An equilibrium approach to indifference pricing

Mark H.A. Davis* and Daisuke Yoshikawa†

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Abstract

The utility indifference framework has received a lot of attention, because it is based on a utility maximization principle, which is one of the most fundamental principles of economics, for pricing a contingent claim. The price based on utility indifference framework is the maximum or minimum (in some cases, threshold) price for each investor. Therefore, the price is the indicator for the investor to join the market of the contingent claim. Our purpose is to expand the view of utility indifference framework, that is, to deduce the equilibrium price in the utility indifference framework. We attain the result that, under the setting of an exponential utility, the equilibrium price will be uniquely evaluated by minimal entropy martingale measure.

1 Introduction

As it is well known, in the theory of the incomplete market, many kinds of frameworks have been studied for pricing contingent claims uniquely and useful tools have been developed such as mean-variance hedging, local-risk minimization as the pricing standard1 and Minimal Martingale Measure2, variance-optimal Martingale3, p-optimal Martingale3 as the pricing measure. Among them, the utility indifference framework is one which has received a lot of attention. This framework has been developed by many authors5 because of the consistency with economics. As basic economics assumes that every investor acts in the market for maximizing their utilities6, the utility indifference framework adopts this assumption. That is, utility indifference framework assumes that an investor tries to maximize his expected utility, whether he holds the contingent claim or not. However, if the expected utility in the case of holding the contingent claim is larger than the expected utility in the case of not holding the contingent claim, it is rational that the investor holds the contingent claim. Consider the case that the investor wants to sell the contingent claim. In this case, the price under which his expected utility selling the contingent claim is larger than the expected utility not selling the contingent claim is clearly valuable for the investor. Utility indifference price is the price which equates the expected utility including the contingent claim with the expected utility not including the contingent claim. That is, in this case, the utility indifference price is the minimum price of the contingent claim for the investor as a seller. On the other hand, when the investor wants to buy the contingent claim, if the price of the contingent claim makes his expected utility larger than the expected utility not holding the contingent claim, this price is advantageous for the investor. Utility indifference price makes the expected utility holding the contingent claim equate with the expected utility not holding the contingent claim. That is, this is the maximum price of the contingent claim for the investor as a buyer. The name of utility

*Imperial College London
†Imperial College London
1Schweizer(1999)[25] is a good summary for this.
2Follmer, et al.(1991)[8]
3Grandits(1999)[13]
4Frittelli(2000)[10], Goll(2001)[11], Monoyios(2007)[19]
6The rigorous formulation of the utility maximization in incomplete market is given by Kramkov, et al.(1999)[6], Schachermayer(2000)[24]
“indifference” price is derived from this context. Furthermore, we can say that the utility indifference price is “fair” for each investor in this situation.

However, although utility indifference price is fair for each investor, the price is not common for every investor, because each investor has different preference, respectively. In fact, when a utility indifference price is fair for an investor who wants to sell the contingent claim, if the utility indifference price is considered expensive to other investors who want to buy the contingent claim, the price is meaningless, because the contingent claim will not be traded in the market. On the other hand, if an offered price of a contingent claim which is fair for an investor who wants to buy the contingent claim is too cheap for other investors who want to sell the contingent claim, the price is meaningless, although the price is fair for the buyer of the contingent claim.

Our purpose is simple; that is, to deduce the price available to stand in the market. We consider such a price exists in the utility indifference framework which we call an equilibrium price.

This paper is constructed as follows. Section 2 describes the review of the utility indifference framework and utility indifference price. In setting out the model, we adopt the exponential utility function\(^7\). Section 3 describes the equilibrium price in the utility indifference framework. In Section 4, we apply the utility indifference framework to a more concrete model, such that the market includes some tradable assets. This model is expanded from the model of Davis(2006)\(^4\).

In this model, we have to solve a multivariate HJB equation and use Cole-Hopf transformation and simple Eigen-decomposition to solve it. This section includes a numerical example and shows the situation how the price of the contingent claims is distributed in the market in the utility indifference framework.

2 Model and the review of Utility Indifference Price

We consider the mathematical framework in given probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\mathcal{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}\). Stochastic process \(X \in \mathbb{R}^d\) is defined as semimartingale, and the expected value of \(X\) is given by the probability measure \(P\). Consider the \(\mathcal{F}_T\)-measurable random variable (or ‘random endowment’) \(B\) which will generate some payoff at time \(T\) (we also write it as \(B_T\) for specifying the time \(T\)). The random variable \(B\) is assumed unbounded from below (Delbaen et al.(2002)\(^5\)). We assume that for some fixed \(\alpha, \epsilon \in (0, \infty)\),

\[
\mathbb{E}[e^{(\alpha+\epsilon)B}] < \infty \quad \text{and} \quad \mathbb{E}[e^{-\epsilon B}] < \infty.
\]

This assumption guarantees that \(B\) is Lebesgue integrable for all martingale measures for which the relative entropy is able to be defined (Becherer(2003)\(^1\)). Utility indifference price is defined as the value or price of the random endowment \(B\) which equates the maximized expected utility of the terminal wealth without the random endowment and the maximized expected utility of the terminal wealth with the random endowment. The maximized expected utility with the random endowment is described as follows,

\[
\sup_{\theta \in \Theta} \mathbb{E} \left[ U(x + p q + \int_0^T \theta_t^T dX_t - B_T q) \right],
\]

where let \(q > 0\) be the quantity of \(B\), \(x + p q\) be initial capital (\(x\) is the money market account, \(p\) is the price of the random endowment \(B\)), \(\theta := \{\theta_t; \ t \in [0, T]\} \in \Theta\) be the \(\mathbb{R}^d\)-valued admissible trading strategy\(^3\), \(\Theta\) be the set of \(X\)-integrable and predictable processes. We can interpret \(X\) as a discount price process of tradable assets. In this paper, we set the quantity \(q\) as positive to make demand and supply to be clearly divided. In fact, by this setting, the relation between demand curve and supply curve is distinctly depicted as seen in Figure 1.

The maximized expected utility without the random endowment is described as follows,

\[
\sup_{\theta \in \Theta} \mathbb{E} \left[ U(x + \int_0^T \theta_t^T dX_t) \right].
\]

\(^3\)HARA utility function or CRRA type utility are also available to use and it is also one of the representative way; e.g. Zariphopoulou(2001)\(^7\), Monoyios(2010)\(^7\).

\(^4\)Davis(2006)\(^4\) uses the model in which there are two assets; the one is tradable asset and the other is intradable one.

\(^5\)That is, \(\theta \in L(X)\) and for some constant \(c \in \mathbb{R}, \int_0^T \theta_t^T dX_t \geq c, \ t \in [0, T]\).
If the price $p$ is the utility indifference price, then it satisfies as follows,

$$
\sup_{\theta \in \Theta} \mathbb{E} \left[ U(x + \int_0^T \theta_t^T dX_t) \right] = \sup_{\theta \in \Theta} \mathbb{E} \left[ U(x + pq + \int_0^T \theta_t^T dX_t - B_T q) \right].
$$

(3)

We call this utility indifference price as utility indifference sell price, because it implies that the investor with the utility $U(\cdot)$ sells the random endowment by the price $p$. This definition implies that the utility indifference sell price is the minimum price of the random endowment $B$; that is, if the market price of the random endowment $B$ is less than utility indifference sell price, then the investor doesn’t sell the random endowment.

It is natural to define the utility indifference buy price $p$ which satisfies as follows,

$$
\sup_{\theta \in \Theta} \mathbb{E} \left[ U(x + \int_0^T \theta_t^T dX_t) \right] = \sup_{\theta \in \Theta} \mathbb{E} \left[ U(x - pq + \int_0^T \theta_t^T dX_t + B_T q) \right].
$$

(4)

Consider the problem (2) which is the left hand side of (3)(4). Using the function $V(\cdot)$ which is the dual function of $U(\cdot)$, this is rewritten as follows,

$$
\sup_{\theta \in \Theta} \mathbb{E} \left[ U(x + \int_0^T \theta_t^T dX_t) \right] = \inf_{\eta \in \mathbb{R}_+, Q \in \mathcal{M}} \left\{ \mathbb{E} \left[ V \left( \frac{dQ}{dP} \right) + \eta \frac{dQ}{dP} \left( x + \int_0^T (\theta_t^*)^T dX_t \right) \right] \right\}
$$

$$
= \inf_{\eta \in \mathbb{R}_+, Q \in \mathcal{M}} \left\{ \mathbb{E} \left[ V \left( \frac{dQ}{dP} \right) + \eta x \right] \right\}
$$

where $\theta^* = \{\theta_t^*; t \in [0,T]\} \in \Theta$ is the solution of the problem (2). $\mathcal{M}$ is a set of $\Theta$-martingale measures satisfying $H(Q|P) < \infty$, and $H(Q|P)$ is relative entropy of $Q$ with respect to $P$, which is defined as follows,

$$
H(Q|P) := \left\{ \int \frac{dQ}{dP} \ln \frac{dQ}{dP} dP \quad Q \ll P \right\}
$$

otherwise.

Note that relative entropy is always non negative (c.f. Theorem 1.4.1 of Ihara(1993)[17]).

Hereafter, we assume that

$$
\mathcal{M} \neq \emptyset,
$$

and we specify the utility function $U(\cdot)$ as

$$
U(x) = -e^{-\gamma x},
$$

where $\gamma \in \mathbb{R}_+$ is risk-aversion. We write the utility indifference sell price and the utility indifference buy price as $p^s(B_T; q)$ and $p^b(B_T; \gamma)$, respectively.

