Large Portfolio Credit Risk Modelling

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Abstract

A model for large portfolio credit risk is developed by using results on the asymptotic behaviour of stochastic networks. An efficient pricing technique is proposed using a newly-introduced quadrature algorithm. Accurate calibration to iTraxx tranche spreads is demonstrated.

Key Words: Stochastic Network, Functional Law of Large Numbers, Functional Central Limit Theorem, Quadratures.

1 Introduction

Rating based models have been widely used for pricing and risk management purposes. Maybe the best known is the Jarrow, Lando and Turnbull model [12]. More recent application of rating models in the context of Markov chains can be found in, for example, Frydman and Schuermann [7] and Lando and Christensen [3].

Even if there are some criticisms of the use of credit ratings for pricing purposes, it is appealing to consider having buckets of obligors with similar characteristics to simplify the mathematics while modelling the likelihood of obligors to modify their credit worthiness. This is particularly useful when handling large portfolios.

The use of latent variables and common factors has been common practice when dealing with large portfolios. Examples well known are the CreditMetrics model [10] and PortfolioRisk+ methodology [4]. Moreover, empirical evidence [8] suggests that the pattern of realized defaults is well represented by a latent variable model that postulates the existence of an exogenous process interpreted as a 2-state (growth/recession) economic variable, although introduction of further states can increase the explanatory power.

Finally, taking an approach of ratings and latent variables related with economic interpretation, it is sensible to assume that obligors move around rating categories at a faster time scale than the economic cycle.

These facts suggest a model in which obligors move around the rating categories at rates depending on the latent process and occasionally default. Moreover, there is an obvious analogy with stochastic networks in which ‘clients’ move around ‘service stations’ for processing. There is a huge literature on the analysis of stochastic networks—see Whitt [21] for a recent textbook account. In particular, work by Choudhury, Mandelbaum et al. [1] studies fluid and diffusion limits for stochastic networks under random environments. We give some modifications of their results in this paper, in a form suitable for application to credit portfolios.

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2 Model Description

Assume a portfolio with \( n \) obligors and \( m \) possible ratings \( 1, \ldots, m \). The initial rating composition of the portfolio is represented by the rating distribution vector \( q^n(0) \in \mathbb{R}^m \). The vector \( q^n(t) \) will represent the random rating distribution of the portfolio at a later time \( t \). This is, \( (q^n(t))_i \) is the number of obligors in the rating category \( i \) at time \( t \) (rather than a proportion).

We assume the transition probabilities are the same for all obligors in the same rating class. Then, \( \sum q^n_k(t) \) is the number of non defaulted obligors at time \( t \). When \( n = 5 \), \( m = 2 \), the state space of \( q^n(t) \) is as shown in figure 2. The figure shows all the possible movements given the current credit ratings: transitions (moves along the diagonal) and defaults (moves to the next diagonal).

![Figure 1: The Sample Space in the Simplest Case](image)

We assume as well there is a finite-state random environment process \( \{\xi_t, t \geq 0\} \) representing some macroeconomic (or sector associated) process that influences the default/transition rates of the obligors. The obligor credit events are independent conditional on the realisation of the environment process and follow a Markov chain with default/transition rates being defined as a function of the environment process. The dependence structure of the defaults in the portfolio comes from this environment process.

The finite state environment process defines different “layers” in which transition parameters are different. These multiple layers are copies of the initial state space shown in Figure 2. Now the process jumps across the layers whenever the random environment process \( \xi_t \) jumps to a different state as illustrated in figure 2. In this model the movements across the grid are expected to be more frequent than the movements across layers, particularly when the number of elements in the system increases. This is why in some literature this structure is referred as a slowly changing environment [15]. All we can observe however is the second coordinate.
of the pair process \(( \xi_t, q^n(t) )\). For the purposes of this paper we will consider the environment process \( \xi_t \) being a Markov chain with an associated matrix \( Q \), however some of the results hold for arbitrary processes.

