Risk Sensitive Benchmarked Asset Management

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Abstract

This paper extends the risk-sensitive asset management theory developed by Bielecki and Pliska and by Kuroda and Nagai to the case where the investor’s objective is to outperform an investment benchmark. The main result is a mutual fund theorem. Every investor following the same benchmark will take positions, in proportions dependent on his/her risk sensitivity coefficient, in two funds: the log-optimal portfolio and a second fund which adjusts for the correlation between the traded assets, the benchmark and the underlying valuation factors.

1 Introduction

The optimal investment problem, in which an investor must select a utility-maximizing asset allocation, is well understood in both discrete and continuous time thanks to the seminal works of Markowitz [9] and Merton [10]. However, relatively little attention has been paid to date to the related problem of benchmarked asset allocation, although it directly concerns the vast majority of institutional investors.

In the benchmarked investment problem, the investor selects an asset allocation to outperform, based on a given performance measure, a given investment benchmark. This results in a number of interesting questions: how can the objective of outperforming a benchmark be reconciled with utility theory? What is the optimal investment strategy? How does the presence of a benchmark manifests itself in the asset allocation? Can any mutual fund theorem be deduced in this case?

In an attempt to answer these questions, we will develop a continuous time optimization model using the risk sensitive control approach introduced by Bielecki and Pliska [2]. This differs from the classical stochastic control approach pioneered by Merton [11] in that the investor’s risk-aversion is characterized explicitly by a risk sensitive parameter rather than implicitly through a utility function. As a result, the objective function in the risk
sensitive control approach is the investors capital penalized by a function of his/her risk aversion.

In so doing, the risk-sensitive approach delivers some significant advantages over the classical stochastic control approach, making it particularly well suited to tackle complex investment problems, such as the development of an Intertemporal Capital Asset Pricing Model, as in Bielecki and Pliska [4] or the resolution of an investment problem with partially observed information as in, for example, Nagai and Peng [12].

Indeed, we will show that a careful formulation of the benchmarked investment problem leads to a simple analytical solution which is no more complex to derive than Kuroda and Nagai’s solution to the optimal investment problem [7]. The mathematical framework of this article is very close to the derivation proposed by Kuroda and Nagai. The contribution of this paper is to extend their analysis to solve the benchmarked investment problem and derive mutual fund theorems.

After presenting the analytical setting in Section 2 of this article, we will introduce our formulation of the investment problem in Section 3 and see that, by construction, this formulation effectively relates utility maximization and benchmark outperformance. The finite-horizon benchmarked investment problem is then solved analytically in Sections 4 and 5 and the solutions is extended asymptotically, in Section 6, to address the infinite horizon problem. As we will see, the optimal asset allocation is not only elegant and economic, but it also leads to the formulation, in Section 7, of two interesting mutual fund theorems. Given an investor-specified benchmark, the optimal asset allocation can be expressed as a portfolio of investments in two mutual funds. The first is the log-utility, or Kelly, portfolio. The second is a correction portfolio related to the comovements of the asset with benchmark and with the comovements of the asset with the underlying valuation factors.

In order to elicit more accurately the impact of the benchmark on the optimal asset allocation, we consider in Section 8 two cases related to traded benchmarks. Our findings are summarized in two corollaries to the mutual fund theorems. Finally, Section 9 applies the insights gained from the Risk Sensitive Benchmark and Asset Management (RSBAM) model to extend the definition of fractional Kelly criterion investment strategies.
2 Analytical Setting

2.1 Asset Only Setting

The growth rates of both assets and benchmark are assumed to depend on \( n \) valuation factors \( X_1(t), \ldots, X_n(t) \) which follow the dynamics given in equation (3) below. As in Kuroda and Nagai’s asset only model, the assets market comprises \( m \) risky securities \( S_i, i = 1, \ldots, m \). In contrast to Kuroda and Nagai, we assume that the money market account process, \( S_0 \) is an affine function of the valuation factors, which enables us to easily model a stochastic short rate. Let \( N = n + m \).

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be the underlying probability space. On this space is defined an \( \mathbb{R}^N \)-valued \( \mathcal{F}_t \)-Brownian motion \( W(t) \) with components \( W_k(t) \). \( S_i(t) \) denotes the price at time \( t \) of the \( i \)th security, with \( i = 0, \ldots, m \), and \( X_j(t) \) denotes the level at time \( t \) of the \( j \)th factor, with \( j = 1, \ldots, n \). We also assume that the factors are observable.

Let \( S_0 \) denote the wealth invested in a money market account. The dynamics of the money market account is given by:

\[
\frac{dS_0(t)}{S_0(t)} = \left( \eta + \zeta' X(t) \right) dt, \quad S_0(0) = s_0
\]  
(1)

where \( \eta \in \mathbb{R} \) is a scalar constant and \( \zeta \in \mathbb{R}^n \) is an \( n \)-element column vector. Note that if we set \( \zeta = 0 \) and \( \eta = r \), then equation (1) can be interpreted as the dynamics of a risk-free rate.

We assume that the dynamics of the \( m \) risky securities and \( n \) factors can be expressed as:

\[
\frac{dS_i(t)}{S_i(t)} = (a + AX(t))dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t), \quad S_i(0) = s_i, \quad i = 1, \ldots, m
\]  
(2)

\[
dX(t) = (b + BX(t))dt + \Lambda dW(t), \quad X(0) = x
\]  
(3)

where \( X(t) \) is the \( \mathbb{R}^n \)-valued factor process with components \( X_j(t) \) and the market parameters \( a, A, b, B, \Sigma := [\sigma_{ij}], i = 1, \ldots, m, j = 1, \ldots, N, \Lambda := [\Lambda_{ij}], i = 1, \ldots, n, j = 1, \ldots, N \) are matrices of appropriate dimensions.

In our analysis, we will make the rather innocuous assumption that

**Assumption 1.** The matrix \( \Sigma \Sigma' \) is positive definite.

Let \( \mathcal{G}_t := \sigma((S(s), L(s), X(s)), 0 \leq s \leq t) \) be the sigma-field generated by the security, liability and factor processes up to time \( t \).
The allocation of wealth among the assets is defined by an $\mathbb{R}^m$-valued stochastic process $h$, where the $i$th component $h_i(t)$ denotes the proportion of current wealth invested in the $i$th risky security at time $t$, $i = 1, \ldots, m$. The proportion invested in the money market account is $h_0(t) = 1 - \sum_{i=1}^{m} h_i(t)$.

**Definition 1.** An investment process $h(t)$ is in class $\mathcal{H}$ if the following conditions are satisfied:

1. $h(t)$ is progressively measurable with respect to $\{\mathcal{B}([0, t]) \otimes \mathcal{G}_t\}_{t \geq 0}$;
2. $P\left(\int_{0}^{T} |h(s)|^2 ds < +\infty\right) = 1, \quad \forall T > 0$.

Taking the budget equation into consideration, the wealth, $V(t)$, of the asset only portfolio, in response to an investment strategy $h \in \mathcal{H}$, follows the dynamics

$$\frac{dV}{V} = \left(\eta + \zeta' X(t)\right) dt + h'(t) \left(a - \eta 1 + (A - 1 \zeta') X(t)\right) dt + h'(t) \Sigma dW_t$$

where $1 \in \mathbb{R}^m$ denotes the $m$-element unit column vector. Defining $\hat{a} := a - \eta 1$ and $\hat{A} := A - 1 \zeta'$, we can express the portfolio dynamics as

$$\frac{dV}{V} = \left(\eta + \zeta' X(t)\right) dt + h'(t) \left(\hat{a} + \hat{A} X(t)\right) dt + h'(t) \Sigma dW_t$$

### 2.2 Benchmark Modelling

We assume that the benchmark evolves according to a similar SDE as the asset prices. Specifically,

$$\frac{dL(t)}{L(t)} = (\alpha + \beta' X(t)) dt + \gamma' dW(t), \quad L(0) = l$$

where $\alpha$ is a scalar constant, $\beta$ is a $n$-element column vector, and $\gamma$ is a $N$-element column vector.