If the utility function is exponential, then $V(y) = \frac{\gamma}{\gamma} \ln \left( \frac{y}{\gamma} \right) - 1$. So, the above equation is calculated in continuation,

$$
\inf_{\eta \in \mathbb{R}_+, Q \in \mathcal{M}} \left\{ \mathbb{E} \left[ V \left( \frac{dQ}{dP} \right) + \eta x \right] \right\}
$$

$$
= \inf_{\eta \in \mathbb{R}_+, Q \in \mathcal{M}} \left\{ \frac{\gamma}{\gamma} \ln \left( \frac{\eta}{\gamma} \right) \mathbb{E} \left[ \frac{dQ}{dP} \right] + \frac{\eta}{\gamma} \mathbb{E} \left[ \frac{dQ}{dP} \ln \left( \frac{dQ}{dP} \right) \right] - \frac{\eta}{\gamma} \mathbb{E} \left[ \frac{dQ}{dP} \right] + \eta x \right\}
$$

$$
= \inf_{\eta \in \mathbb{R}_+} \left\{ \frac{\gamma}{\gamma} \ln \left( \frac{\eta}{\gamma} \right) - \frac{\eta}{\gamma} + \eta x + \frac{\eta}{\gamma} \inf_{Q \in \mathcal{M}} H(Q|P) \right\}
$$

$$
= \inf_{\eta \in \mathbb{R}_+} \left\{ \frac{\gamma}{\gamma} \ln \left( \frac{\eta}{\gamma} \right) - \frac{\eta}{\gamma} + \eta x + \frac{\eta}{\gamma} H(Q^0|P) \right\}.
$$

On line 3, we use $\eta \in \mathbb{R}_+$. We write the solution of $\inf_{Q \in \mathcal{M}} H(Q|P)$ as $Q^0 \in \mathcal{M}$ which we call the minimal entropy martingale measure (hereafter MEMM). The solution of $\inf_{\eta \in \mathbb{R}_+} \left\{ \frac{\gamma}{\gamma} \ln \left( \frac{\eta}{\gamma} \right) - \frac{\eta}{\gamma} + \eta x + \frac{\eta}{\gamma} H(Q^0|P) \right\}$ is given as follows,

$$
\eta^0 = \gamma e^{-\gamma(x + \frac{1}{\gamma} H(Q^0|P))}.
$$
Therefore, the maximized utility is described as follows,

$$\sup_{\theta \in \Theta} \mathbb{E} \left[ U(x + \int_0^T \theta_t^\top dX_t) \right] = \mathbb{E} \left[ V \left( \eta^0 \frac{dQ^0}{dP} \right) + \eta^0 \frac{dQ^0}{dP} \left( x + \int_0^T (\theta_t^\top)^\top dX_t \right) \right] = -e^{-\gamma(x + 1/\gamma [H(Q^0|P])} \right) \). (5)

Likewise, we can solve the utility maximization problem (1) which is the right hand side of (3).

$$\sup_{\theta \in \Theta} \mathbb{E} \left[ U(x + pq + \int_0^T \theta_t^\top dX_t - B_T q) \right] = \inf_{\eta \in \mathbb{R}^+, Q \in \mathcal{M}} \left\{ \mathbb{E} \left[ V \left( \eta^0 \frac{dQ^0}{dP} \right) + \eta^0 \frac{dQ^0}{dP} \left( x + pq + \int_0^T \hat{\theta}_t^\top dX_t - B_T q \right) \right] \right\} \right) (6)

\begin{align*}
\text{where } \hat{\theta} := \{ \hat{\theta}_t; t \in [0, T] \} \in \Theta \text{ is the solution of (1), } \hat{Q} \text{ is the solution of } \\
\inf_{\eta \in \mathbb{R}^+} \left\{ \frac{\eta^0}{\gamma} \ln \left( \frac{\eta^0}{\gamma} \right) - \frac{\eta^0}{\gamma^0} + \eta^0 (x + pq) + \frac{\eta^0}{\gamma^0} \inf_{Q \in \mathcal{M}} \left\{ H[Q|P] - \gamma q \text{ } E^Q [B_T] \right\} \right\} \right) \right),
\end{align*}

Note that the existence of \( \hat{Q} \) is shown by Proposition 2.2 of Becherer(2003)[1]. Then, the optimal \( \eta \) is given as follows,

$$\hat{\eta} = \gamma e^{-\gamma(x + 1/\gamma [H(Q^0|P)] + q [p - E^Q [B_T]])} \right) \).$$

Therefore, the maximized utility is given such that,

$$\sup_{\theta \in \Theta} \mathbb{E} \left[ U(x + pq + \int_0^T \theta_t^\top dX_t - q B_T) \right] = -e^{-\gamma(x + 1/\gamma [H(Q^0|P)] + q [p - E^Q [B_T]])} \right) \). (7)

From the definition of the utility indifference price and (5)(7), \( p^*(B_T; q) \) satisfies,

$$-e^{-\gamma(x + 1/\gamma [H(Q^0|P)])} = -e^{-\gamma(x + 1/\gamma [H(Q^0|P)] + p^*(B_T; \gamma) q - q E^Q [B_T])} \right) \).$$

Consequently, it holds that

$$p^*(B_T; q) = E^Q [B_T] - \frac{1}{\gamma} \left( H(Q^0|P) - H(Q^0|P) \right) \right) \right) \right) \right) \right) \right), (8)

This result is consistent with the main result of Delbaen et al(2002)[5].

**Remark 2.1**

From the definition of dual function, for any \( \theta \in \Theta \), it holds,

$$\mathbb{E} \left[ U(x + pq + \int_0^T \theta_t^\top dX_t - B_T q) \right] \leq \inf_{\eta \in \mathbb{R}^+, Q \in \mathcal{M}} \left\{ \mathbb{E} \left[ V \left( \eta^0 \frac{dQ^0}{dP} \right) + \eta^0 \frac{dQ^0}{dP} \left( x + pq + \int_0^T \theta_t^\top dX_t - B_T q \right) \right] \right\} \right).$$

In section 7 of Davis(2006)[4], it is pointed out that the equality of the above equation holds if and only if

$$x + pq + \int_0^T \theta_t^\top dX_t - B_T q = -\frac{1}{\gamma} \ln \left( \frac{\hat{\eta}^0}{\gamma} \right) \). (9)
Here, we show that the condition (9) is equivalent \( \theta = \hat{\theta} \), which means that the equality of (6) holds.

As seen above, \( \hat{\eta} \) is given by \( \hat{\eta} = \gamma e^{-\gamma(x+pq+\frac{1}{\gamma} \inf_{Q \in \mathcal{M}} \{H[Q|P]-\gamma qE^Q[B_r]\})} \). It implies that

\[
-\frac{1}{\gamma} \ln \left( \frac{\hat{\eta} d\hat{Q}}{dP} \right) = x + pq + \frac{1}{\gamma} H[\hat{Q}|P] - qE^\hat{Q}[B_r] - \frac{1}{\gamma} \ln \left( \frac{d\hat{Q}}{dP} \right)
\]

Therefore, (9) is calculated as follows,

\[
x + pq + \left[ \int_0^T \theta_i^T dX_t - B_r q \right] = x + pq + \frac{1}{\gamma} H[\hat{Q}|P] - qE^\hat{Q}[B_r] - \frac{1}{\gamma} \ln \left( \frac{d\hat{Q}}{dP} \right)
\]

\[
\leftrightarrow e^{-\gamma \left( \int_0^T \theta_i^T dX_t + q(p-B_r) \right)} = e^{-H[\hat{Q}|P]-\gamma q(p-E^\hat{Q}[B_r]) + \frac{1}{\gamma} \ln \left( \frac{d\hat{Q}}{dP} \right)}
\]

\[
\leftrightarrow E \left[ e^{-\gamma \left( \int_0^T \theta_i^T dX_t + q(p-B_r) \right)} \right] = E \left[ e^{-H[\hat{Q}|P]-\gamma q(p-E^\hat{Q}[B_r]) + \frac{1}{\gamma} \ln \left( \frac{d\hat{Q}}{dP} \right)} \right] = e^{-H[\hat{Q}|P]-\gamma q(p-E^\hat{Q}[B_r])}
\]

From line 2 and line 3, \( \frac{d\hat{Q}}{dP} \) has to the form as follows,

\[
\frac{d\hat{Q}}{dP} = \frac{e^{-\gamma \left( \int_0^T \theta_i^T dX_t + q(p-B_r) \right)}}{e^{-H[\hat{Q}|P]-\gamma q(p-E^\hat{Q}[B_r]) + \frac{1}{\gamma} \ln \left( \frac{d\hat{Q}}{dP} \right)}} = \frac{e^{-\gamma \left( \int_0^T \theta_i^T dX_t + q(p-B_r) \right)}}{E \left[ e^{-\gamma \left( \int_0^T \theta_i^T dX_t + q(p-B_r) \right)} \right]}
\]

(10)

Note that, from the convexity of the expected utility, \( \hat{\theta} \), the solution of (1), is unique and it has to satisfy,

\[
E \left[ -e^{-\gamma(x+pq+f_0^T \hat{\theta}_i dX_t - B_r q)} \left( \int_0^T \theta_i^T dX_t \right) \right] = 0, \quad \text{for all } \theta' \in \Theta.
\]

(11)

It implies that the martingale property generated by \( \hat{\theta} \). That is, from the uniqueness of \( \hat{\theta} \), it holds \( \theta = \hat{\theta} \).

\( \square \)

**Remark 2.2**

Although the equation (11) is an expanded result of Section 3.1 of Frittelli(2000)[10], where the discrete model is discussed, we can directly deduce it. If \( \theta \in \Theta \), for any \( \kappa > 0 \), it holds \( \kappa \theta \in \Theta \). Since \( \hat{\theta} \) is the solution of the problem (1), it holds for any \( \theta \in \Theta \) and any \( \kappa > 0 \),

\[
E \left[ -e^{-\gamma(x+pq+f_0^T \hat{\theta}_i dX_t - B_r q)} \right] \geq E \left[ -e^{-\gamma(x+pq+f_0^T (\theta_i^* + \kappa \theta_i') dX_t - B_r q)} \right]
\]

\[
= E \left[ -e^{-\gamma \int_0^T \theta_i^T dX_t} e^{-\gamma(x+pq+f_0^T \hat{\theta}_i dX_t - B_r q)} \right]
\]

\[
= E \left[ \left( 1 - \gamma \kappa \int_0^T \theta_i^T dX_t + \gamma^2 \kappa^2 \int_0^T \frac{1}{2} (\theta_i^T)^2 d(X)_t \right) \left( e^{-\gamma(x+pq+f_0^T \hat{\theta}_i dX_t - B_r q)} \right) \right]
\]

\[
\leftrightarrow E \left[ \left( \int_0^T \theta_i^T dX_t \right) \left( e^{-\gamma(x+pq+f_0^T \hat{\theta}_i dX_t - B_r q)} \right) \right] \leq E \left[ \left( \frac{1}{2} \gamma \kappa \int_0^T (\theta_i^T)^2 d(X)_t \right) \left( e^{-\gamma(x+pq+f_0^T \hat{\theta}_i dX_t - B_r q)} \right) \right].
\]

Likewise, for same \( \theta' \),

\[
E \left[ -e^{-\gamma(x+pq+f_0^T \hat{\theta}_i dX_t - B_r q)} \right] \geq E \left[ -e^{-\gamma(x+pq+f_0^T (\theta_i^* - \kappa \theta_i') dX_t - B_r q)} \right]
\]

\[
= E \left[ \left( 1 + \gamma \kappa \int_0^T \theta_i^T dX_t + \frac{1}{2} \gamma^2 \kappa^2 \int_0^T (\theta_i^T)^2 d(X)_t \right) \left( e^{-\gamma(x+pq+f_0^T \hat{\theta}_i dX_t - B_r q)} \right) \right]
\]

\[
\leftrightarrow E \left[ \left( -\int_0^T \theta_i^T dX_t \right) \left( e^{-\gamma(x+pq+f_0^T \hat{\theta}_i dX_t - B_r q)} \right) \right] \leq E \left[ \left( \frac{1}{2} \gamma \kappa \int_0^T (\theta_i^T)^2 d(X)_t \right) \left( e^{-\gamma(x+pq+f_0^T \hat{\theta}_i dX_t - B_r q)} \right) \right].
\]
That is, for some $M \in \mathbb{R}$, it holds,

\[
E \left[ \int_0^T \theta'_i dX_t \right] \left( e^{-\gamma(x+pq+\int_0^T \theta'_i dX_t - B_T q)} \right) \leq \kappa M.
\]

Since the above inequality holds for any $\kappa$, martingale property is proved.

