) \) will represent the random rating distribution of the portfolio at a later time \( t \).

We define the index set of transition events \( I = \{(i, j) | i, j = 1, \ldots, k\} \) and denote the transition rate from rating \( i \) to rate \( j \), \( i, j = 1, \ldots, k \), by \( \mu_{(i,j)}(x) = \mu_{ij}(x) \); default rates are denoted by \( \mu_{(i,i)}(x) \equiv \mu_i(x) \). Associated to these credit
events we define the set of vectors \( \{ \mathbf{v}_{(i,j)} \in \mathbb{R}^k | (i, j) \in I \} \) that define the changes in the rating distribution vector in case of a credit event. This is

\[
\mathbf{v}_{(i,j)} = \begin{cases} 
\mathbf{e}_j - \mathbf{e}_i & i \neq i \\
-\mathbf{e}_i & i = j
\end{cases}
\]

where \( \mathbf{e}_i \) is the \( i \)-th canonical vector in \( \mathbb{R}^k \).

Under this assumptions the credit events of the obligors are identically distributed and occur according to the first jump of a Poisson process with rates

\[
\hat{\mu}_i(x) = \sum_{j=1}^{k} \mu_{ij}(x)
\]

Once a credit event occurs at time \( t \), the obligor defaults with probability \( \mu_i(X_t)/\hat{\mu}_i(X_t) \) while a transition to rate \( j \neq i \) has probability \( \mu_{ij}(X_t)/\hat{\mu}_i(X_t) \).
The occurrence of a credit event for an obligor with credit rating \( i \) is given by the first jump of a Poisson process. The rate of occurrence depends on the environment random process but is independent of time. This is, for a set of positive real numbers \( \{ \mu_{ij}^x | x \in X, (i, j) \in I \} \) we define the transition default rates as

\[
\mu_{(i,j)}(t, X_t) = \sum_{x \in X} \mu_{ij}^x I\{X_t=x\}
\]

for \((i, j) \in I\).
16 The Fluid Limit

With the notation used above, we have the set of rate functions

\[ \alpha^n_{(i,j)}(t, y, x) = y_i \mu_{(i,j)}(t, x) \]

and since the rate function does not depend on \( n \) it is clear that

\[ \alpha^{(0)}_{(i,j)}(t, y, x) = y_i \mu_{(i,j)}(t, x) \]

satisfy the conditions of theorem 2. Using vector notation

\[ \alpha^{(0)}(t, y, x) = A_t(x)y \]

where \( A \) is the infinitesimal generator of the process.

We can verify that by assuming

\[ \lim_{n} \frac{1}{n} Q^n(0) = Q^{(0)}(0) \]
for all \( n \geq 0 \) all conditions of theorem 2 hold and the fluid limit process \( Q^{(0)}(t) \) is the solution to the deterministic PDE system

\[
\frac{d}{dt} Q^{(0)}(t) = A_t Q^{(0)}(t) \quad (15)
\]

and whenever \( A = A_t \) is time independent (no environment influence) the solution is given by

\[
Q^{(0)}(t) = e^{tA} Q^{(0)}(t)
\]

otherwise, since \( X_t \) has at most finite jumps in any bounded interval of time, we can define a countable set of jump times of \( X_t \) \( t_0 = 0, t_1 < ... \) and define \( Q^{(0)}(t) \) recursively

\[
Q^{(0)}(t) = e^{(t-t_i)A(X_{t_i})} Q^{(0)}(t) \quad (16)
\]

for \( t_i < t_{i+1} \).
The diffusion limit

Since \( \alpha_n^{n,i,j} = \alpha^{(0)}_{i,j} \) for all \( n \geq 0 \) and \( (i,j) \in I \) we can verify that by defining \( \alpha^{(1)}_{i,j} = 0 \) for all \( (i,j) \in I \) the conditions in theorem 3 hold. Therefore assuming

\[
\lim_{n \to \infty} \sqrt{n} \left[ \frac{Q_n^{(0)}}{n} - Q^{(0)}(0) \right] =^d Q^{(1)}(0) \tag{17}
\]

implies

\[
\lim_{n \to \infty} \sqrt{n} \left[ \frac{Q_n^{(1)}}{n} - Q^{(0)}(t) \right] =^d Q^{(1)}(t) \tag{18}
\]

where \( Q^{(1)}(t) \) satisfies a SDE which may be expressed as the following integral equation

\[
Q^{(1)}(t) = Q^{(1)}(0) + \int_0^t A_t Q^{(1)}(t) + \sum_{l=1}^k \int_0^t \left( Q^{(0)}_l(t) \right)^{1/2} B_l dW_t^{(l)}
\]
where $W^l$ is a $k$-dimensional vector of independent standard Brownian motions for $l = 1, ..., k$. The matrices $B_l$ have components

$$(B_l(t))_{ij} = \begin{cases} 
-\mu_{i,j}^{1/2} & \text{if } i = j \\
\mu_{i,j}^{1/2} & \text{if } l = i \neq j \\
0 & \text{otherwise}
\end{cases}$$