Exposure to the short-term interest rate lies buried in equation (4). This can easily be seen by expressing this equation as

$$\frac{dL(t)}{L(t)} = \left(\hat{\alpha} + \hat{\beta}' X(t) + \hat{\kappa} (\eta + \zeta' X(t))\right) dt + \gamma' dW_k(t), \quad L(0) = l$$

where $\hat{\alpha} := \alpha - \hat{\kappa} \eta$ and $\hat{\beta} := \beta - \hat{\kappa} \zeta$ for some scalar $\hat{\kappa}$ reflecting the exposure of the benchmark to the short-term interest rate, $r(t) = \eta + \zeta' X(t)$.

This formulation is wide enough to encompass a multitude of different situations such as:
• **the single benchmark case**, in which, for example, an equity investor’s performance is evaluated against an equity index such as the S&P500 or the FTSE.

• **the single benchmark plus alpha.** In this case, the manager’s performance is evaluated against the performance of a prespecified index plus a given excess return target, called “alpha”. Hedge fund managers typically fall in this category. It is customary for them to have a target based on a short-term interest rate plus alpha, implying that \( \beta = \zeta, \gamma = 0 \) and that \( \alpha = \hat{\alpha} + \eta \), where \( \hat{\alpha} \) is the excess return target.

• **the composite benchmark case.** The investor’s performance is assessed against a benchmark representing a portfolio of indexes, such as 5% cash, 35% Citigroup World Government Bond Index, 25% S&P 500 and 35% MSCI EAFE.

• **the composite benchmark plus alpha**, a situation generally faced by endowment fund managers. In this case, the \( \alpha \) parameter contains an adjustment reflecting a targeted level of excess return.

### 3 Problem Setup

#### 3.1 Optimization Criterion

In our view, the objective of benchmarked investors is to maximize the risk adjusted growth of their assets relative to the benchmark. We will express this objective through the definition of a new optimization criterion, representing the log excess return of the asset portfolio over its benchmark, \( F(t) \), defined as:

\[
F(t) = \ln \frac{V(t)}{L(t)} = \ln V(t) - \ln L(t)
\]

By Itô, the log of excess return in response to a strategy \( h \) is

\[
F(t) = \ln \frac{v}{l} + \int_{0}^{t} d \ln V(s) - \int_{0}^{t} d \ln L(s) \\
= \ln \frac{v}{l} + \int_{0}^{t} \left( \eta + \zeta' X(s) \right) + h(s)' \left( \hat{\alpha} + \hat{A} X(s) \right) ds - \frac{1}{2} \int_{0}^{t} h(s)' \Sigma \Sigma' h(s) ds \\
+ \int_{0}^{t} h(s)' \Sigma dW(s) - \int_{0}^{t} (\alpha + \beta' X(s)) ds + \frac{1}{2} \int_{0}^{t} \gamma' \gamma ds - \int_{0}^{t} \gamma' dW(s)
\]

The use of the log excess return presents a number of advantages. First, this ratio is invariant to scaling of \( v \) and \( l \). This is a particularly useful property in a control framework as it ensure that the optimal investment strategy will not depend on the initial level of wealth and benchmark, but
only depends on the relative position of assets with respect to liabilities. Second, the log of the ratio can be interpreted as the cumulative tracking error of the portfolio with respect to its benchmark. This definition is consistent with the ultimate objective of benchmarked investors, which is to grow their assets compared to their benchmark by generating a positive cumulative tracking error. Third, the mathematical treatment is also simplified as, for any given investment strategy, we consider a SDE with a drift affine in $X$ and a constant diffusion.

In our analysis, we will consider two related classes of risk sensitive control problems$^1$. Firstly, we consider the class of problems $P_\theta$: for $\theta \in ]0, +\infty[$, maximize the risk sensitive expected log excess return per unit of time

$$J_\theta(v, x; h) := \liminf_{t \to \infty} \left(\frac{-2}{\theta}\right) t^{-1} \ln \mathbb{E} e^{-\frac{\theta}{2} F(t; h)}$$

(6)

Secondly, we consider the class of problems $P_{\theta,T}$: for $\theta \in ]0, +\infty[$, maximize the risk sensitive expected log excess return over a time horizon $T$

$$J_{\theta,T}(v, x; h) := \left(\frac{-2}{\theta}\right) \ln \mathbb{E} e^{-\frac{\theta}{2} F(t; h)}$$

(7)

The class of admissible strategies for problem $P_{\theta,T}$ is $A(T) \subset \mathcal{H}$ defined in Section 4.2 below. A strategy $h \in \mathcal{H}$ is in $A(T)$ if the technical condition (11) is satisfied. The class of admissible strategies for problem $P_\theta$ is $A$, where $h \in A$ if and only if the restriction of $h$ to $[0, T]$ is in $A(T)$.

Although this formulation seems rather ad-hoc and remote from utility theory, it can in fact be interpreted as a utility maximization problem, as shown by Bielecki and Pliska [3]. Indeed, the criterion

$$U_\theta(F(t; h)) = \mathbb{E} e^{-\frac{\theta}{2} \ln F(t; h)} = \mathbb{E} F(t; h)^{-\frac{\theta}{2}}$$

represents the maximization of the expected utility, under the power utility function, derived from the relative position of the investor’s portfolio with respect to its benchmark at time $t$. Thus, for $\theta > 0$, $J_\theta(v, x; h)$ can be interpreted as the long-term relative growth rate of the expected utility derived from the relative position of the investor’s portfolio with respect to its benchmark under the power utility function.

$^1$The case $\theta \in ]-2, 0[$ could be considered. However, we will omit it since it represents a risk-seeking behaviour.
3.2 Risk Sensitive Asset Management as a Special Case of RSBAM

To illustrate the relationship between the risk sensitive benchmarked asset problem we propose and Bielecki and Pliska’s risk sensitive asset management, we will consider the case of a non-zero constant benchmark \( l \in \mathbb{R} \setminus \{0\} \).

In the case of a benchmark with constant value \( l \), the log excess return becomes

\[
F(t) = \ln \frac{V(t)}{l} = \ln V(t) - \ln l
\]

and the infinite horizon risk sensitive criterion can be expressed as

\[
\left( -\frac{2}{\theta} \right) \left[ \ln \mathbb{E} e^{-\frac{\theta}{2} F(t;h)} \right] = \left( -\frac{2}{\theta} \right) \left[ \ln \mathbb{E} e^{-\frac{\theta}{2} \ln V(t;h)} \right] - \ln l
\]

Consequently, the problem originally considered by Bielecki and Pliska can be interpreted as a benchmark problem in which the benchmark to beat is the starting level of wealth.