We can also consider the problem (4). Left hand side of (4) is same as (5). Right hand side of (4) is given as follows,

\[
\sup_{\theta \in \Theta} E \left[ U(x - pq + \int_0^T \theta'_i dX_t + B_T q) \right] = -e^{-\gamma \left( x + \frac{1}{\gamma} H[\tilde{Q}|P] - pq + \gamma \tilde{E}[B_T] \right)},
\]

where $\tilde{Q}$ is the solution of the problem $\inf_{Q \in \mathcal{M}} \left\{ H[Q|P] + \gamma q \tilde{E}[B_T] \right\}$. Therefore, utility indifference buy price is given as follows,

\[
p^b(B_T; q) = \tilde{E}[B_T] + \frac{1}{\gamma q} \left( H[\tilde{Q}|P] - H[Q^0|P] \right)
\]

\[
= \inf_{Q \in \mathcal{M}} \left\{ \tilde{E}[B_T] + \frac{1}{\gamma q} \left( H[Q|P] - H[Q^0|P] \right) \right\}
\]

\[
= - \sup_{Q \in \mathcal{M}} \left\{ -\tilde{E}[B_T] - \frac{1}{\gamma q} \left( H[Q|P] - H[Q^0|P] \right) \right\}.
\]

\section{The equilibrium of utility indifference price}

In this section, we consider the equilibrium of the market. First, we give a definition of the equilibrium on the market of the random endowment $B$ (c.f. Mas-Collel et.al. [21]).

\begin{definition}
Let an economy specify the investors’ preferences which is described by the utility function $U := \{U_i(\cdot); U_i(x) := -e^{-\gamma x}, i = 1, \ldots, I \}$. An allocation $q^* := \{q_i^*, i = 1, \ldots, I\}$ and a price $p$ of the random endowment $B$ constitutes a price equilibrium if there is an assignment such that

1. \textbf{Offer price condition:} For any investor with utility function $\{U_i, i = 1, \ldots, I\}$, when the investor sells $q_i^*$-units of the random endowment, $(p, q^*)$ is preferred to all other allocations $(p, (q^*_i)'')$; that is, an expected utility corresponding to the allocation $(p, q^*_i)$ is larger than another expected utility corresponding to the allocation $(p, (q^*_i)'')$.

2. \textbf{Bid price condition:} For any investor with utility function $\{U_{I+j}, j = 1, \ldots, J\}$, when the investor buys $q_j^*$-units of the random endowment, $(p, q_j^*)$ is preferred to all other allocations $(p, (q_j^*)''')$; that is, an expected utility corresponding to the allocation $(p, q_j^*)$ is larger than another expected utility corresponding to the allocation $(p, (q_j^*)''')$.

3. \textbf{Market cleared condition} $\sum_{i=1}^{I} q_i^* = \sum_{j=1}^{J} q_j^*$.

\end{definition}

\begin{theorem}
If investors in the market of the random endowment $B_T$ act according to the utility maximization, there is an equilibrium price $p^*$, such that

\[
p^* = \tilde{E}[Q^0|B_T].
\]

Before proving above Theorem, we show a few Lemmas and define the set of the Radon-Nikodym derivative of equivalent martingale measure for physical measure $P$, that is,

\[\mathcal{N} := \left\{ \frac{dQ}{dP}(\omega) | \omega \in \Omega, Q \in \mathcal{M} \right\} .\]
Furthermore, we define the set of the terminal wealth for strategy $\Theta$,

$$
\mathcal{G} := \left\{ G(\theta) \mid \theta \in \Theta, G(\theta) = \int_0^T \theta_t^i \, dX_t < \infty \right\}.
$$

Let $\mathcal{H}$ be a Hilbert space. It is clear that the set $\mathcal{G} \subset \mathcal{H}$. We set $G(\Theta) \in \mathcal{G}$ as terminal wealths via all admissible strategies $\Theta$ for a fixed $\omega \in \Omega$.

**Lemma 3.3**

For MEMM $Q^0$, Radon-Nikodym derivative $\frac{dQ^0}{dP}(\omega)$ is given as follows,

$$
\frac{dQ^0}{dP} = \frac{e^{-\gamma G(\theta^0)}}{\int e^{-\gamma G(\theta^0)} \, dP},
$$

where $\gamma$ is a risk-aversion and $\theta^0$ is the solution of the problem (2) in the case that the utility function is specified as $U(x) := -e^{-\gamma x}$.

**Proof**

MEMM is given as the solution of the problem

$$
\inf_{Q \in \mathcal{M}} \left\{ H(Q|P) \right\}.
$$

Since $Q \in \mathcal{M}$, for any admissible strategy $\theta \in \Theta$, it holds that

$$
\int_{\Omega} G(\theta)(\omega) \frac{dQ}{dP}(\omega) \, dP(\omega) = 0, \quad \text{for all } \frac{dQ}{dP} \in \mathcal{N}.
$$

It means that $\frac{dQ}{dP} \in \mathcal{N}$ has $\Theta$-martingale property and this martingale property is the constraint of the optimization problem. Therefore, for a unique Lagrange multiplier $\lambda \in \mathcal{H}$, we can rewrite the optimization problem (14) as follows,

$$
\inf_{\frac{dQ}{dP} \in \mathcal{N}, \lambda \in \mathcal{H}} \left\{ \int_{\Omega} dQ \ln \frac{dQ}{dP}(\omega) \, dP(\omega) + \int_{\Omega} \lambda(\theta, G(\Theta))(\omega) \frac{dQ}{dP}(\omega) \, dP(\omega) \right\},
$$

(15)

where we used the Frechét differentiability of the relative entropy functional $\int_{\Omega} \frac{dQ}{dP}(\omega) \ln \frac{dQ}{dP}(\omega) \, dP(\omega)$ and the continuous linear functional $\int_{\Omega} G(\theta)(\omega) \frac{dQ}{dP}(\omega) \, dP(\omega)$ in order to apply the Lagrange multiplier method (Favretti(2005)[7], Cochrane and Magcgregor(1978)[3] and Botelho(2009)[10]). For some $\frac{dQ}{dP} \in \mathcal{N}$, $\epsilon \in \mathbb{R}_+$ and some function $\eta(\omega)$ such that $\int_{\Omega} \eta(\omega) \, dP(\omega) = 0$, we set as follows,

$$
\frac{dQ}{dP}(\omega) = \frac{dQ'}{dP}(\omega) + \epsilon \eta(\omega).
$$

Let $\lambda^*$ be the optimizer for the problem (15). By a variational method using $\frac{dQ}{dP} = \frac{dQ'}{dP} + \epsilon \eta(\omega)$, the first order condition of the optimization is deduced as follows,

$$
\int_{\Omega} \eta(\omega) \left\{ \ln \frac{dQ'}{dP}(\omega) + 1 + \lambda^* G(\Theta) \right\} \, dP(\omega) = 0.
$$

Therefore, for any $\omega \in \Omega$, $\ln \frac{dQ'}{dP}(\omega) + 1 + \lambda^* G(\Theta) = \text{constant}$. Since it is required that $\int \frac{dQ}{dP} \, dP = 1$, $\frac{dQ}{dP}$ has the form as follows,

$$
\frac{dQ}{dP}(\omega) = \frac{e^{-(\lambda^* G(\Theta))}(\omega)}{\int_{\Omega} e^{-(\lambda^* G(\Theta))}(\omega) \, dP(\omega)}.
$$

The optimizer $\lambda^*$ has to satisfy the martingale property,

$$
\int_{\Omega} G(\theta'')(\omega) \frac{dQ}{dP}(\omega) \, dP(\omega) = \int_{\Omega} G(\theta'')(\omega) \frac{e^{-(\lambda^* G(\Theta))}(\omega) \, dP(\omega)}{\int_{\Omega} e^{-(\lambda^* G(\Theta))}(\omega) \, dP(\omega)} = 0 \text{ for all } \theta'' \in \Theta.
$$

\[\text{From the definition of } G(\Theta), \text{ the linear functional } \int_{\omega \in \Omega} G(\theta) \frac{dQ}{dP}(\omega) \, dP(\omega) \text{ is bounded. Therefore, it is continuous.}\]
Since \( \langle \lambda^*, G(\Theta) \rangle \) is given by the linear combination of \( \lambda^* \) and \( G(\Theta) \), it holds \( \langle \lambda^*, G(\Theta) \rangle \in \mathcal{G} \). That is, for some \( \theta^* \in \Theta \), it holds that \( \langle \lambda^*, G(\Theta) \rangle = G(\theta^*) \).

Since the solution of (1), say \( \theta^0 \), has to satisfy the following equation,

\[
\mathbb{E} \left[ -\gamma G(\theta^0) e^{-\gamma (x + G(\theta^0))} \right] = 0 \text{ for all } \theta' \in \Theta,
\]

it holds

\[ \theta^* = \gamma \theta^0. \]

Therefore,

\[
\frac{dQ}{dP}(\omega) = \frac{e^{-\gamma G(\theta^0)(\omega)}}{\int_{\Omega} e^{-\gamma G(\theta^0)(\omega)} dP(\omega)}.
\]

Q.E.D.

Proposition 3.2 of Becherer(2003)[1] shows that \( \lim_{q \downarrow 0} p^x(B_T; q) = \mathbb{E}_{Q^0} [B_T] \). This is the case of utility indifference sell price. We can easily show that the case of utility indifference buy price, too; that is \( \lim_{q \downarrow 0} p^b(B_T; q) = \mathbb{E}_{Q^0} [B_T] \). Furthermore, we can show the convergence of the utility indifference price on the quantity \( q \).

