A NOTE ON THE VIX FORMULA

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If \((S_t, t \in [0, T])\) denotes the price of a financial asset, the realized variance is defined as

\[
RV_T = \sum_{i=1}^{n} \left( \log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2,
\]

where \(t_i\) is a pre-specified sequence of times \(0 = t_0 < t_1 < \cdots < t_n = T\), in practice daily sampling. A variance swap is a forward contract in which cash amounts equal to \(A \times RV_T\) and \(A \times PRV_T\) are exchanged at time \(T\), where \(A\) is the dollar value of one variance point and \(PRV_T\) is the variance swap ‘price’, agreed at time 0.

In the classical approach, if we model \(S_t\) under a risk-neutral measure \(Q\) as

\[
S_t = F_t \exp(X_t - \frac{1}{2} \langle X \rangle_t),
\]

where \(F_t\) is the forward price (assumed to be continuous and of bounded variation) and \(X_t\) is a continuous martingale with quadratic variation process \(\langle X \rangle_t\), then \(S_t\) is a continuous semi-martingale and

\[
\log \frac{S_t}{S_{t_{i-1}}} = (g(t_i) - g(t_{i-1})) + (X_{t_i} - X_{t_{i-1}}),
\]

with \(g(t) = \log(F_t) - \frac{1}{2} \langle X \rangle_t\). For \(m = 1, 2, \ldots\) let \(\{s^m_i, i = 0, \ldots, k_m\}\) be the ordered set of stopping times in \([0, T]\) containing the times \(t_i\) together with times \(\tau^m_0 = 0, \tau^m_k = \inf\{t > \tau^m_{k-1} : |X_t - X_{\tau^m_{k-1}}| > 2^{-m}\}\) wherever these are smaller than \(T\). If \(RV^m_T\) denotes the realized variance computed as in (1) but using the times \(s^m_i\), then \(RV^m_T \rightarrow \langle X \rangle_T = \langle \log S \rangle_T\) almost surely, see Rogers and Williams, Theorem IV.30.1. For this reason, most of the pricing literature on realized variance studies the continuous-time limit \(\langle X \rangle_T\) rather than the finite sum (1).

In this note we take, for convenience, interest rates and dividend yields as zero, so that \(F_t = S_0\) and \(S_t\) is a continuous martingale. A standing assumption is that

\[
\mathbb{E} \int_0^T \frac{1}{S_t} d\langle S \rangle_t < \infty,
\]

where \(\mathbb{E}\) denotes the \(Q\)-expectation.

1. THE VIX FORMULA

The key insight into the analysis of variance derivatives in the continuous limit was provided by Neuberger (1994). With \(S_t\) as above and \(f\) a \(C^2\) function, it follows from the general Itô formula that \(Y_t = f(S_t)\) is a continuous semimartingale with \(d\langle Y \rangle_t = (f'(S_t))^2 d\langle S \rangle_t\). In particular, with the above notation

\[
d(\log S_t) = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} d\langle S \rangle_t = \frac{1}{S_t} dS_t - \frac{1}{2} d(\log S)_t,
\]

so that

\[
\langle \log S \rangle_T = 2 \int_0^T \frac{1}{S_t} dS_t - 2 \log(S_T/S_0)
\]

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This shows that the realized variation is replicated by a portfolio consisting of a self-financing trading strategy that invests a constant $2 in the underlying asset $S$ together with a European option whose exercise value at time $T$ is $-2 \log(S_T/S_0)$. Assuming that the stochastic integral is a martingale, (3) shows that the risk-neutral value of the variance swap rate $P_T^{RV}$ is

$$P_T^{RV} = \mathbb{E}\left[\langle\log S \rangle_T\right] = -2\mathbb{E}[\log(S_T/F_T)] = 2P_{log},$$

that is, the variance swap rate is equal to the forward value $P_{log}$ of two ‘log contracts’—European options with exercise value $-\log(S_T/S_0)$, a convex function of $S_T$. (4) gives us a way to evaluate $P_T^{RV}$ in any given model. The next step is to rephrase $P_{log}$ in terms of call and put prices only.