4 Derivation of the Bellman Equation

4.1 Criterion Under the Expectation

Multiplying by \(-\frac{\theta}{2}\) and taking the exponential on both sides of (5), we get

\[
e^{-\frac{\theta}{2} F(t)} = f_0^{-\frac{\theta}{2}} \exp \left\{ \frac{\theta}{2} \int_0^t g(X_s, h(s); \theta) ds - \frac{\theta}{2} \int_0^t (h(s)' \Sigma - \gamma') dW_s \right. \\
\left. - \frac{1}{2} \left( \frac{\theta}{2} \right)^2 \int_0^t (h(s)' \Sigma - \gamma')(h(s)' \Sigma - \gamma')' ds \right\}
\]

where \( f_0 = \frac{v}{l} \) and

\[
g(x, h; \theta) = \frac{1}{2} \left( \frac{\theta}{2} + 1 \right) h' \Sigma \Sigma' h - \eta - \zeta' x - h' (\hat{a} + \hat{A}x) - \frac{1}{2} \frac{\theta}{2} (h' \Sigma \gamma + \gamma' \Sigma' h) \\
+ (\alpha + \beta' x) + \frac{1}{2} \left( \frac{\theta}{2} - 1 \right) \gamma' \gamma
\]

4.2 Change of Measure

Let \( \mathbb{P}_h^\theta \) be the measure on \((\Omega, \mathcal{F})\) defined as

\[
\chi_t := \frac{d\mathbb{P}_h^\theta}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = \exp \left\{ -\frac{\theta}{2} \int_0^t (h(s)' \Sigma - \gamma') dW_s - \frac{1}{2} \left( \frac{\theta}{2} \right)^2 \int_0^t (h(s)' \Sigma - \gamma')(h(s)' \Sigma - \gamma')' ds \right\}, \forall t \geq 0
\]
We denote by $\mathcal{A}(T)$ the set of investment strategies $h \in \mathcal{H}$ on $[0, T]$ such that $\mathbb{P}^\theta_h$ is a probability measure, i.e.

$$
\mathbb{E} \left[ \exp \left\{ -\frac{\theta}{2} \int_0^t (h(s)\Sigma - \gamma) \, dW_s - \frac{1}{2} \left( \frac{\theta}{2} \right)^2 \int_0^t (h(s)\Sigma - \gamma)'(h(s)\Sigma - \gamma)' \, ds \right\} \right] = 1
$$

(11)

For $h \in \mathcal{A}(T)$,

$$
W^\theta_t = W_t + \frac{\theta}{2} \int_0^t (\Sigma'h(s) - \gamma) \, ds
$$

is a standard Brownian motion under $\mathbb{P}^\theta_h$ and $X_t$ satisfies the SDE:

$$
dX_s = \left( b + B X_s - \frac{\theta}{2} \Lambda (\Sigma' h(s) - \gamma) \right) \, ds + \Lambda dW^\theta_s
$$

We can now introduce the auxiliary criterion function under the measure $\mathbb{P}^\theta_h$:

$$
I(f_0, x; h; t, T) = \ln f_0 - \frac{2}{\theta} \ln \mathbb{E}^\theta \left[ \exp \left\{ \frac{\theta}{2} \int_0^{T-t} g(X_s, h(s); \theta) \, ds \right\} \right]
$$

(12)

where $\mathbb{E}^\theta [\cdot]$ denotes the expectation taken with respect to measure $\mathbb{P}^\theta_h$.

### 4.3 The HJB Equation

Let $\Phi$ be the value function for the auxiliary criterion function $I(f_0, x; h; t, T)$. Then $\Phi$ is defined as

$$
\Phi(t, x) = \sup_{h \in \mathcal{A}(T-t)} I(f_0, x; h; t, T)
$$

and it satisfies the HJB PDE

$$
\frac{\partial \Phi}{\partial t} + \sup_{h \in \mathbb{R}^m} L^h_t \Phi = 0
$$

(13)

where

$$
L^h_t \Phi = \left( b + B x - \frac{\theta}{2} \Lambda (\Sigma' h - \gamma) \right)' D \Phi + \frac{1}{2} \text{tr} \left( \Lambda \Lambda' D^2 \Phi \right) - \frac{\theta}{4} (D \Phi)' \Lambda \Lambda' D \Phi - g(x, h; \theta)
$$

(14)
5 Solving the Finite Time Horizon Problem

5.1 Solving the HJB PDE

We start by considering the terms inside the supremum:

\[
\sup_{h \in \mathbb{R}^m} L^h_t \Phi
= \sup_{h \in \mathbb{R}^m} \left\{ (b + Bx)' D\Phi - \frac{\theta}{2} (h' \Sigma - \gamma') \Lambda' D\Phi + \frac{1}{2} \text{tr} (\Lambda \Lambda' D^2\Phi) - \frac{\theta}{4} (D\Phi)' \Lambda \Lambda' D\Phi \\
- \frac{1}{2} \left( \frac{\theta}{2} + 1 \right) h' \Sigma' h + \eta + \zeta' x + h'(\dot{a} + \dot{A}x) + \frac{1}{2} \theta^2 (h' \Sigma \gamma + \gamma' \Sigma' h) - (\alpha + \beta' x) \\
- \frac{1}{2} \left( \frac{\theta}{2} - 1 \right) \gamma' \gamma \right\}
= (b + Bx)' D\Phi + \frac{\theta}{2} \gamma' \Lambda' D\Phi + \frac{1}{2} \text{tr} (\Lambda \Lambda' D^2\Phi) - \frac{\theta}{4} (D\Phi)' \Lambda \Lambda' D\Phi - (\alpha + \beta' x)
- \frac{1}{2} \left( \frac{\theta}{2} - 1 \right) \gamma' \gamma + \eta + \zeta' x
+ \sup_{h \in \mathbb{R}^m} \left\{ - \frac{1}{2} \left( \frac{\theta}{2} + 1 \right) \left[ h - \frac{2}{\theta + 2} (\Sigma \Sigma')^{-1} \left( \dot{a} + \dot{A}x - \frac{\theta}{2} \Sigma \Lambda' D\Phi + \frac{\theta}{2} \Sigma \gamma \right) \right]' (\Sigma \Sigma') \left[ h - \frac{2}{\theta + 2} (\Sigma \Sigma')^{-1} \left( \dot{a} + \dot{A}x - \frac{\theta}{2} \Sigma \Lambda' D\Phi + \frac{\theta}{2} \Sigma \gamma \right) \right] \\
+ \frac{1}{\theta + 2} \left( \dot{a} + \dot{A}x - \frac{\theta}{2} \Sigma \Lambda' D\Phi + \frac{\theta}{2} \Sigma \gamma \right)' (\Sigma \Sigma')^{-1} \left( \dot{a} + \dot{A}x - \frac{\theta}{2} \Sigma \Lambda' D\Phi + \frac{\theta}{2} \Sigma \gamma \right) \right\}
\]

Under Assumption 1, the quadratic form (15) attains a supremum of

\[
\sup_{h \in \mathbb{R}^m} L^h_t \Phi
= (b + Bx)' D\Phi + \frac{\theta}{2} \gamma' \Lambda' D\Phi + \frac{1}{2} \text{tr} (\Lambda \Lambda' D^2\Phi) - \frac{\theta}{4} (D\Phi)' \Lambda \Lambda' D\Phi - (\alpha + \beta' x)
- \frac{1}{2} \left( \frac{\theta}{2} - 1 \right) \gamma' \gamma + \eta + \zeta' x
+ \frac{1}{\theta + 2} \left( \dot{a} + \dot{A}x - \frac{\theta}{2} \Sigma \Lambda' D\Phi + \frac{\theta}{2} \Sigma \gamma \right)' (\Sigma \Sigma')^{-1} \left( \dot{a} + \dot{A}x - \frac{\theta}{2} \Sigma \Lambda' D\Phi + \frac{\theta}{2} \Sigma \gamma \right) \\
\times \left( \dot{a} + \dot{A}x - \frac{\theta}{2} \Sigma \Lambda' D\Phi + \frac{\theta}{2} \Sigma \gamma \right)
\]

for the optimal investment strategy

\[
h^* = \frac{2}{\theta + 2} (\Sigma \Sigma')^{-1} \left( \dot{a} + \dot{A}x - \frac{\theta}{2} \Sigma \Lambda' D\Phi + \frac{\theta}{2} \Sigma \gamma \right)
\]