**Lemma 3.4**

On the utility indifference price based on the exponential utility \( U(x) = -e^{-\gamma x} \), it holds that

\[
\lim_{q \downarrow 0} p^x(B_T; q) = \lim_{q \downarrow 0} p^b(B_T; q) = \mathbb{E}_{Q^0} [B_T].
\]

**Proof** Consider the problem (12) for the utility indifference buy price, that is,

\[
p^b(B_T; q) = \inf_{Q \in \mathcal{M}} \left\{ \mathbb{E}_{Q} [B_T] + \frac{1}{\gamma q} \left( H(Q|P) - H(Q^0|P) \right) \right\}.
\]

Let \( \tilde{Q} \) be the solution of this problem. Likewise the Lemma 3.3, the Radon-Nikodym derivative \( \frac{d\tilde{Q}}{dP} \) is given as follows,

\[
\frac{d\tilde{Q}}{dP}(\omega) = \frac{e^{-\gamma q(B(\omega) + \langle \lambda, G(\Theta) \rangle(\omega))}}{\int_{\Omega} e^{-\gamma q(B(\omega) + \langle \lambda, G(\Theta) \rangle(\omega))} dP(\omega)},
\]

where unique optimizer \( \lambda \in \mathcal{H} \) and \( G(\Theta) \in \mathcal{G} \) are given for satisfying as follows,

\[
\int_{\omega \in \Omega} G(\theta')(\omega) \frac{e^{-\gamma q(B(\omega) + \langle \lambda, G(\Theta) \rangle(\omega))}}{\int_{\Omega} e^{-\gamma q(B(\omega) + \langle \lambda, G(\Theta) \rangle(\omega))} dP(\omega)} dP(\omega) = 0, \text{ for all } \theta' \in \Theta.
\]

For the utility maximization problem (4), the optimal solution \( \bar{\theta} \) satisfies \( \int_{\Omega} G(\theta')(\omega) e^{-\gamma (q(B(\omega) + G(\bar{\theta})(\omega))} dP(\omega) = 0, \) for all \( \theta' \in \Theta. \) From the uniqueness of \( \lambda \), it holds that

\[
q(\lambda, G(\Theta)) = G(\bar{\theta}).
\]

That is,

\[
\frac{d\tilde{Q}}{dP}(\omega) = \frac{e^{-\gamma (q(B(\omega) + G(\bar{\theta}))(\omega))}}{\int_{\Omega} e^{-\gamma (q(B(\omega) + G(\bar{\theta}))(\omega))} dP(\omega)}.
\]

Next, we show that the convergence \( \tilde{Q} \to Q^0 \), when \( q \to 0 \).

\[
\lim_{q \downarrow 0} \sup_{\theta' \in \Theta} \mathbb{E} \left[ -e^{-\gamma \langle f_0^T \theta'_i \rangle dX_t + B_T q} \right] = \lim_{q \downarrow 0} \mathbb{E} \left[ -e^{-\gamma \langle f_0^T \bar{\theta}_i \rangle dX_t + B_T q} \right] 
\]

\[
\geq \lim_{q \downarrow 0} \mathbb{E} \left[ -e^{-\gamma \langle f_0^T (\theta'_i) \rangle dX_t + B_T q} \right] 
\]

\[
= \mathbb{E} \left[ -e^{-\gamma \langle f_0^T (\bar{\theta}_i) \rangle dX_t} \right].
\]
On the other hand, using duality,

\[
\limsup_{q\downarrow 0} \mathbb{E} \left[ -e^{-\gamma (f^*_T \theta^*_T dX_t + B_T q)} \right] = \liminf_{q\downarrow 0} \inf_{\eta \in \mathbb{R}^+} \left\{ \mathbb{E} \left[ \mathbb{V} \left( \frac{d\mathbb{Q}}{d\mathbb{P}} \right) + \eta \frac{d\mathbb{Q}}{d\mathbb{P}} \left( \int_0^T \theta_t^T dX_t + B_T q \right) \right] \right\} \\
= \liminf_{q\downarrow 0} \inf_{\eta \in \mathbb{R}^+} \left\{ \eta \ln \eta - \eta + \frac{\eta}{\gamma} \inf_{Q \in \mathcal{M}} \left\{ H(Q|P) + \gamma q \mathbb{E}^{Q^0}[B_T] \right\} \right\} \\
\leq \liminf_{q\downarrow 0} \inf_{\eta \in \mathbb{R}^+} \left\{ \eta \ln \eta - \eta + \frac{\eta}{\gamma} \left( H(Q^0|P) + \gamma q \mathbb{E}^{Q^0}[B_T] \right) \right\}
\]

The solution of \( \inf_{Q \in \mathbb{R}^+} \left\{ \frac{\eta}{\gamma} \ln \frac{\eta}{\gamma} - \frac{\eta}{\gamma} + \frac{\eta}{\gamma} \left( H(Q^0|P) + \gamma q \mathbb{E}^{Q^0}[B_T] \right) \right\} \) is given by, \( \gamma e^{-\gamma \left( \frac{1}{\gamma} H(Q^0|P) + \gamma q \mathbb{E}^{Q^0}[B_T] \right)} \).

Therefore,

\[
\limsup_{q\downarrow 0} \mathbb{E} \left[ -e^{-\gamma (f^*_T \theta^*_T dX_t + B_T q)} \right] = \liminf_{q\downarrow 0} \inf_{\eta \in \mathbb{R}^+} \left\{ \mathbb{E} \left[ -e^{-\gamma (f^*_T \theta^*_T dX_t)} \right] \right\} = \mathbb{E} \left[ -e^{-\gamma (f^*_T \theta^*_T dX_t)} \right] .
\]

That is,

\[
\lim_{q\downarrow 0} \mathbb{E} \left[ -e^{-\gamma (f^*_T \theta^*_T dX_t + B_T q)} \right] = \mathbb{E} \left[ -e^{-\gamma (f^*_T \theta^*_T dX_t)} \right].
\]

From the uniqueness of the solution of the utility maximization problem, when \( q \downarrow 0 \),

\[
qB + G(\hat{\theta}) \to G(\theta^0).
\]

Therefore, when \( q \to 0 \), the measure \( \hat{Q} \) converges to MEMM, that is, \( \hat{Q} \to Q^0 \), which implies that \( H(\hat{Q}|P) \to H(Q^0|P) \).

Since \( Q^0 \) is MEMM, \( H(Q|P) \geq H(Q^0|P) \) for any \( Q \in \mathcal{M} \). So, \( \frac{H(Q^0|P) - H(Q^0|P)}{q} \geq 0 \), for \( q > 0 \). Assume that \( \lim_{q\downarrow 0} \frac{H(Q^0|P) - H(Q^0|P)}{q} > 0 \). Then, for any \( Q \in \mathcal{M} \) and \( q > 0 \), some \( \epsilon > 0 \) exists, such that \( \frac{H(Q^0|P) - H(Q^0|P)}{q} > \epsilon \). So, \( H(Q|P) > H(Q^0|P) + \epsilon q \). This implies that, for \( \bar{Q} \), \( \lim_{q\downarrow 0} H(\bar{Q}|P) = H(Q^0|P) + \epsilon \lim_{q\downarrow 0} q \). That is, \( \lim_{q\downarrow 0} H(\bar{Q}|P) > H(Q^0|P) \). However, \( \lim_{q\downarrow 0} H(\bar{Q}|P) = H(Q^0|P) \). Therefore, \( \lim_{q\downarrow 0} \frac{H(Q^0|P) - H(Q^0|P)}{q} = 0 \). The case of the utility indifference buy price is proved. By the same way, we can show the case of the utility indifference sell price.

Q.E.D.

Frittelli(2000b)[9] and Rouge(2000)[23] show that the utility indifference sell price is non-decreasing, when the risk-aversion increases. Since \( p^b(B_T; q) = -p^s(-B_T; q) \) (Becherer(2003)[1]), it is easily shown that utility indifference buy price is non-increasing, when the risk-aversion increases.

**Lemma 3.5**

For the increasing buying (selling) quantity of random endowment \( B \), the utility indifference buy price is non-increasing (the utility indifference sell price is non-decreasing).

**Proof** It is sufficient to prove the case of the utility indifference buy price, since we can use same way in the case of the utility indifference sell price.

The utility indifference buy price for the risk-aversion \( \gamma \) and the quantity \( q \) is given by,

\[
p^b(B_T; q) = \inf_{Q \in \mathcal{M}} \left\{ \mathbb{E}^Q[B] + \frac{1}{\gamma q} (H(Q|P) - H(Q^0|P)) \right\} \\
= \mathbb{E}^Q[B] + \frac{1}{\gamma q} (H(\hat{Q}|P) - H(Q^0|P)) ,
\]
where \( \hat{Q} \) is the solution of the problem \( \inf_{Q \in \mathcal{M}} \{ H[Q|P] + \gamma q E^Q[B_T] \} \). For \( \epsilon > 0 \), consider the utility indifference buy price \( p' \) for the risk-aversion \( \gamma \) and the quantity \( q + \epsilon \). It is given by,

\[
p' = \inf_{Q \in \mathcal{M}} \left\{ \mathbb{E}^Q [B] + \frac{1}{\gamma (q + \epsilon)} \left( H(Q|P) - H(Q^0|P) \right) \right\}
\]

\[
\leq \mathbb{E}^\hat{Q} [B] + \frac{1}{\gamma (q + \epsilon)} \left( H(\hat{Q}|P) - H(Q^0|P) \right)
\]

\[
= \mathbb{E}^\hat{Q} [B] + \frac{1}{\gamma q} \left( H(\hat{Q}|P) - H(Q^0|P) \right) - \frac{1}{\gamma q} \left( H(\hat{Q}|P) - H(Q^0|P) \right) + \frac{1}{\gamma (q + \epsilon)} \left( H(\hat{Q}|P) - H(Q^0|P) \right)
\]

\[
= p^b(B; q) - \left( \frac{1}{\gamma q} - \frac{1}{\gamma (q + \epsilon)} \right) \left( H(\hat{Q}|P) - H(Q^0|P) \right)
\]

\[
\leq p^b(B; q)
\]

Q.E.D.

**Corollary 3.6**

The lower bound of a utility indifference buy price is given as follows,

\[
p^b(B_T; q) \geq \frac{1}{\gamma q} \ln \mathbb{E}^{Q_0} [e^{\gamma B_T}].
\]

The upper bound of a utility indifference sell price is given as follows,

\[
p^s(B_T; q) \leq \frac{1}{\gamma q} \ln \mathbb{E}^{Q_0} [e^{-\gamma B_T}].
\]

**Proof** From the definition of the utility indifference price, it holds

\[
\mathbb{E} \left[ e^{-\gamma (x + \int_0^T (\theta_t^0)^T dX_t)} \right] = \mathbb{E} \left[ e^{-\gamma (x + p^b(B_T; q) + \int_0^T \theta_t^0 dX_t - B_T - q)} \right].
\]

where \( \theta_0 \) and \( \hat{\theta} \) are solutions of left-hand side and right hand side of (3), respectively. From above equation,

\[
\mathbb{E} \left[ e^{-\gamma (x + \int_0^T (\theta_t^0)^T dX_t)} \right] \geq \mathbb{E} \left[ e^{-\gamma (x + p^s(B_T; q) + \int_0^T (\theta_t^0)^T dX_t - B_T - q)} \right]
\]

\[
\Leftrightarrow e^{\gamma p^s(B_T; q)} \geq \frac{\mathbb{E} \left[ e^{-\gamma (x + \int_0^T (\theta_t^0)^T dX_t - B_T)} \right]}{\mathbb{E} \left[ e^{-\gamma (x + \int_0^T (\theta_t^0)^T dX_t)} \right]} = \mathbb{E}^{Q_0} [e^{\gamma B_T}]
\]

\[
\Leftrightarrow p^s(B_T; q) \geq \frac{1}{\gamma q} \ln \mathbb{E}^{Q_0} [e^{\gamma B_T}].
\]

Likewise, we can show the case of utility indifference buy price.