For a $C^2$ function $f$ the exact 2nd-order Taylor formula can be expressed as

$$f(x) = f(x_0) + f_+'(x_0)(x-x_0) + \int_{x_0}^{x} (y-x)^+f''(dy) + \int_{x_0}^{\infty} (x-y)^+f''(dy).$$

In particular

$$\log(x/x_0) = \frac{1}{x_0}(x-x_0) + \int_{0}^{x_0} \frac{y-x}{y^2}dy + \int_{x_0}^{\infty} \frac{(x-y)}{y^2}dy.$$ 

Applying this formula with $x = S_T, x_0 = S_0$ and combining with (3) gives

$$\frac{1}{2}(X)_T = \int_{0}^{T} \frac{1}{S_t}dS_t + \frac{1}{S_0}(S_T - S_0) + \int_{0}^{S_0} \frac{y-S_T}{y^2}dy + \int_{S_0}^{\infty} \frac{(S_T - y)}{y^2}dy.$$ 

The first two terms on the right of (7) have expectation zero, due to standing assumption (2). Assuming that puts and calls maturing at $T$ are available for all strikes $K \in \mathbb{R}^+$ at traded prices $P(T, K), C(T, K)$, (7) provides a perfect hedge for the realized variance in terms of self-financing dynamic trading in the underlying (the first four terms) and a static portfolio of puts and calls and hence uniquely specifies the variance swap rate $P_T^{RV}$ as

$$P_T^{RV} = \mathbb{E}[\langle X \rangle_T] = 2\int_{0}^{\infty} \frac{1}{y^2}\{P(T, y)1_{(y \leq S_0)} + C(T, y)1_{(y > S_0)}\}dy.$$ 

We define the corresponding implied volatility $\sigma$ by

$$\mathbb{E}[\langle X \rangle_T] = \sigma^2 T,$$

so that

$$\sigma^2 = \frac{2}{T} \int_{0}^{\infty} \frac{1}{y^2}\{P(T, y)1_{(y \leq S_0)} + C(T, y)1_{(y > S_0)}\}dy.$$ 

For practical reasons it may be necessary to split the integrand at some strike level $K_0$ not equal to the forward price $S_0$. Taking $K_0 < S_0$, we have

$$\int_{0}^{S_0} = \int_{0}^{K_0} + \int_{K_0}^{S_0}, \quad \int_{S_0}^{\infty} = \int_{K_0}^{\infty} - \int_{K_0}^{S_0}.$$ 

Hence the last two terms in (7) are equal to

$$\int_{0}^{K_0} \frac{(y-S_T)^+}{y^2}dy + \int_{K_0}^{\infty} \frac{(S_T - y)^+}{y^2}dy + \int_{K_0}^{S_0} \frac{(S_T - y)^+ - (S_T - y)^+}{y^2}dy.$$ 

The expectation of the last term is

$$\int_{K_0}^{S_0} \frac{P(y, T) - C(y, T)}{y^2}dy = \int_{K_0}^{S_0} \frac{(y - S_0)}{y^2}dy = \log \left( \frac{S_0}{K_0} \right) - \frac{S_0 - K_0}{K_0},$$
where the first equality uses put-call parity. Thus, from (9), we have the final implied volatility formula

\[
\sigma^2 = \frac{2}{T} \left[ \int_{0}^{\infty} \frac{1}{y^2} \{ P(T, y) 1_{(y \leq K_0)} + C(T, y) 1_{(y > K_0)} \} dy + \log \left( \frac{S_0}{K_0} \right) - \frac{S_0 - K_0}{K_0} \right]
\]

In the definition of the VIX formula, \( K_0 \) is the highest traded strike less than the forward and the proportional gap \( x = (S_0 - K_0)/K_0 \) is small. The last two terms in (10) are \( \log(1 + x) - x \approx \frac{1}{2} x^2 \). Now suppose we have traded \( T \)-maturity call and put options with strikes \( K_i \) and mid prices \( Q_i \) (puts for \( K_i \leq K_0 \), calls for \( K_i > K_0 \)). Defining \( \Delta K_i = (K_{i+1} - K_{i-1})/2 \), we replace the integral in (10) by a finite-difference approximation to give

\[
\hat{\sigma}^2 = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} Q_i - \frac{1}{T} \left( \frac{S_0}{K_0} - 1 \right)^2.
\]

The sum is easily seen to be the trapezoidal approximation to the integral. By definition the VIX is

\[
\text{VIX} = 100 \times \hat{\sigma}.
\]

Note this is not an ‘approximation’. It is the formula used by the CBOE to compute the VIX index in terms of traded asset prices (mid-market values are used). We stated above that \( \hat{\sigma}^2 \) is an approximation to \( \sigma^2 \) given by (10), but it would be more accurate to say that \( \sigma^2 \) is an approximation to \( \hat{\sigma}^2 \).