We now verify that \( \Phi \) has the form

\[
\Phi(x, t) = \frac{1}{2} x' Q(t) x + q'(t) x + k(t)
\]
where $Q$ is an $n \times n$ symmetric matrix, $q$ is an $n$-element column vector and $k$ is a scalar. After tedious calculations, we deduce that $\Phi$ given by equation (18) solves the HJB PDE provided that the following system of differential equations can be solved:

- **a Riccati equation** related to the coefficient of the quadratic term and used to determine the symmetric non-negative matrix $Q(t)$,

$$
\dot{Q}(t) - Q(t)K_0Q(t) + K_1'Q(t) + Q(t)K_1 + \frac{2}{\theta + 2} \hat{A}'(\Sigma\Sigma')^{-1}\hat{A} = 0 \quad (19)
$$

for $t \in [0, T]$, with *terminal condition* $Q(T) = 0$ and with

$$
K_0 = \frac{\theta}{2} \left[ \Lambda \left( I - \frac{\theta}{\theta + 2} \Sigma'(\Sigma\Sigma')^{-1}\Sigma \right) \Lambda' \right] \quad (20)
$$

$$
K_1 = B - \frac{\theta}{\theta + 2} \Lambda\Sigma'(\Sigma\Sigma')^{-1}\hat{A}
$$

- **a linear ordinary differential equation** related to the coefficient of the linear term and used to determine the vector $q(t)$,

$$
\dot{q}(t) + (K_1' - Q(t)K_0)q(t) + Q(t)b + \frac{\theta}{2} Q'(t)\Lambda\gamma + \zeta - \beta \\
+ \frac{1}{\theta + 2} \left( 2\hat{A}' - \theta Q'(t)\Lambda\Sigma' \right) (\Sigma\Sigma')^{-1} \left( \hat{a} + \frac{\theta}{2} \Sigma\gamma \right)
= 0 \quad (21)
$$

with *terminal condition* $q(T) = 0$.

and with the time dependent coefficient $k(t)$ in equation (18) defined as

$$
k(s) = \ln f_0 + \int_s^T l(t)dt \quad (22)
$$

for $0 \leq s \leq T$ and where

$$
l(t) = \frac{1}{2} \text{tr} (\Lambda\Lambda'Q(t)) - \frac{\theta}{4} q'(t)\Lambda\Lambda'q(t) + b'q(t) \\
+ \frac{1}{\theta + 2} \hat{a}'(\Sigma\Sigma')^{-1}\hat{a} + \frac{\theta^2}{4} \frac{1}{\theta + 2} q'(t)\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'q(t) \\
- \frac{\theta}{\theta + 2} q'(t)\Lambda\Sigma'(\Sigma\Sigma')^{-1}\hat{a} - \frac{1}{\theta + 2} \frac{\theta^2}{2} q'(t)\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\gamma \\
+ \frac{\theta}{2} \gamma'\Lambda'q(t) - \frac{1}{2} \left( \frac{\theta}{2} - 1 \right) \gamma'\gamma + \frac{\theta}{\theta + 2} \hat{a}'(\Sigma\Sigma')^{-1}\Sigma\gamma \\
+ \frac{1}{\theta + 2} \frac{\theta^2}{4} \gamma'(\Sigma\Sigma')^{-1}\Sigma\gamma + \eta - \alpha
$$

(23)
Remark: The differential equations obtained in the RSBAM case are a straightforward extension of the differential equations that would be derived in the asset only case. First and most importantly, the Riccati equations used to determine the coefficient matrix of the quadratic term in the RSBAM and the asset only case are identical. Moreover, the equations (21) and (18) contain terms involving the benchmark parameters $\alpha$, $\beta$ and $\zeta$. The equations for the asset-only case are recovered by setting $\alpha = 0$, $\beta = 0$ and $\zeta = 0$.

The matrix $I - \frac{\theta}{\theta + 2} \Sigma' \Sigma^{-1} \Sigma$ appearing in the definition of $K_0$ in equation (20) can be rewritten as:

$$ (I - \Sigma' \Sigma^{-1} \Sigma) + \frac{2}{\theta + 2} \Sigma' \Sigma^{-1} \Sigma $$

(24)

Now, $\Sigma' \Sigma^{-1} \Sigma$ can be interpreted as the projection on the column space of the matrix $\Sigma$ and therefore $I - \Sigma' \Sigma^{-1} \Sigma$ is also a projection. Thus, $\Sigma' \Sigma^{-1} \Sigma$ and $I - \Sigma' \Sigma^{-1} \Sigma$ are both positive definite matrix. Then, if $\theta > 0$, $I - \frac{\theta}{\theta + 2} \Sigma' \Sigma^{-1} \Sigma$ is positive definite and therefore the Riccati equation (19) has a unique solution $Q(t) \geq 0$ defined for all $t \leq T$. (See Davis [5], Proposition 4.4.2.)

5.2 Formalizing the Solution

The following Theorem extends Theorem 2.1 in Kuroda and Nagai [7] to the assets and benchmark case and can be proved in a similar fashion.

Theorem 1. The investment strategy $h^*(t)$ defined by

$$ h^*(t) = \frac{2}{\theta + 2} (\Sigma\Sigma')^{-1} \left( \hat{a} + \frac{\theta}{2} \Sigma\gamma - \frac{\theta}{2} \Sigma\Lambda' q(t) + \left( \hat{A} - \frac{\theta}{2} \Sigma\Lambda' Q(t) \right) X_t \right) $$

(25)

where $q$ is a solution of equation (19) and $q$ is a solution of equation (21), belongs to $A(T)$ and is optimal in $A(T)$ for the finite time horizon problem

$$ J_{\theta,T}(v, x; h) := \left( \frac{-2}{\theta} \right) \ln \mathbb{E} e^{-\frac{2}{\theta} \mathcal{F}(t;h)} = \frac{1}{2} x' Q(0) x + q'(0) x + k(0) $$

where $k$ is given by equation (22).

Proof. The proof is articulated around two main ideas. First, we need to verify that indeed $h^*(t) \in A(T)$. This follows from an argument proposed by Bensoussan [1]. Then, we must prove the optimality of $h^*$. The argument needed here can be found in Kuroda and Nagai [7].

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6 Asymptotic Behaviour of the Solution as $t \to +\infty$

6.1 The Riccati Equation

Since the Riccati equation has not changed from the asset only case, we can simply apply the results obtained by Kuroda and Nagai [7], keeping in mind our different treatment of the money market account process.

As in Kuroda and Nagai [7], we define for $\theta > 0$

$$C := \sqrt{\frac{2}{\theta + 2} \Sigma'(\Sigma\Sigma')^{-1}\hat{A}}$$

$$N^{-1} := \frac{\theta}{2} \left( I - \frac{\theta}{\theta + 2} \Sigma'(\Sigma\Sigma')^{-1}\Sigma \right)$$

so that the Ricatti equation (19) becomes

$$\dot{Q}(t) + K_1' Q(t) + Q(t) K_1 - Q(t) \Lambda N^{-1} \Lambda' Q(t) + CC' = 0, \quad Q(T) = 0 \quad (26)$$

Now, consider the linear ODE for a given $N \times n$ matrix $K$:

$$\dot{P} + (K_1 - \Lambda K)' P + P(K_1 - \Lambda K) + C'C + K'N K = 0, \quad P(T) = 0 \quad (27)$$

whose solution is

$$P(t) = \int_t^T e^{(s-t)(K_1 - \Lambda K)'} (C'C + K'N K)e^{(s-t)(K_1 - \Lambda K)} ds$$

then by Lemma 4.1 in Kuroda and Nagai [7], $Q(t) \leq P(t)$, $t \in [0,T]$. In particular, define

$$K := \frac{2}{\theta + 2} \Sigma'(\Sigma\Sigma')^{-1}\hat{A}$$

If $G := B - \Lambda \Sigma' (\Sigma\Sigma')^{-1}\hat{A}$ is stable then $K_1 - \Lambda K = B - \Lambda \Sigma' (\Sigma\Sigma')^{-1}\hat{A}$ is stable.