Q.E.D.

**Proof of Theorem 3.2** Assume that there are \( I + J \) investors with risk-aversion \( \{ \gamma_i; i = 1, \ldots, I + J \} \). Let the allocation \( q^s := \{ q_i^s; i = 1, \ldots, I \} \), for some \( i, q_i^s > 0 \), \( q^b := \{ q_j^b; j = 1, \ldots, J \} \), for some \( j, q_j^b > 0 \) be selling quantities and buying quantities, respectively.

Assume that some \( p > \mathbb{E}^{Q_0} [B_T] \) and the allocation \( (q^s, q^b) \) is the equilibrium. From Lemma 3.4 and Lemma 3.5, the lower (upper) bound of the utility indifference sell price (utility indifference buy price) is \( \mathbb{E}^{Q_0} [B_T] \). Therefore, any allocation \( (q^s, q^b) \) is not preferred for the buy-side of the random endowment to not buying the random endowment. That is, the allocation \( (q^s, q^b) \) is impossible. Conversely, if \( p < \mathbb{E}^{Q_0} [B_T] \), then any allocation \( (q^s, q^b) \) is not preferred for the sell-side of the random endowment to not selling the random endowment. Therefore, the price \( p^* = \mathbb{E}^{Q_0} [B_T] \) is only available price in which some allocation is available. Furthermore, the available allocation is given as \( \{ (q_i^s)^*; i = 1, \ldots, I \} \), for all \( i, (q_i^s)^* = 0 \}, \{ (q_j^b)^*; j = 1, \ldots, J \} \), for all \( j, (q_j^b)^* = 0 \} \).

Q.E.D.
Investor’s information bias In the previous part, every investor has the same information which is represented as a filtration $\mathcal{F}$. However, in the real market, there might be some information bias; that is, some investors have much information, and some investors have less information on the market. We consider the information bias by the sub-filtration (Gombani et al. (2007)) [12], Becherer(2003)[1]).

We define filtrations such that these satisfy the usual conditions of completeness and right-continuity: $\mathbb{F}^i := \{ \mathcal{F}^i_t; t \in [0,T] \}$ and $\mathbb{F} := \{ \mathcal{I}_t; t \in [0,T] \}, i \in \mathbb{N}$. At time $0$, these are trivial $\sigma$-algebras. Furthermore, we assume that $\sigma$-algebras $\mathcal{F}^i_0$, $\mathcal{I}^0_0$, $\mathcal{I}^1_0$, $\cdots$ are independent under $P$, where $\sigma$-algebra $\mathcal{F}^i_0$ is generated by the observable price processes. Therefore, the information generated by $\mathcal{F}^i_0$ is common for all investors. Other independent $\sigma$-algebra $\mathcal{I}^i_0$, $\cdots$ is generated by the additional information which is not necessarily observable by all investors. Let $\mathcal{I} := \{ \mathcal{I}_t; t \in [0,T] \}$ be given by $\mathcal{I}_t := \bigvee_{i=0}^\infty \mathcal{I}^i_t$. As the same way, we construct the other filtration $\hat{\mathcal{I}}$ with $\mathcal{I}_t = \hat{\mathcal{I}}_t$ and such that $\mathcal{I} \subseteq \hat{\mathcal{I}}$. Let $\mathcal{F} := \mathbb{F}^0 \vee \mathcal{I}$ and $\hat{\mathcal{F}} := \mathbb{F}^0 \vee \hat{\mathcal{I}}$. Furthermore, we set $\mathcal{M}(\mathbb{F})$ and $\mathcal{M}(\hat{\mathcal{F}})$ to be the set of martingale measures corresponding to the different filtrations.

The proof of Proposition 4.7 of Becherer(2003)[1] shows that

$$\mathcal{M}(\mathbb{F}) \supseteq \mathcal{M}(\hat{\mathcal{F}}).$$

For any $\mathbb{H} \in \{ \mathbb{F}, \hat{\mathcal{F}} \}$,

Following Becherer(2003)[1], we assume a unique $Q^* \in \mathcal{M}$ exists such that its density $\frac{dQ^*}{dP}$ is $\mathcal{F}^i_T$-measurable, i.e.

$$\{ Q^* \} := \left\{ Q \in \mathcal{M} \left| \frac{dQ}{dP} \text{ is } \mathcal{F}^i_T \text{-measurable} \right. \right\}.$$ 

Becherer(2003)[1] proves that the measure $Q^*$ minimizes the relative entropy over $\mathcal{M}(\mathbb{H})$. That is, $Q^*$ is MEMM and this is common for every filtration.

From these results, (8) and (12), it holds that

$$p^*(B_T; \mathbb{F}) \geq p^*(B_T; \hat{\mathbb{F}}) \quad p^b(B_T; \mathbb{F}) \leq p^b(B_T; \hat{\mathbb{F}})$$

Since MEMM $Q^*$ is common for every investor, it implies that the equilibrium price of the random endowment $B$ is given by $p^* = \mathbb{E}^Q [B_T]$, even if there is information bias among investors.

4 Applying to the multi assets basis model

In this section, we apply the above results to the basis model of Davis(2006)[4] and show how the utility indifference buy (sell) price are distributed in the market. We consider the market which is constructed of tradable assets $X = \{ X^i_t, 0 \leq t \leq \infty, i = 1, \cdots, d \}$ and intradable assets $Y = \{ Y^j_t, 0 \leq t \leq \infty, j = 1, \cdots, k \}$. For simplicity, let $X$ and $Y$ be discounted price processes. On the given probability space, we assume stochastic processes for these assets as below: let $w = \{ w^i_t, 0 \leq t \leq \infty \}, i = 1, \cdots, d + k$ be Brownian Motion, and $\mathbb{E}[dw^i_t dw^j_t] = \rho_{i,j} dt$ for constant $\rho_{i,j} > 0$, $i, j = 1, \cdots, d + k$, and all other coefficients be constant.$^{11}$

$$\begin{pmatrix} dX_t / X_t \\ dY_t / Y_t \end{pmatrix} = \mathbf{m} dt + \Sigma d\mathbf{w}_t,$$

\hspace{1cm} (16)

where $dX_t = (dX^1_t / X^1_t, \cdots, dX^d_t / X^d_t)^\top$, $dY_t = (dY^1_t / Y^1_t, \cdots, dY^k_t / Y^k_t)^\top$, $\mathbf{m} = (\mu_1, \cdots, \mu_{d+k})^\top$, $d\mathbf{w}_t = (dw^1_t, \cdots, dw^{d+k}_t)^\top$, $\Sigma = (\sigma_{i,j})_{i,j=1,\cdots,d+k}$ and $\sigma_{i,j} = \begin{cases} \sigma_i & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$.

We consider the contingent claim $B_T$ on the intradable assets $Y$ such that, $B_T := h(Y_T).$

$^{11}$We can formulate the model such that the volatility matrix $\Sigma$ is not a diagonal matrix. Such a model setting directly express that each asset is driven by every Brownian motions $\{ w^i, i = 1, \cdots, n+k \}$. However, under the setting which sets the volatility matrix as the diagonal matrix, each assets is driven by every Brownian motion via the correlation between each Brownian motions. So, in this paper, we adopt the diagonal volatility matrix.
where \( \{h^i(\cdot), \ i = 1, \cdots, k\} \) are the European pay-off functions and \( \{b_i \in \mathbb{R}, \ i = 1, \cdots, k\} \). Many authors\footnote{e.g. Henderson(2002)[14], Musiela and Zariphopoulou(2004)[22], Monoyios(2004)[18]} considered the case of \( k = 1 \). We expand the case of \( k \geq 1 \).

We try to reformulate from Brownian motions \( \{w^i, \ i = 1, \cdots, d + k\} \) dependent each other to independent Brownian motions \( \{\tilde{w}^i, \ i = 1, \cdots, d + k\} \). For the correlation defined as \( d\langle w^i, w^j \rangle_t = \rho_{i,j} dt \), the correlation matrix is set by \( R := (\rho_{i,j})_{i,j = 1, \cdots, d+k} \). We can decompose it (by Cholesky decomposition) as follows,

\[
R = AA^\top,
\]

where \( A := (a_{i,j})_{i,j = 1, \cdots, d+k} \) is lower triangular matrix; \( a_{i,j} = 0 \) if \( i < j \). Using this matrix, for independent Brownian motions \( w^i = \{w^i_t; \ t \in [0,T], \ i = 1, \cdots, d + k\} \), we can set the relationship as follows,

\[
dw_t = Adw'_t
\]

Therefore, the dynamics of underlying assets is rewritten as follows,

\[
\left(\begin{array}{c}
\frac{dX_1}{X_1} \\
\frac{dY_1}{Y_1}
\end{array}\right) = mdt + \Sigma Adw', \tag{17}
\]

Next, we find the local martingale measure for the process \( X \). Generally, if \( Q \) is a probability measure equivalent to \( P \) on \( \mathcal{F}_t \), then there exist adapted processes \( g := \{g^i_t; t \in [0,T], \ i = 1, \cdots, d + k\} \in G \), where \( g^i_0 \) is square integrable on the time satisfying \( \int_0^T g^i_t dt < \infty \), a.s. and \( G \) is the set of all adapted processes of square integrable on the time, and

\[
\frac{dQ}{dP} = \exp \left( \sum_{i=1}^{d+k} \int_0^T g^i_t d(w^i)'_t - \frac{1}{2} \sum_{i=1}^{d+k} \int_0^T (g^i_t)^2 dt \right).
\]

For \( g_t := (g^1_t, \cdots, g^{d+k}_t)^\top \), we can find that the transformation

\[
d\tilde{w}_t = dw'_t - g_t dt
\]

where \( \tilde{w} = \{\tilde{w}^i_t; t \in [0,T], \ i = 1, \cdots, d + k\} \) is a Brownian motion under the measure \( Q \). We give \( (d+k, 1) \)-vector \( g_t = (g^1_t, \cdots, g^{d+k}_t, 0, \cdots, 0)^\top \) by \( g_t = -A^{-1}\Sigma^{-1}m_t \), where \( (d + k,1) \) vector \( m_t \) is defined as \( m_t := (\mu_1, \cdots, \mu_d, 0, \cdots, 0)^\top \). It is clear that \( g_t \) is uniquely defined and we set \( g^i_t; i = 1, \cdots, d \) as \( g^i_t \equiv g^i \), which makes the asset \( X \) local martingale under the measure \( Q \). That is, except for \( (g^{d+1}_t, \cdots, g^{d+k}_t) \), every factor of \( g_t \) is given by \( g_1 \).