It is always possible rewrite (19) as

$$Q^{(1)}(t) = Q^{(1)}(0) + \int_0^t A_t Q^{(1)}(t) + \int_0^t B(s)d\hat{W}^{(l)}_t$$

where $\hat{W}_t$ is a $k$-dimensional vector of independent standard Brownian motions. In the case $k = 2$ the SDE is

\begin{align*}
    dQ^{(1)}_1(t) &= -Q^{(1)}_1(t)(\mu_1(t) + \mu_{12}(t))dt + Q^{(1)}_2(t)\mu_{12}(t)dt \\
    &\quad - (Q^{(0)}_1(t))^{1/2}(\mu_1^{1/2}(t)dW^{(1)}_{1,t} + \mu_{12}^{1/2}(t)dW^{(1)}_{2,t}) + (Q^{(0)}_2(t)\mu_{21}(t))^{1/2}dW^{(2)}_{1,t} \\
    dQ^{(1)}_2(t) &= -Q^{(1)}_2(t)(\mu_2(t) + \mu_{21}(t))dt + Q^{(1)}_1(t)\mu_{21}(t)dt \\
    &\quad - (Q^{(0)}_2(t))^{1/2}(\mu_2^{1/2}(t)dW^{(2)}_{2,t} + \mu_{21}^{1/2}(t)dW^{(2)}_{1,t}) + (Q^{(0)}_1(t)\mu_{12}(t))^{1/2}dW^{(1)}_{2,t}
\end{align*}
That is equivalent to the the following SDE system

\[ dQ^{(1)}(t) = A_t Q^{(1)}(t) dt + B(t) d\hat{W}_t \]

where

\[ B(t) = \begin{pmatrix} \sigma_1(t) & 0 \\ \rho(t)\sigma_2(t) & \sqrt{1 - \rho^2(t)\sigma_2(t)} \end{pmatrix} \]

\[ \sigma_1^2(t) = Q_1^{(0)}(t)(\mu_1(t) + \mu_{12}(t)) + Q_2^{(0)}(t)\mu_{21}(t) \]

\[ \sigma_2^2(t) = Q_2^{(0)}(t)(\mu_2(t) + \mu_{21}(t)) + Q_1^{(0)}(t)\mu_{12}(t) \]

\[ \rho(t)\sigma_1(t)\sigma_2(t) = -Q_1^{(0)}(t)\mu_{12}(t) - Q_2^{(0)}(t)\mu_{21}(t) \]

and

\[ \hat{W}_t = (\hat{W}_t^{(1)}, \hat{W}_t^{(2)})^t \]

is a bi dimensional standard Brownian motion with independent components.
Assuming $A_t = A$ time independent, the solution of the SDE is given by

$$Q^{(1)}(t) = e^{tA}Q^{(1)}(0) + \int_0^t e^{(t-s)A}B(s)d\hat{W}_s$$

with $Q^{(1)}(0) = 0$. The process is a stable Gaussian system with covariance matrix

$$M(t) = \text{Cov}[Q^{(1)}(t), Q^{(1)}(t)] = \int_0^t e^{(t-s)A}B(s)B(s)^{tr}(e^{(t-s)A})^{tr}ds$$

that can be calculated numerically by solving the Lyapunov equation

$$\frac{d}{dt}M(t) = AM(t) + M(t)A^{tr} + B(t)B^{tr}(t), \quad M(0) = 0.$$ 

In the general case of $X_t$ taking values in $\mathcal{X}$ the process is defined similarly to the case of the fluid limit, this is

$$Q^{(1)}(t) = e^{(t-t_i)A_{t_i}}Q^{(1)}(t_i) + \int_{t_i}^t e^{(t-s)A_{t_i}}B(s)d\hat{W}_t$$

for $t_i < t < t_{i+1}$ where $t_i$ is the time of the $i$-th jump of $X_t$. 

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18 Some numerical results

We assume a two state (two credit ratings) system, portfolio sizes 20, 50 and 100 and a two state external random environment where jumps occur according to a standard Poisson process (rate 1). By sampling 1000 times the random environment we construct both the diffusion approximation and the exact distribution. The latter is obtained by integrating the Kolmogorov forward equation associated to the process using Runge-Kutta. The parameters are shown in the table.

<table>
<thead>
<tr>
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<th>$X(t) = 0$</th>
<th>$X(t) = 1$</th>
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<tbody>
<tr>
<td>$\mu_1$</td>
<td>0.1</td>
<td>0.2</td>
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<tr>
<td>$\mu_2$</td>
<td>0.2</td>
<td>0.4</td>
</tr>
<tr>
<td>$\mu_{12}$</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>$\mu_{21}$</td>
<td>0.2</td>
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</tr>
</tbody>
</table>
The exact and approximated distributions for the case of 20 elements and initial distribution 50/50. We can observe that the fitting of the marginal distributions is outstanding despite the relatively small number of elements.
We consider the approximation to the number of survivors for different number of elements and initial distribution in Graphs 2.
Graphs 2. Approximation to the number of survivors
19 Computing the loss distribution

The rescaled process is a *piecewise-deterministic Markov process*: moves along integral curves of a vector field with jumps between ‘layers’:
Deterministic motion comprises fluid limit

\[ \frac{dQ^{(0)}(t)}{dt} = AQ^{(0)}(t) \]

plus Lyapunov equation

\[ \frac{d}{dt} M(t) = AM(t) + M(t)A^{tr} + B(t)B^{tr}(t). \]
To compute $P\{\text{Loss} \in [\alpha, \beta]\}$, we have to calculate the probability mass in a ‘slice’:

However, this is a 1-dimensional problem: just need the marginal distribution in the $(1, 1)$ direction.

Overall method: Monte Carlo simulation of factor process plus solution of Lyapunov equation etc for each sample path.
20 Concluding Remarks

- The key question in credit risk remains how to get an effective and scientifically justifiable handle on the ‘correlation’ question.

- For pricing applications ($Q$-measure), calibration to iTraxx tranche quotes provides a market-driven approach. However, this only covers a small part of the universe.

- For risk management applications ($P$-measure), empirical change-of-rating data supports the use of low-dimensional latent variables (see Giampieri, Davis & Crowder, QF 2005).

- Quite possibly, ‘firm value’ models à la Merton will provide a well-founded link between equity correlation and default correlation.