6.2 Main Result

The first main result and its proof are analogous to Proposition 2.2 in Kuroda and Nagai [7].

Proposition 1. (i). If

$$G := B - \Lambda \Sigma' (\Sigma\Sigma')^{-1}\hat{A} \text{ is stable} \quad (28)$$
then \( Q(0) = Q(0; T) \) converges as \( T \to +\infty \) to a nonegative definite matrix \( \tilde{Q} \), which is a solution of the algebraic Riccati equation:
\[
K_1' \tilde{Q} + \tilde{Q}K_1 - \tilde{Q}K_0 \tilde{Q} + \frac{2}{\theta + 2} \hat{A}'(\Sigma \Sigma')^{-1} \hat{A} = 0 \tag{29}
\]
Moreover, \( \tilde{Q} \) satisfies the estimate
\[
0 \leq \tilde{Q} \leq 2 \int_0^{+\infty} e^{sG} \hat{A}'(\Sigma \Sigma')^{-1} \hat{A} e^{sG} ds \tag{30}
\]

(ii). In addition, \( q(0) = q(0; T) \) converges as \( T \to +\infty \) to a constant vector \( \tilde{q} \), which satisfies
\[
\left( K_1' - \tilde{Q}(t)K_0 \right) \tilde{q}(t) + \tilde{Q}(t)b + \frac{2}{\theta} \tilde{Q}'(t) \Lambda \gamma + \varsigma - \beta \\
= \frac{1}{\theta + 2} \left( 2\hat{A}' - \theta \tilde{Q}'(t) \Lambda \Sigma' \right) (\Sigma \Sigma')^{-1} \left( \tilde{a} + \frac{\theta}{2} \Sigma \gamma \right) = 0 \tag{31}
\]
Moreover, \( \frac{q(0; T)}{T} \) converges to a constant \( \rho(\theta) \) defined by
\[
\rho(\theta) = \frac{1}{2} \text{tr} \left( \Lambda \Lambda' \tilde{Q}(t) \right) - \frac{\theta}{4} \tilde{q}'(t) \Lambda \Lambda' \tilde{q}(t) + b' \tilde{q}(t) + \frac{1}{\theta + 2} \tilde{q}'(t) (\Sigma \Sigma')^{-1} \tilde{a} \\
= \frac{1}{\theta + 2} \frac{\theta^2}{4} \tilde{q}'(t) \Lambda \Sigma' (\Sigma \Sigma')^{-1} \Sigma \Lambda' \tilde{q}(t) - \frac{\theta}{\theta + 2} \tilde{q}'(t) \Lambda \Sigma' (\Sigma \Sigma')^{-1} \tilde{a} \\
- \frac{1}{\theta + 2} \frac{\theta^2}{8} \tilde{q}'(t) \Lambda \Sigma' (\Sigma \Sigma')^{-1} \Sigma \gamma + \frac{\theta}{2} \gamma' \Lambda \tilde{q}(t) \\
= \frac{1}{2} \left( \frac{\theta}{2} - 1 \right) \gamma' + \frac{\theta}{\theta + 2} \tilde{a}'(\Sigma \Sigma')^{-1} \Sigma \gamma + \frac{1}{\theta + 2} \frac{\theta^2}{4} \gamma' (\Sigma \Sigma')^{-1} \Sigma \gamma \\
+ \eta - \alpha \tag{32}
\]

(iii). If, in addition to condition (28) we assume that
\[
(B', \hat{A}'(\Sigma \Sigma')^{-1} \Sigma) \text{ is controllable} \tag{33}
\]
then the solution \( \tilde{Q} \) of equation (29) is strictly positive definite.

6.3 Ergodic Bellman Equation

From the Proposition, we deduce the ergodic Bellman PDE:
\[
\rho = \sup_{h \in \mathbb{R}^m} L^h_t \Phi, \quad x \in \mathbb{R}^n \tag{34}
\]
where
\[
L^h_t \Phi = \left( b + BX - \frac{\theta}{2} \Lambda (\Sigma' h - \gamma) \right)' \Phi + \frac{1}{2} \text{tr} \left( \Lambda \Lambda' D^2 \Phi \right) \Phi - \frac{\theta}{4} (D \Phi)' \Lambda \Lambda' D \Phi - g(X, h; \theta)
\]
has a solution given by the pair of functions \( w = \frac{1}{2}x'\tilde{Q}x + \tilde{q}x \) and \( \rho(\theta) \) defined above.

As in Kuroda and Nagai [7], we define \( \mathcal{A} \) as the set of investment strategies \( h \) such that \( h \in \mathcal{A}(T) \forall T \). Such strategy is said to be admissible. We can now extend Theorem 2.3.

**Theorem 2.** Assume \( \theta > 0 \)

(i). Assuming equation (28), we have

\[
\sup_{h \in \mathcal{A}} J(v, x; h) \leq \rho(\theta)
\]

(ii). If, in addition to equation (28), we assume condition (33), that

\[
(B', \Lambda) \text{ is controllable}
\]

and also assume that

\[
\dot{Q}\Lambda \Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' \dot{Q} < \dot{A}'(\Sigma\Sigma')^{-1}\dot{A}
\]

where \( \dot{Q} = \frac{\theta}{2}\tilde{Q} \) and \( \tilde{Q} \) is the nonegative definite solution of the algebraic Riccati equation (29), then the investment strategy \( \tilde{h}_t \) defined by

\[
\tilde{h}_t = \frac{2}{\theta + 2}(\Sigma\Sigma')^{-1} \left( \hat{a} + \frac{\theta}{2}\Sigma\gamma - \frac{\theta}{2}\Sigma\Lambda'\tilde{q}(t) + \left( \dot{A} - \frac{\theta}{2}\Sigma\Lambda'\tilde{Q}(t) \right) X_t \right)
\]

is optimal for the control problem:

\[
\sup_{h \in \mathcal{A}} J(v, x; h) = J(v, x; \tilde{h}) = \rho(\theta)
\]

**Proof.** The proof follows from Kuroda and Nagai [7] and from the proof of Theorem 1 presented earlier in this paper. \( \square \)

### 6.4 Asymptotic Behaviour of the Solution as \( \theta \to 0 \)

As \( \theta \to 0 \), then

\[
K_0 \to 0 \\
K_1 \to B
\]
Over a finite time horizon, the coefficients of the value function given in equations (19), (21) and (22) become

\[ \dot{Q}(t) + B'Q(t) + Q(t)B + \hat{A}'(\Sigma\Sigma')^{-1}\hat{A} = 0, \]
\[ \dot{q}(t) + B'q(t) + Q(t)b + \zeta - \beta + \hat{A}'(\Sigma\Sigma')^{-1}\hat{a} = 0, \]
\[ k(s) = \ln f_0 + \int_s^T \frac{1}{2} \text{tr} \left( \Lambda\Lambda' \hat{Q}(t) \right) + b'\hat{q}(t) + \frac{1}{2} \hat{a}'(\Sigma\Sigma')^{-1}\hat{a} + \frac{1}{2} \gamma'\gamma + \eta - \alpha = 0, \quad s \leq T \]

and the investment strategy \( h^* \in A(T) \) can be expressed as

\[ h^*(t) = (\Sigma\Sigma')^{-1} \left( \hat{a} + \hat{A}X(t) \right) \]