For \( i = 1, \cdots, k \), \( Y \) is rewritten as follows,

\[
dY^i_t/Y^i_t = (\rho_{d+i} + \sigma_{d+i}\tilde{a}^i_t, g_t)dt + \sigma_{d+i}dW^i_t,
\]

where \( \tilde{a}_{d+i} := (a_{d+i,1}, \cdots, a_{d+i,d+i-1}, 0, \cdots, 0)^\top \) is \( (d + i) \)-th row vector of \( A \), and \( W^i = \{W^i_t, t \in [0,\infty)\} \) is defined as follows,

\[
dW^i_t = (a_{d+i,1}d\tilde{a}^i_1 + \cdots + a_{d+i,d+i}d\tilde{a}^i_{d+i}).
\]

That is, \( W^i \) is a Brownian motion under \( Q \) such that \( d\langle W^i, W^j \rangle_t = \tilde{a}^i_t \tilde{a}^j_t dt + \rho_{d+i,d+j} dt \). PDE on \( Y \) is solved as,

\[
Y^i_t = y_i e^{\int_0^T (\rho_{d+i} + \sigma_{d+i}(\tilde{a}^i_t, g_t) + \frac{1}{2} \sigma_{d+i}^2) dt + \int_0^T \sigma_{d+i} dW^i_t}, \quad Y^i_t = y_i \in \mathbb{R}
\]

First, we deduce the MEMM, that is, we consider the problem \( \inf_{Q \in \mathcal{M}} H[Q|P] \).

\[
H[Q^0|P] = \inf_{Q \in \mathcal{M}} H[Q|P]
\]

\[
= \inf_{Q \in \mathcal{M}} \int \frac{dQ}{dP} \ln \frac{dQ}{dP} dP = \inf_{Q \in \mathcal{M}} \int \ln \frac{dQ}{dP} dQ
\]

\[
= \inf_{g_t \in G, i = d+1, \cdots, d+k} \frac{1}{2} \sum_{i=1}^{d+k} \int_0^T (g^i_t)^2 dt = \inf_{g_t \in G, i = d+1, \cdots, d+k} \frac{1}{2} \left( \sum_{i=1}^{d} (g^i)^2 T + \sum_{i=d+1}^{d+k} \int_0^T (g^i)^2 dt \right)
\]

\[
= \frac{1}{2} \sum_{i=1}^{d} (g^i)^2 T
\]
Next, we deduce the utility indifference sell price.

\[
p^i(B_T; t) = \sup_{Q \in M} \left\{ \mathbb{E}^Q [B_T] - \frac{1}{\gamma_q} \left( H(Q|P) \right) \right\} + \frac{1}{\gamma_q} H(Q^0|P) \\
= \sup_{Q \in M} \left\{ \mathbb{E}^Q [h(Y_T)] - \frac{1}{\gamma_q} \mathbb{E}^Q \left[ \ln \left( \frac{dQ}{dP} \right) \right] \right\} + \frac{1}{\gamma_q} 2 \sum_{i=1}^d (g_i')^2 T \\
= \sup_{Q \in M} \mathbb{E}^Q \left\{ h(Y_T) - \frac{1}{\gamma_q} \left( \sum_{i=1}^{d+k} \int_0^T g_i' d\bar{w}_i + \frac{1}{2} \sum_{i=d+1}^{d+k} \int_0^T (g_i')^2 dt \right) \right\}
\]

We define the variable \( Z := \{ Z^1, \cdots, Z^k \} \), where \( Z^i := \{ Z^i_t, t \in [0, \infty) \} \) is given as follows,

\[
Z^i_t := \frac{1}{\sigma_{d+i}} \ln Y^i_t,
\]

and for \( i = 1, \cdots, k, \)

\[
f(z_1, \cdots, z_k) := h(e^{\sigma_{d+1} z_1}, \cdots, e^{\sigma_{d+k} z_k}).
\]

Then, we can construct, for \( s \in (t, T), \)

\[
dZ^i_s = \frac{1}{\sigma_{d+i}} \left( \mu_{d+i} - \frac{1}{2}\sigma_{d+i}^2 + \sigma_{d+i} a_{d+i}^T g_t \right) ds + dW_s^i, \quad Z^i_t = z_i \in \mathbb{R}
\]

We think a value function \( V(t, z) \) as,

\[
V(t, z) := \sup_{Q \in M} \mathbb{E}^Q \left\{ f(Z_T) - \frac{1}{\gamma_q} \left( \sum_{i=1}^{d+k} \int_t^T g_i' d\bar{w}_i + \frac{1}{2} \sum_{i=d+1}^{d+k} \int_t^T (g_i')^2 ds \right) \right\}
\]

\[
= \sup_{Q \in M} \mathbb{E}^Q \left\{ - \frac{1}{\gamma_q} \left( \sum_{i=1}^{d+k} g_i' d\bar{w}_i \right) - \frac{1}{2 \gamma_q} \sum_{i=d+1}^{d+k} (g_i')^2 dt + V(t, z) \right\}
\]

where \( z = (z_1, \cdots, z_k)^T \). The HJB equation is deduced as follows,

\[
V_t + \frac{1}{2} \sum_{i,j=1, \cdots, k} V_{z_i z_j} \rho_{d+i,d+j} + \sum_{i=1}^k V_{z_i} \left( \mu_{d+i} - \frac{1}{2}\sigma_{d+i}^2 \right) + \sup_{g_i' \in G, \quad j=d+1, \cdots, d+k} \left\{ \sum_{i=1}^k V_{z_i} a_{d+i}^T g_t - \frac{g^2_t}{2 \gamma_q} \right\} = 0, \quad (18)
\]

\[
V(T, z) = f(z),
\]

where \( g^*_2 = (g_{d+1}^{d+1}, \cdots, g_{d+k}^{d+k})^T \). The optimal \( g^*_2 \) is given by,

\[
g^*_2 = \gamma q \bar{A}^T V_z,
\]

where \( V_z := (V_{z_1}, \cdots, V_{z_k})^T \) and \( \bar{A} := (a_{d+i,d+j})_{i,j=1, \cdots, k} \) is lower triangular matrix; \( a_{d+i,d+j} = 0 \), if \( i < j \).

Therefore, (18) is rewritten as,

\[
V_t + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^k V_{z_i z_j} \rho_{d+i,d+j} + \sum_{i=1}^k V_{z_i} \frac{1}{\sigma_{d+i}} \left( \mu_{d+i} - \frac{1}{2}\sigma_{d+i}^2 + \sigma_{d+i} a_{d+i}^T g_1 \right) + \frac{\gamma q}{2} V_z^T \bar{R} V_z = 0,
\]

where \( \bar{R} := (\bar{\rho}_{d+i,d+j})_{i,j=1, \cdots, k} \) is given by

\[
\bar{R} = \bar{A} \bar{A}^T.
\]

Since \( \bar{R} \) is symmetric,

\[
\bar{R} = E \Lambda E^T,
\]

\[
E = \text{diag}(e_i), \quad i = 1, \cdots, k
\]
where $\Lambda = (\lambda_{i,j})_{i,j=1,\ldots,k}$ and $E := (e_1, \cdots, e_k)$. The components of $\Lambda$ and $E$ are eigenvalues and eigenvectors; $\lambda_{i,j} = \begin{cases} 
abla_i & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$ are eigenvalues and $\{e_i = (e_{1,i}, \cdots, e_{k,i})^\top, i = 1, \cdots, k\}$ are eigenvectors such that,

$$e_i^\top e_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Define $\mathbf{z}' := (z_1', \cdots, z_k')^\top$ as follows,

$$
\left( \begin{array}{c} z_1 \\ \vdots \\ z_k \\
\end{array} \right) = \left( \begin{array}{cccc} e_{1,1} & \cdots & e_{1,k} \\ \vdots & \ddots & \vdots \\ e_{k,1} & \cdots & e_{k,k} \\
\end{array} \right) \left( \begin{array}{c} z_1' \\ \vdots \\ z_k' \\
\end{array} \right) = E \mathbf{z}',
$$

Let $V'(t, z_1', \cdots, z_k')$ be $V(t, z_1, \cdots, z_k) := V'(t, e_1^\top \mathbf{z}, \cdots, e_k^\top \mathbf{z})$. Then, it holds that $V_{z_i} = e_{i,1}V'_{z_1} + e_{i,k}V'_k = \tilde{e}_i V_k$, where $\tilde{e}_i := (e_{i,1}, \cdots, e_{i,k})^\top$. That is, $V_k = EV_k'$. And, it also holds

$$V_{z_i, z_j} = e_{i,k} \left( e_{j,1}V'_{z_1, z_i} + e_{j,k}V'_{z_k, z_i} \right) + e_{i,k} \left( e_{j,1}V'_{z_1, z_k} + e_{j,k}V'_{z_k, z_k} \right)$$

$$= (e_{i,1}, \cdots, e_{i,k}) \left( \begin{array}{cccc} V'_{z_1, z_1} & \cdots & V'_{z_1, z_k} \\ \vdots & \ddots & \vdots \\ V'_{z_k, z_1} & \cdots & V'_{z_k, z_k} \\
\end{array} \right) \left( \begin{array}{c} e_{j,1} \\ \vdots \\ e_{j,k} \\
\end{array} \right)$$

$$= \tilde{e}_i V'_{z_i, z_k} \tilde{e}_j = \sum_{l=1}^k \sum_{m=1}^k e_{i,l} V'_{z_i, z_m} e_{j,m}.$$

Therefore HJB equation (18) is rewritten again,

$$V'_t + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^k \rho_{d+i, d+j} \sum_{l=1}^k e_{i,l} V'_{z_i, z_l} e_{j,m} + \frac{1}{2} \sum_{i=1}^k \sum_{l=1}^k e_{i,l} V'_{z_i, z_l} \frac{1}{\sigma_{d+i}} \left( \mu_{d+i} - \frac{1}{2} \sigma_{d+i}^2 + \sigma_{d+i} \alpha_{d+i, g_l}^\top \right) + \frac{\gamma q}{2} (V'_k)^\top AV_k' = 0$$

$$V'(t, z_1', \cdots, z_k') = f(E \mathbf{z}')$$

$$0 = V'_t + \frac{1}{2} \sum_{i=1}^k \sum_{m=1}^k V'_{z_i, z_m} e_{i,1} R_d e_m + \frac{1}{2} \sum_{l=1}^k e_{i,l} R_d e_l + \frac{\gamma q}{2} (V'_k)^\top AV_k'$$

$$= V'_t + \frac{1}{2} \sum_{i=1}^k \sum_{m=1}^k V'_{z_i, z_m} e_{i,1} R_d e_m + \frac{1}{2} \sum_{l=1}^k e_{i,l} R_d e_l + \frac{\gamma q}{2} (V'_k)^\top AV_k'.$$

where $\alpha_i := \frac{1}{\sigma_{d+i}} \left( \mu_{d+i} - \frac{1}{2} \sigma_{d+i}^2 + \sigma_{d+i} \alpha_{d+i, g_l}^\top \right)$ and $\alpha = (\alpha_1, \cdots, \alpha_k)^\top$. Let $V'(t, \mathbf{z}') := \sum_{i=1}^k \ln U^i(t, z_i') e_{i,1}^\top \frac{\gamma q}{\sigma_{d+i}}$. Then, we can calculate in continuation,

$$0 = \sum_{i=1}^k e_{i,1}^\top R_d e_l \left( U^i_t + \frac{1}{2} e_{i,1}^\top R_d e_l U^i_{z_i'} + e_{i,1}^\top \alpha U^i_{z_i'} \right),$$

$$\sum_{i=1}^k \ln U^i(T, z_i') e_{i,1}^\top \frac{\gamma q}{\sigma_{d+i}} = f(E \mathbf{z}').$$
Setting some functions \( \{c^i(t); i = 1, \cdots, k\} \) (independent of \( \{y_i; i = 1, \cdots, k\} \)) as such

\[
e^{\frac{1}{2} \gamma q_i \lambda_i^2} \left( U^i_t + \frac{1}{2} \lambda_i^2 \right) e^{\gamma q_i \lambda_i^2} u^{\gamma q_i \lambda_i^2} U^i_{zz_i} + e^{\gamma q_i \lambda_i^2} U^i_{z_i} \right) = c^i(t), \ i = 1, \cdots, k,
\]

and \( \sum_{i=1}^{k} c^i(t) = 0 \) for any \( t \in [0, T] \). In addition, we set

\[
U^i(t, z_i^0) = e^{-\frac{1}{2} \gamma q_i \lambda_i^2 z_i^2 - \frac{1}{2} \lambda_i^2 \gamma q_i \lambda_i^2 \tau} e^{\gamma q_i \lambda_i^2 \int_0^\tau c^i(s)ds} u^i(\tau, z_i^0)
\]

so the HJB equation is rewritten as, \( \forall i = 1, \cdots, k \),

\[
u^i_t = \frac{e^{\frac{1}{2} \gamma q_i \lambda_i^2}}{2} \nu^i_{z_i^2} + \sum_{i=1}^{k} e^{\frac{1}{2} \gamma q_i \lambda_i^2} \lambda_i^2
\]

And, terminal condition is described as,

\[
\sum_{i=1}^{k} \ln \left( e^{-\frac{1}{2} \gamma q_i \lambda_i^2} u^i(0, z_i^0) \right) = f(Ex')
\]

That is,

\[
\prod_{i=1}^{k} u^i(0, z_i^0) \frac{e^{\gamma q_i \lambda_i^2 \lambda_i^2}}{\nu^i_{z_i^2}} = \prod_{i=1}^{k} e^{\frac{1}{2} \gamma q_i \lambda_i^2 z_i^2 + \gamma q f(Ex')}
\]

Note that, the interval is changed as,

\( \{t \leq T\} \rightarrow \{\tau \geq 0\} \).

We have to solve these partial differential equations,

\[
u^1_t = \frac{e^{\frac{1}{2} \gamma q_i \lambda_i^2}}{2} \nu^1_{z_i^2} + \sum_{i=1}^{k} e^{\frac{1}{2} \gamma q_i \lambda_i^2} \lambda_i^2
\]

\[
\prod_{i=1}^{k} u^i(0, z_i^0) \frac{e^{\gamma q_i \lambda_i^2 \lambda_i^2}}{\nu^i_{z_i^2}} = \prod_{i=1}^{k} e^{\frac{1}{2} \gamma q_i \lambda_i^2 z_i^2 + \gamma q f(Ex')}
\]

It is well known that the solution of \( \nu^i_t = \frac{e^{\frac{1}{2} \gamma q_i \lambda_i^2}}{2} \nu^i_{z_i^2} \), is given as \( u^i(\tau, z_i^0) = \int K^i(z_i^0 - \bar{y}_i, \tau)u(0, \bar{y}_i) d\bar{y}_i \),

where (as it is also well known that) the fundamental solution \( K(\cdot) \) is given as follows,

\[
K^i(z_i^0, \tau) = \frac{1}{\sqrt{2\pi e^{\gamma q_i \lambda_i^2}}} e^{-\frac{(z_i^0)^2}{2 e^{\gamma q_i \lambda_i^2}}}
\]

However, the terminal condition is given by the combination of variables \( z_i^0 \). Therefore, it is difficult to express each \( u(0, \bar{y}_i) \) independently. So, we consider the relationship below,

\[
\prod_{i=1}^{k} u^i(\tau, z_i^0) \frac{e^{\gamma q_i \lambda_i^2 \lambda_i^2}}{\nu^i_{z_i^2}} \leq \prod_{i=1}^{k} \left( \int K^i(z_i^0 - \bar{y}_i, \tau)u(0, \bar{y}_i) d\bar{y}_i \right) \frac{e^{\gamma q_i \lambda_i^2 \lambda_i^2}}{\nu^i_{z_i^2}}
\]

\[
\leq \prod_{i=1}^{k} K^i(z_i^0 - \bar{y}_i, \tau) \frac{e^{\gamma q_i \lambda_i^2 \lambda_i^2}}{\nu^i_{z_i^2}} G(\bar{y}_1, \cdots, \bar{y}_k) d\bar{y}_k
\]
where \( G(\tilde{y}_1, \ldots, \tilde{y}_k) = \prod_{i=1}^{k} u^i(0, \tilde{y}_i) \) expresses terminal condition for \( u^i \). Furthermore, \( \int \prod_{i=1}^{k} K^i(\tilde{z}_i - \tilde{y}_i, \tau)G(\tilde{y})d\tilde{y} \) is given as follows,

\[
\int \prod_{i=1}^{k} K^i(\tilde{z}_i - \tilde{y}_i, \tau) \frac{\sigma^T R_{d+1} y_i}{\sigma_{d+1}} G(\tilde{y}_1, \ldots, \tilde{y}_k)d\tilde{y} = \prod_{i=1}^{k} \left( \frac{1}{\sqrt{2\pi}} \frac{\sigma^T R_{d+1} y_i}{\sigma_{d+1}} \right) e^{-\frac{(\tilde{z}_i - \tilde{y}_i)^2}{2\sigma^2_{d+1}}} \prod_{i=1}^{k} u^i(0, \tilde{y}_i) \frac{\sigma^T R_{d+1} y_i}{\sigma_{d+1}} d\tilde{y}
\]

For \( \tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_k)^T \), we define \( \tilde{y}_a = \left( \frac{\ln \tilde{y}_1}{\sigma_{d+1}}, \ldots, \frac{\ln \tilde{y}_k}{\sigma_{d+k}} \right) : = E\tilde{y} \). Then, we can calculate in continuation as follows,

above equation = \( \int \prod_{i=1}^{k} \left( \frac{1}{\sqrt{2\pi}} \frac{\sigma^T R_{d+1} y_i}{\sigma_{d+1}} \right) e^{-\frac{(\tilde{z}_i - \tilde{y}_i)^2}{2\sigma^2_{d+1}}} \prod_{i=1}^{k} e^{\frac{1}{\sigma_{d+1}} \tilde{y}_i + \gamma qf(E\tilde{y})} J(\tilde{y})d\tilde{y} \),

where \( J(\tilde{y}) \) is Jacobian such that,

\[
J(\tilde{y}) = \begin{vmatrix}
\frac{1}{\sigma_{d+1}} & \ldots & \frac{1}{\sigma_{d+k}} \\
\vdots & \ddots & \vdots \\
\frac{1}{\sigma_{d+1}} & \ldots & \frac{1}{\sigma_{d+k}}
\end{vmatrix}
= \begin{vmatrix}
e_{11} & \ldots & e_{kk}
e_{11} & \ldots & e_{kk}
e_{1k} & \ldots & e_{kk}
\end{vmatrix}
\]

Therefore,

\[
\prod_{i=1}^{k} u^i(\tau, \tilde{z}_i) \frac{\sigma^T R_{d+1} y_i}{\sigma_{d+1}} \leq \int \prod_{i=1}^{k} K^i(\tilde{z}_i - \tilde{y}_i, \tau) \frac{\sigma^T R_{d+1} y_i}{\sigma_{d+1}} G(\tilde{y}_1, \ldots, \tilde{y}_k)d\tilde{y}
\]

\[
= \int \prod_{i=1}^{k} \left( \frac{1}{\sqrt{2\pi}} \frac{\sigma^T R_{d+1} y_i}{\sigma_{d+1}} \right) e^{-\frac{(\tilde{z}_i - \tilde{y}_i)^2}{2\sigma^2_{d+1}}} \prod_{i=1}^{k} e^{\frac{1}{\sigma_{d+1}} \tilde{y}_i + \gamma qf(y)} \frac{1}{\sigma_{d+1}} d\tilde{y}
\]

Let \( C(t, y_1, \ldots, y_k) = V \left(t, \frac{1}{\sigma_{d+1}} \ln y_1, \ldots, \frac{1}{\sigma_{d+k}} \ln y_k\right) \).

Then, it is clear that \( p^e(B_T; q) = C(t, y) \).

\[
p^e(B_T; q) = C(t, y) = V \left(t, \frac{1}{\sigma_{d+1}} \ln y_1, \ldots, \frac{1}{\sigma_{d+k}} \ln y_k\right).
\]

Define \( y_{\gamma} = \left( \frac{\ln y_1}{\sigma_{d+1}}, \ldots, \frac{\ln y_k}{\sigma_{d+k}} \right)^T \). From the definition of \( V(t, z_1, \ldots, z_k) = V(t, e_1^T z, \ldots, e_k^T z) \), it
holds \( V(t, \frac{1}{\sigma_{d+k}} \ln y_1, \cdots, \frac{1}{\sigma_{d+k}} \ln y_k) := V'(t, e_1^\top y_\sigma, \cdots, e_k^\top y_\sigma) \). We can calculate in continuation,

\[
p^* (B_T; q) = V'(t, e_1^\top y_\sigma, \cdots, e_k^\top y_\sigma) = \sum_{i=1}^k \ln U^0(t, e_i^\top y_\sigma) + \sum_{i=1}^k \ln \left( e^{-\frac{\gamma y_i^\top y_\sigma}{\lambda_i} - \frac{1}{2} \frac{\gamma y_i^\top y_\sigma}{\lambda_i^2}} \int_\mathbb{C} e^{-\frac{1}{2} y_i^\top y_\sigma} d\gamma \right)
\]

\[
\leq \sum_{i=1}^k \ln \left( e^{-\frac{\gamma y_i^\top y_\sigma}{\lambda_i} - \frac{1}{2} \frac{\gamma y_i^\top y_\sigma}{\lambda_i^2}} \int_\mathbb{C} e^{-\frac{1}{2} y_i^\top y_\sigma} d\gamma \right)
\]

\[
\leq \sum_{i=1}^k \ln \left( e^{-\frac{\gamma y_i^\top y_\sigma}{\lambda_i} - \frac{1}{2} \frac{\gamma y_i^\top y_\sigma}{\lambda_i^2}} \right)
\]