Over an infinite time horizon, the coefficients of the value function given in equations (29), (31) and (32) become

\[ B'\tilde{Q} + \tilde{Q}B + \hat{A}'(\Sigma\Sigma')^{-1}\hat{A} = 0 \]
\[ B'\tilde{q}(t) + \tilde{Q}(t)b + \zeta - \beta + A'(\Sigma\Sigma')^{-1}\hat{a} = 0 \]
\[ \rho(\theta) = \frac{1}{2} \text{tr} \left( \Lambda\Lambda' \tilde{Q}(t) \right) + b'\tilde{q}(t) + \frac{1}{2} \hat{a}'(\Sigma\Sigma')^{-1}\hat{a} + \frac{1}{2} \gamma'\gamma + \eta - \alpha \]

while the investment strategy \( \tilde{h} \in A \) remains defined as

\[ \tilde{h}(t) = (\Sigma\Sigma')^{-1} \left( \hat{a} + \hat{A}X(t) \right) \]

Over both a finite and the infinite time horizon, we note that the optimal investment strategy is now fully independent from the benchmark and is equal to the asset only optimal investment strategy in the case when \( \theta = 0 \). This finding confirms the conventional wisdom that for a log utility investor, benchmarks are irrelevant. Log utility investors aim at maximizing the growth rate of the portfolio in absolute term rather than relative to a benchmark. In fact, since Bielecki and Pliska [2] noted that the infinite-time horizon risk-sensitive control criterion is optimal sure in the sense of Foldes [6] and that the objective of a log utility investor is to maximize the growth rate of the portfolio, it is not surprising that the investment strategy for the infinite-time horizon problem is the same as the investment strategy for the finite-time horizon problem.

7 Mutual Fund Theorems

For a given time \( t \) and state vector \( X(t) \), the efficient frontier can be entirely parameterized using the risk-sensitivity, \( \theta \). This ideas is illustrated in the following Mutual Fund Theorems.
7.1 Asset Only Case

A mutual fund theorem can be established in the asset only case solved by Kuroda and Nagai [7].

**Theorem 3.** *(Asset Only Mutual Fund Theorem).* Given a time $t$ and a state vector $X(t)$, any portfolio can be expressed as a linear combination of investments into two “mutual funds” with respective risky asset allocations:

\[
\begin{align*}
    h^K(t) &= (\Sigma\Sigma')^{-1} \left( \hat{a} + \hat{A}X(t) \right) \\
    h^C(t) &= - (\Sigma\Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t)X(t))
\end{align*}
\]

and respective allocation to the money market account given by:

\[
\begin{align*}
    h^K_0(t) &= 1 - 1' (\Sigma\Sigma')^{-1} \left( \hat{a} + \hat{A}X(t) \right) \\
    h^C_0(t) &= 1 + 1' (\Sigma\Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t)X(t))
\end{align*}
\]

Moreover, if an investor has a risk sensitivity $\theta$, then the respective weights of each mutual fund in the investor’s portfolio are equal to $\frac{2}{\theta + 2}$ and $\frac{\theta}{\theta + 2}$.

**Proof.** In the asset only case, the optimal risk-sensitive asset allocation is given by:

\[
    h^*(t) = \frac{2}{\theta + 2} (\Sigma\Sigma')^{-1} \left( \hat{a} - \frac{\theta}{2} \Sigma \Lambda q(t) + \left( \hat{A} - \frac{\theta}{2} \Sigma \Lambda' Q(t) \right) X_t \right)
\]

Now, we denote by

\[
\begin{align*}
    h^K(t) &= (\Sigma\Sigma')^{-1} \left( \hat{a} + \hat{A}X(t) \right) \\
    h^C(t) &= - (\Sigma\Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t)X(t))
\end{align*}
\]

the risky asset allocation of funds $K$ and $C$. We see the risky allocation of any optimal portfolio is a linear combination of investments in mutual funds $K$ and $C$, with respective asset allocation $\frac{2}{\theta + 2}$ and $\frac{\theta}{\theta + 2}$.

Now, since

\[
    h(t) = \frac{2}{\theta + 2} h^K(t) + \frac{\theta}{\theta + 2} h^C(t)
\]

then, by the budget equation

\[
\begin{align*}
    h_0(t) &= 1 - 1'h(t) \\
    &= \frac{2}{\theta + 2} (1 - 1'h^K(t)) + \frac{\theta}{\theta + 2} (1 - 1'h^C(t)) \\
    &= \frac{2}{\theta + 2} h^K_0(t) + \frac{\theta}{\theta + 2} h^C_0(t)
\end{align*}
\]
where

\[
\begin{align*}
h^K_0(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\
h^C_0(t) &= 1 + \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\Lambda'(q(t) + Q(t)X(t))
\end{align*}
\]

**Remark 1:** as was expected, the asset allocation within funds \(K\) and \(C\) is independent from the investor’s risk aversion. As we saw in Subsection 6.4, when \(\theta \to 0\) the optimal portfolio becomes fund \(K\). Also, as \(\theta \to +\infty\) the optimal portfolio becomes fund \(C\). Fund \(C\) can be interpreted as a trading strategy trading on the comovement of assets and valuation factors.

**Remark 2:** when we assume that there are no underlying valuation factors, the risky securities follow geometric Brownian motions with drift vector \(\mu\) and the money market account becomes the risk-free asset (i.e. \(\eta = r\) and \(\zeta = 0\)). In this case \(\Sigma\Lambda' = 0\) and we can then easily see that fund \(C\) is fully invested in the risk-free asset. As a result, we recover Merton’s Mutual Fund Theorem for \(m\) risky assets and a risk-free asset.

### 7.2 Benchmarked Case

**Theorem 4.** *(Benchmark and Assets Mutual Fund Theorem).* Given a time \(t\) and a state vector \(X(t)\), any portfolio can be expressed as a linear combination of investments into two “mutual funds” with respective risky asset allocations:

\[
\begin{align*}
h^K(t) &= (\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\
h^C(t) &= (\Sigma\Sigma')^{-1}\left[\Sigma\gamma - \Sigma\Lambda'(q(t) + Q(t)X(t))\right]
\end{align*}
\]

and respective allocation to the money market account given by:

\[
\begin{align*}
h^K_0(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\
h^C_0(t) &= 1 + \mathbf{1}'(\Sigma\Sigma')^{-1}\left[\Sigma\gamma - \Sigma\Lambda'(q(t) + Q(t)X(t))\right]
\end{align*}
\]

Moreover, if an investor has a risk sensitivity \(\theta\), then the respective weights of each mutual fund in the investor’s portfolio are equal to \(\frac{2}{\theta + 2}\) and \(\frac{\theta}{\theta + 2}\).

**Proof.** The proof for this theorem follows naturally from the proof of theorem 3.

**Corollary 1.** *(Geometric Brownian Motion.)* When the risky assets follow a Geometric Brownian Motion with drift vector \(\mu\) and the money market account is risk-free (i.e. \(\eta = r\) and \(\zeta = 0\)), then any optimal portfolio can be
expressed as a linear combination of investments into two “mutual funds” with respective asset allocations

\begin{align*}
  h^K(t) &= (\Sigma'\Sigma')^{-1}(\mu - r1) \\
  h^C(t) &= (\Sigma'\Sigma')^{-1}\Sigma\gamma
\end{align*}

and respective allocation to the money market account given by:

\begin{align*}
  h^K_0(t) &= 1 - 1'(\Sigma'\Sigma')^{-1}(\mu - r1) \\
  h^C_0(t) &= 1 - 1'(\Sigma'\Sigma')^{-1}\Sigma\gamma
\end{align*}

Moreover, if an investor has a risk sensitivity \( \theta \), then the respective weights of each mutual fund in the investor’s portfolio are equal to \( \frac{2}{\theta+2} \) and \( \frac{\theta}{\theta+2} \).

### 7.3 The Role of The Risk Sensitivity Parameter \( \theta \)

One of the main differences between these mutual fund theorems and Merton’s mutual fund theorem is in the role played by the risk sensitivity parameter, \( \theta \). In our mutual fund theorems, the respective asset allocations of the mutual funds are fixed, up to a change in the state vector \( X(t) \). Moreover, the respective allocation of each mutual fund in the investor’s portfolio is directly and solely determined by the risk sensitivity parameter \( \theta \).

However, while in the asset only case \( \theta \) represents the sensitivity of an investor to total risk, in the RSBAM, \( \theta \) seems rather to represent the investor’s sensitivity to active risk. When \( \theta \) is low, the investor will take more active risk by investing larger amounts into the log-utility portfolio. On the other hand, when \( \theta \) is high, the investor will divert most of his/her funds to the correction fund, which is dominated by the term \( (\Sigma'\Sigma')^{-1}\Sigma\gamma \), a term designed to track the index. Hence, in the RSBAM, the investor already takes the benchmark risk as granted. The main unknown is therefore how much additional risk the investor is willing to take in order to outperform the benchmark.

To confirm this intuition and further clarify the role of the benchmark, we will next consider the case when the benchmark is itself a portfolio of traded assets.

### 8 Special Case: Traded Benchmark

So far, we have modelled the benchmark independently from the asset market. An interesting question arises in the case when the benchmark is itself a traded asset.
8.1 Benchmark as a Portfolio of Risky Assets

We will first consider the case when the benchmark is a constant proportion strategy invested in a combination of traded assets. The benchmark dynamics can be expressed as:

\[
\frac{dL_t}{L_t} = \nu'(a + AX(t))dt + \nu'\Sigma dW_t
\]

where \( \nu \) is a \( m \)-element allocation vector satisfying the budget equation:

\[ 1'\nu = 1 \]

The corresponding optimal asset allocation for the finite-time horizon problem is

\[
h^*(t) = \frac{2}{\theta + 2} (\Sigma \Sigma')^{-1} \left( \hat{a} + \frac{\theta}{2} \Sigma \Sigma' \nu - \frac{\theta}{2} \Sigma \Lambda' q(t) + \left( \hat{A} - \frac{\theta}{2} \Sigma \Lambda' \Sigma Q(t) \right) X_t \right)
\]

As \( \theta \to 0 \),

\[
h^*(t) \to (\Sigma \Sigma')^{-1} \left( \hat{a} + \hat{A} X_t \right)
\]

and we recover the log utility optimal portfolio.

As \( \theta \to +\infty \),

\[
h^*(t) \to \nu - (\Sigma \Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t)X_t)
\]

The resulting investment strategy can be decomposed into two elements. The first one, \( \nu \), replicates the index by holding the constituting securities in the proportion dictated by the definition of the index and encoded in the vector \( \nu \). The second element is a risk adjustment trade, which, when combined with the allocation to the money market account can be interpreted as a “long-short macro hedge fund” with zero net weight, so that

\[
h^*_0(t) - 1'(\Sigma \Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t)X_t) = 0
\]
The \( n \)-element vector \( q(t) \) satisfies the linear ordinary differential equation

\[
\dot{q}(t) + (K_1' - Q(t)K_0)q(t) + Q(t)b + \frac{\theta}{\theta + 2} Q(t)\Lambda\Sigma'\nu - \frac{2}{\theta + 2} \hat{A}'\nu \\
+ \frac{1}{\theta + 2} \left( 2\hat{A}' - \theta Q'(t)\Lambda\Sigma' \right) (\Sigma\Sigma')^{-1}\hat{a} \\
= 0
\]

with terminal condition \( q(T) = 0 \).

For \( \theta \in [0, +\infty[ \), we can extend Theorem 4 to show that the optimal investment strategy can now be separated into a mutual fund, an index fund and a “long-short hedge fund”:

**Corollary 2. (Fund Separation Theorem with a Constant Proportion Benchmark (I)).** Given a time \( t \) and a state vector \( X(t) \), any portfolio can be expressed as a linear combination of investments into a “mutual funds”, an index fund and a “long-short hedge fund” with respective risky asset allocations:

\[ h^K(t) = (\Sigma\Sigma')^{-1} \left( \hat{a} + \hat{A}X(t) \right) \]
\[ h^I(t) = \nu \]
\[ h^H(t) = -(\Sigma\Sigma')^{-1}\Sigma\Lambda' (q(t) + Q(t)X(t)) \]  

and respective allocation to the money market account given by:

\[ h^K_0(t) = 1 - 1'(\Sigma\Sigma')^{-1} \left( \hat{a} + \hat{A}X(t) \right) \]
\[ h^I_0(t) = 0 \]
\[ h^H_0(t) = 1'(\Sigma\Sigma')^{-1}\Sigma\Lambda' (q(t) + Q(t)X(t)) \]  

Moreover, if an investor has a risk sensitivity \( \theta \), then the respective weights of each mutual fund in the investor’s portfolio are equal to \( \frac{2}{\theta + 2} \), \( \frac{\theta}{\theta + 2} \) and \( \frac{\theta}{\theta + 2} \).

**Proof.** This Corollary can be proved in a similar fashion to Theorem 4 after taking into consideration our findings related to the asymptotic behaviour of the optimal strategy when \( \theta \to 0 \) and \( \theta \to +\infty \). \( \square \)

### 8.2 Benchmark as a Static Portfolio of Risky Assets and the Bank Account

The other case to consider is that of a benchmark defined as constant proportion strategy invested in risky assets and in the money market account.
In this case, the benchmark dynamics can be expressed as:

\[
\frac{dL_t}{L_t} = (\eta + \zeta' X(t)) + \nu'(t)(\hat{a} + \hat{A}X(t))dt + \nu'(t)\Sigma dW_t \\
= \left[ (1 - \nu' 1) \eta + \nu' a + ((1 - \nu' 1)\zeta' + \nu' A) X(t) \right] dt + \nu'(t)\Sigma dW_t
\]

where \( \nu \) is a \( m \)-element allocation vector satisfying the budget equation:

\[
1' \nu = 1 - h_0^L
\]

and \( h_0^L \) is the allocation left in the money market account.

The optimal risky asset allocation for the finite-time horizon problem is

\[
h^*(t) = \frac{2}{\theta + 2} (\Sigma \Sigma')^{-1} \left( \hat{a} + \hat{A}X_t \right) + \frac{\theta}{\theta + 2} \nu \\
- \frac{\theta}{\theta + 2} (\Sigma \Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t)X_t)
\]

As \( \theta \to 0 \),

\[
h^*(t) \to (\Sigma \Sigma')^{-1} \left( \hat{a} + \hat{A}X_t \right)
\]

and we recover the log utility optimal portfolio.

As \( \theta \to +\infty \),

\[
h^*(t) \to \nu - (\Sigma \Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t)X_t)
\]

The resulting investment strategy can be decomposed into two elements. The first one, \( \nu \) replicates the risky component of the index by holding the constituting securities in the proportion dictated by the definition of the index and encoded in the vector \( \nu \). Applying the budget equation, we note that

\[
h_0^L = 1 - 1' \nu
\]

so that when \( 1' \nu = 1 \), \( h_0^L = 0 \) and we recover the “static portfolio of risky asset” case considered earlier.

Studying the allocation to the money market account shows that the second element,

\[-(\Sigma \Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t)X_t)\]

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can be interpreted as a risk adjustment or “long-short macro hedge fund” with zero net weight. Indeed, its allocation to the bank account, \( h_0^L(t) \) is given by

\[
h_0^L(t) = 1 - 1' \left[ I_m \nu + (\Sigma \Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t) X_t) \right] \\
= 1 - 1 + h_0^L - 1' (\Sigma \Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t) X_t)
\]

i.e.

\[
1' (\Sigma \Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t) X_t) = 0
\]

Hence, as we found out in previous case, as \( \theta \to +\infty \), the optimal allocation becomes fully invested in the benchmark plus a zero net weight risk adjustment portfolio. The difference here is that the zero net weight risk adjustment portfolio is entirely comprised of risky assets.