Here,

\[
\sum_{i=1}^k \frac{1}{\gamma q \lambda_i} (e_i^\top (y_\sigma - y_\sigma - \tau \alpha))^2 = \sum_{i=1}^k \frac{1}{\gamma q \lambda_i} (y_\sigma - y_\sigma - \tau \alpha)^\top e_i e_i^\top (y_\sigma - y_\sigma - \tau \alpha)
\]

\[
\sum_{i=1}^k \frac{1}{\lambda_i} e_i e_i^\top = \sum_{i=1}^k \frac{1}{\lambda_i} \begin{pmatrix} e_{1,i} & \cdots & e_{k,i} \\ \vdots & \ddots & \vdots \\ e_{k,i} & \cdots & e_{1,i} \end{pmatrix} = \sum_{i=1}^k \frac{1}{\lambda_i} \begin{pmatrix} e_{1,i} e_{1,i} & \cdots & e_{1,i} e_{k,i} \\ \vdots & \ddots & \vdots \\ e_{k,i} e_{1,i} & \cdots & e_{k,i} e_{k,i} \end{pmatrix} = E^\top \Lambda^{-1} E = \tilde{R}^{-1}
\]

Therefore,

\[
p^* (B_T; q) \leq \frac{1}{\gamma q} \ln \prod_{i=1}^k \left( \frac{1}{\sqrt{2\pi}} \frac{\gamma q \lambda_i}{\sqrt{\gamma q \lambda_i}} \right) \int e^{-\frac{1}{2} (y_\sigma - y_\sigma - \tau \alpha)^\top \tilde{R}^{-1} (y_\sigma - y_\sigma - \tau \alpha) + h(y)} \frac{1}{\prod_{i=1}^k \sigma_{d+i} \bar{y}_i} d\gamma
\]

\[
= \frac{1}{\gamma q} \ln \prod_{i=1}^k \frac{1}{\sqrt{2\pi}} \frac{\gamma q \lambda_i}{\sqrt{\gamma q \lambda_i}} \int \frac{(2\pi)^{k/2}}{(2\pi)^{k/2}} \frac{1}{\prod_{i=1}^k \sqrt{\lambda_i} \sigma_{d+i} \bar{y}_i} e^{-\frac{1}{2} (y_\sigma - y_\sigma - \tau \alpha)^\top \tilde{R}^{-1} (y_\sigma - y_\sigma - \tau \alpha) + h(y)} d\gamma
\]

\[
= \frac{1}{\gamma q} \ln \prod_{i=1}^k \left( \frac{1}{\sqrt{2\pi}} \frac{\gamma q \lambda_i}{\sqrt{\gamma q \lambda_i}} \right)^{1/2} \int \frac{(2\pi)^{k/2}}{(2\pi)^{k/2}} \frac{1}{\prod_{i=1}^k \sqrt{\lambda_i} \sigma_{d+i} \bar{y}_i} e^{-\frac{1}{2} (y_\sigma - y_\sigma - \tau \alpha)^\top \tilde{R}^{-1} (y_\sigma - y_\sigma - \tau \alpha) + h(y)} d\gamma
\]

\[
= \frac{1}{\gamma q} \ln \prod_{i=1}^k \frac{1}{\sqrt{2\pi}} \frac{\gamma q \lambda_i}{\sqrt{\gamma q \lambda_i}} \int \frac{(2\pi)^{k/2}}{(2\pi)^{k/2}} \frac{1}{\prod_{i=1}^k \sqrt{\lambda_i} \sigma_{d+i} \bar{y}_i} e^{-\frac{1}{2} (y_\sigma - y_\sigma - \tau \alpha)^\top \tilde{R}^{-1} (y_\sigma - y_\sigma - \tau \alpha) + h(y)} d\gamma
\]
where $g$ is multivariate lognormal density. Since $p^b(B_T; q) = -p^s(-B_T; q)$, the utility indifference buy price is given as follows,

$$p^b(B_T; q) = -\sum_{i=1}^{k} \frac{1}{\gamma q \lambda_i} \ln \mathbb{E} \left[ e^{-\gamma q \lambda_i b_i h^i (\tilde{Y}_T)} \right].$$  \hspace{1cm} (20)

Furthermore, the equilibrium price is deduced from above equation as follows,

$$p^s(B_T; \gamma) = \lim_{q \to 0} p^s(B_T; q) = \lim_{q \to 0} \sum_{i=1}^{k} \frac{1}{\gamma q \lambda_i} \ln \mathbb{E} \left[ e^{\gamma q \lambda_i b_i h^i (\tilde{Y}_T)} \right]$$

$$= \lim_{q \to 0} \sum_{i=1}^{k} \frac{1}{\lambda_i} \mathbb{E} \left[ \lambda_i b_i h^i (\tilde{Y}_T) e^{\gamma q \lambda_i b_i h^i (\tilde{Y}_T)} \right]$$

$$= \sum_{i=1}^{k} b_i \mathbb{E} \left[ h^i (\tilde{Y}_T) \right].$$ \hspace{1cm} (21)

### 4.1 Numerical Example

In this section, we give a numerical example which shows the situation described in the Theorem 3.2. We consider the random endowment $B_T$ such that,

$$B_T := h^1(Y_T^1) + \cdots + h^3(Y_T^3),$$

where $\{h^i(\cdot); i = 1, 2, 3\}$ is the pay-off function of put option. Basic parameters required to calculate (20)-(21) are given in Table 1. In this case, there are three intradable assets. The price of the random endowment on the three intradable assets are given by the other three tradable assets. The exercise prices of each put option are common. We calculate the utility indifference price of this random endowment by using the results (20)-(21), where we use Monte Carlo simulation through 500,000 sample paths to calculate expectation. We assume that there are four sellers of the random endowment with risk-aversion 0.1, 0.5, 1.0, and 2.0 and there are three buyers of the random endowment with risk-aversion 0.1, 0.3 and 1.5.

The result is given in Table 2. Each column shows the utility indifference price for corresponding risk-aversion and the quantity of the random endowment. We can see that the utility indifference sell price is increasing with the quantity and the risk-aversion. On the other hand, utility indifference buy price is decreasing with the quantity and the risk-aversion. Figure 1 shows this situation more clearly. The lines on this figure made by the Bezier interpolation. Furthermore, from this result, we can construct the aggregate demand curve and the aggregate supply curve. In fact, when we calculate each investor’s selling quantities of the random endowment to some utility indifference sell price and sum up these quantities, the summed quantity represents the market aggregate supply of the random endowment for this utility indifference sell price. After we do this calculation for all utility indifference sell prices, we can construct the aggregate supply curve. From Figure 1, we can easily estimate that the aggregate supply curve will be increasing with the quantity of the random endowment. Likewise, the aggregate demand curve can be deduced, and the form will be decreasing with the quantity. The fact that the form of the aggregate supply curve is increasing and the form of the aggregate demand curve is decreasing matches to our intuition of economics. However, the intersection of the supply curve and demand curve is the point where the quantity of the random endowment is zero. Therefore, the equilibrium is zero for some positive equilibrium price (27.31 in this numerical example). Theorem 3.2 claims such a situation.
### Table 1: Market Parameters

<table>
<thead>
<tr>
<th>assets number</th>
<th>tradable assets</th>
<th>interest rate</th>
<th>expiry</th>
<th>exercise price</th>
<th>payoff type</th>
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</thead>
<tbody>
<tr>
<td>6</td>
<td>100</td>
<td>0.05</td>
<td>1</td>
<td>100</td>
<td>put option</td>
</tr>
<tr>
<td>price</td>
<td>0.25</td>
<td>0.3</td>
<td>0.35</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>volatility</td>
<td>0.05</td>
<td>0.07</td>
<td>0.09</td>
<td>0.12</td>
<td>0.12</td>
</tr>
<tr>
<td>drift</td>
<td></td>
<td></td>
<td></td>
<td>correlation matrix</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.96</td>
<td>0.95</td>
<td>0.94</td>
<td>0.86</td>
</tr>
<tr>
<td></td>
<td>0.96</td>
<td>1</td>
<td>0.95</td>
<td>0.94</td>
<td>0.86</td>
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<tr>
<td></td>
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<td>0.86</td>
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<td>0.86</td>
<td>0.86</td>
<td>0.86</td>
<td>0.86</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0.85</td>
<td>0.85</td>
<td>0.85</td>
<td>0.85</td>
<td>0.85</td>
</tr>
</tbody>
</table>

### Table 2: Utility indifference prices

<table>
<thead>
<tr>
<th>quantity</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>utility indifference sell price</th>
<th>utility indifference buy price</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>27.3199</td>
<td>27.3199</td>
<td>27.3199</td>
<td>27.3199</td>
<td>27.3199</td>
<td>27.3199</td>
</tr>
<tr>
<td>1</td>
<td>33.1102</td>
<td>62.9274</td>
<td>91.5274</td>
<td>126.672</td>
<td>23.0138</td>
<td>17.4881</td>
</tr>
<tr>
<td>2</td>
<td>40.4708</td>
<td>91.2545</td>
<td>125.633</td>
<td>160.014</td>
<td>19.8002</td>
<td>12.8162</td>
</tr>
<tr>
<td>3</td>
<td>48.1325</td>
<td>112.077</td>
<td>147.066</td>
<td>174.636</td>
<td>17.4197</td>
<td>10.1038</td>
</tr>
<tr>
<td>4</td>
<td>55.7899</td>
<td>123.873</td>
<td>155.735</td>
<td>184.607</td>
<td>15.548</td>
<td>8.31115</td>
</tr>
<tr>
<td>5</td>
<td>62.8009</td>
<td>137.994</td>
<td>168.231</td>
<td>191.151</td>
<td>14.05</td>
<td>12.8162</td>
</tr>
<tr>
<td>6</td>
<td>69.3489</td>
<td>144.735</td>
<td>174.035</td>
<td>195.016</td>
<td>12.8385</td>
<td>6.14092</td>
</tr>
<tr>
<td>7</td>
<td>75.4546</td>
<td>151.537</td>
<td>180.033</td>
<td>198.685</td>
<td>11.7399</td>
<td>5.40671</td>
</tr>
<tr>
<td>8</td>
<td>81.414</td>
<td>157.430</td>
<td>182.150</td>
<td>194.370</td>
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<td>4.85598</td>
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<tr>
<td>9</td>
<td>86.849</td>
<td>162.681</td>
<td>185.395</td>
<td>202.827</td>
<td>10.1240</td>
<td>4.38175</td>
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<tr>
<td>9.9</td>
<td>90.6584</td>
<td>165.611</td>
<td>189.259</td>
<td>203.211</td>
<td>9.49149</td>
<td>4.04483</td>
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</tbody>
</table>
“Sell price 1” depicts that the relation between the price of the random endowment and the supply of the random endowment by the investor with risk-aversion 0.1. Likewise, “sell price 2”, “sell price 3” and “sell price 4” depicts this relation by the investor with risk-aversion 0.5, 1 and 2, respectively. And “buy price 1”, “buy price 2” and “buy price 3” depicts that the relation between the price and the demand by the investor with risk-aversion 0.1, 0.3 and 1.5.

References