Here again, for \( \theta \in [0, +\infty] \), we can extend Theorem 4 to show that the optimal investment strategy can be separated into a mutual fund, an index fund and a “long-short hedge fund”:

**Corollary 3.** (Fund Separation Theorem with a Constant Proportion Benchmark (II)). Given a time \( t \) and a state vector \( X(t) \), any portfolio can be expressed as a linear combination of investments into a “mutual funds”, an index fund and a “long-short hedge fund” with respective risky asset allocations:

\[
\begin{align*}
h^K(t) &= (\Sigma \Sigma')^{-1} \left( \hat{a} + \hat{A} X(t) \right) \\
h^I(t) &= \nu \\
h^H(t) &= - (\Sigma \Sigma')^{-1} \Sigma \Lambda' (q(t) + Q(t) X(t))
\end{align*}
\]

and respective allocation to the money market account given by:

\[
\begin{align*}
h^K_0(t) &= 1 - 1' (\Sigma \Sigma')^{-1} \left( \hat{a} + \hat{A} X(t) \right) \\
h^I_0(t) &= 1 - 1' \nu \\
h^H_0(t) &= 0
\end{align*}
\]

Moreover, if an investor has a risk sensitivity \( \theta \), then the respective weights of each mutual fund in the investor’s portfolio are equal to \( \frac{2}{\theta + 2}, \frac{\theta}{\theta + 2} \) and \( \frac{\theta}{\theta + 2} \).

**Proof.** This Corollary can be proved in a similar fashion to Theorem 4 after taking into consideration our findings related to the asymptotic behaviour of the optimal strategy when \( \theta \to 0 \) and \( \theta \to +\infty \).

**Remark:** when there are no valuation factors and the risky securities follow geometric Brownian motion, what happens if the investor’s benchmark is the log utility portfolio \( K \)? Applying Corollary 3 to \( \nu = h^K(t) \) we
deduce that the investor’s asset allocation is a 100% investment in the log utility portfolio. Indeed, since the log utility portfolio is growth maximizing, no investor can hope to outperform it. Therefore, the only possible strategy for an investor benchmarked against the log utility portfolio is to fully invest in the benchmark.

9 Application of the RSBAM to Investment Management

At this stage, the RSBAM model is still a stylized model which cannot be used directly to solve the sort of asset allocation problems arising in the financial industry. Indeed, it does not consider constraints, transaction costs or taxes and assumes that the investment strategy can be rebalanced continuously, which is, of course, greatly impractical in reality.

However, as an analytical model, the RSBAM can be used to gain insights into the economics of concrete investment management problems. Not only can it be used to obtain an idealized solution to an asset allocation problem in a fairly complex setting with an arbitrary number of assets and underlying valuation factors, but it can also be used to extend asset management theory and try to address some current issues. In particular, we can apply the insights gained from the RSBAM model to extend the definition of fractional Kelly strategies.

A full Kelly strategy is an investment strategy aiming at maximizing the long-term growth rate of wealth. It therefore invests in the log-utility optimal portfolio. As pointed out by MacLean et al. [8], such a strategy is very risky. An alternative is to consider the class of fractional Kelly strategies. In a fractional Kelly strategy, a proportion $k$ of the wealth is invested in the log utility optimal portfolio, while the remaining $1 - k$ is invested in the risk-free asset (for an overview of the properties of fractional Kelly strategies, refer to [13]).

Fractional Kelly strategies have a certain theoretical and practical appeal. The main appeal is that they keep most of the nice properties of a log utility investment while significantly reducing the risk associated with a log utility investment strategy. In addition, fractional Kelly strategies arise naturally in the Merton investment model. Indeed, one of Merton’s mutual fund theorems guarantees that the optimal strategy can be split in an allocation to the risk-free asset and an allocation to a mutual fund investing in the log utility optimal portfolio. In Merton’s lognormal world, fractional Kelly strategies are therefore optimal investment strategies. However, McLean
and Ziemba acknowledge that the optimality of fractional Kelly strategy is the exception rather than the rule, and that, in the Merton model, optimality is entirely due to the assumption of lognormality of asset prices.

What happens in the risk-sensitive world? Theorem 4 proves that any optimal portfolio can be split into an allocation to the log utility optimal (or Kelly) portfolio and a correction fund (related to the benchmark and to a risk-adjustment trade). As a result of the presence of the correction fund, fractional Kelly strategies are suboptimal. Moreover, the inadequacy of fractional Kelly strategies increases as the investor’s risk aversion, \( \theta \), increases. Indeed, while for small \( \theta \) the optimal portfolio will be mostly invested in the log utility optimal portfolio, with only a relatively small allocation to the correction fund, as \( \theta \) becomes large the situation changes and most of the allocation is made to the correction fund.

If we restrict the risk-sensitive model to a lognormal model, then Corollary 1 introduces a split of the optimal strategy into an allocation to the log utility optimal portfolio and an allocation to a correction fund solely related to the relationship between assets and benchmark. Fractional Kelly strategies are therefore not optimal in this case either.

However, read differently, Theorem 4 hints that the basic concept of fractional Kelly strategies, i.e. combining the log-utility portfolio with another type of investment, could be extended in the risk-sensitive case. Indeed we could introduce a “benchmark adjusted fractional Kelly strategy” splitting the optimal portfolio between a fraction \( k \) (with \( k = \frac{2}{\theta + 2} \)) invested in the log-utility portfolio and a fraction \( 1 - k \) invested in the correction fund rather than the risk-free asset.

To conclude, this widened definition of fractional Kelly strategies has three main advantages. First, benchmark-adjusted fractional Kelly strategies are consistent with the definition of fractional Kelly strategies when investors do not have a benchmark and assume that there is no underlying valuation factor. Second, by construction and by Theorem 4, benchmark-adjusted fractional Kelly strategies yield optimal portfolios in the general case. Third, our definition of benchmark-adjusted fractional Kelly strategies straightforwardly links investor’s risk sensitivity, \( \theta \), with the fraction \( k \) used to parameterize the family of fractional Kelly strategies.

10 Conclusion

Risk-sensitive control provides a powerful theoretical framework to help address complex portfolio selection problems, such as the benchmarked invest-
ment question.

Through a careful formulation of the benchmarked investment problem we have developed a Risk Sensitive Benchmarked Asset Management model yielding an elegant analytical solution no more difficult to derive than the solution to the classical optimal investment problem considered by Bielecki and Pliska [2] or Kuroda and Nagai [7].

The insights gained from the RSBAM successfully address the four questions we originally set out to solve. Our formulation of the problem shares a financial and a utility-based interpretation. Moreover, the optimal strategy can intuitively be decomposed via mutual fund theorems into a linear combination of the log-utility risky portfolio and a correction portfolio. In addition, when the benchmark is traded, the correction portfolio can be further decomposed into an “indexed” component, replicating the benchmark, and a “risk adjustment” component, trading the comovement between assets and factors.

These two levels of decomposition further emphasize the importance of risk aversion in a manager’s investment strategy. A risk-averse manager will tend to track the benchmark very closely, giving up appreciation potential in order to lower the chance of falling behind the index. On the other hand, a risk insensitive manager will tend to ignore the benchmark and look for a maximization of the long-term growth of the portfolio. This finding therefore highlights the importance of ensuring that a manager’s risk-aversion is in line with the risk-aversion of his/her investors.

References