Each credit events occurs as the first jump of a Poisson process with rate dependent of the state of the environment process \( \xi_t \). Transition rates from \( i \) to \( j \) are denoted \( \mu_{ij}(\xi) \), \( i, j = 1, ..., m \), \( i \neq j \). Default rates are denoted \( \mu_i(\xi) = \mu_{ii}(\xi) \), \( i = 1, ..., m \). Then, a credit event occurs according to the first jump of a Poisson process with rate

\[
\hat{\mu}_i(\xi) = \sum_{j=1}^{m} \mu_{ij}(\xi)
\]

(2.1)

Once a credit event occurs at time \( t \), the obligor defaults with probability \( \frac{\mu_i(\xi)}{\hat{\mu}_i(\xi)} \), while a transition to rate \( j \neq i \) has probability \( \frac{\mu_{ij}(\xi)}{\hat{\mu}_i(\xi)} \).

From the description above, for \( n > 0 \) the process \( q^n(t) \) can be expressed as the solution to the following integral equation

\[
q^n(t) = q^n(0) + \sum_{i=1}^{m} \sum_{j=1}^{m} N_{ij} \left( \int_{0}^{t} \mu_{ij}(\xi_s)q^n_i(s)ds \right) v_{ij}
\]

(2.2)

where for \( i, j = 1, ..., m \), \( N_{ij} \) is a collection of independent Poisson processes, and \( v_{ij} \) is the set of vectors defined as

\[
v_{ij} = \begin{cases} 
  e_j - e_i & j \neq i \\
  -e_i & i = j 
\end{cases}
\]

(2.3)

The generator matrix \( A(\xi) \) associated to the process is defined as

\[
A_{ij}(\xi) = \begin{cases} 
  \mu_{ij}(\xi) & j \neq i \\
  -\hat{\mu}_i(\xi) & i = j 
\end{cases}
\]

(2.4)

2.1 Formal Definition of the Model

The random environment process \( \xi_t \) is a finite state process in continuous time taking values in a set of indices \( \Xi \) and having at most a finite number of jumps in any bounded interval of \([0, \infty)\).

To construct the process \( q^n(t) \) we consider a collection of mutually independent Poisson processes \( \{N_i\}_{i \in I} \), for a set of indices denoted by \( I \). We define a collection of vectors \( \{v_i\} \) in \( \mathbb{R}^m \),
$m \in \mathbb{N}$, and a collection of non-negative functions of the form $\mu_{i,n} (\cdot, \cdot, \xi) : [0, \infty) \times \mathbb{R}^m \rightarrow [0, \infty)$ for all $i \in I$ and $\xi \in \Xi$. We assume each $\mu_{i,n} (t, \cdot, \xi)$ is continuous with respect to the second argument. More generally we can assume the function being Lipschitz bounded (see [5]), however for the purpose of this paper continuity is always assumed.

We define the probability space $(\Omega^1, \mathcal{F}^1, P^1)$ associated to the random environment process $\xi_t$, where $\Omega^1 = D([0, \infty), \Xi)$ the space of Right Continuous with Left Limits (RCLL) functions from $[0, \infty)$ into $\Xi$, $\mathcal{F}^1$ is a $\sigma$-algebra in $\Omega^1$ and $P^1$ is some probability measure. The measure space associated to the collection $\{A_i\}$ is defined by the state space $\Omega^2 = \Omega_1 \times \cdots \times \Omega_p$ where $\Omega_i = D([0, \infty), \mathbb{R}^n)$ and the minimum sigma-algebra $\mathcal{F}^2$ generated by $\Omega^2$. The probability measure $P^2$ is defined as the product measure. Finally, we assume independence between the environment process and each Poisson process $A_i$, therefore the probability space is defined as $(\Omega, \mathcal{F}, P)$ with $\Omega = \Omega^1 \times \Omega^2$, $\mathcal{F} = \sigma(\Omega)$ and $P = P^1 \times P^2$. We will denote by $\omega = (\omega_1, \omega_2)$ the elements of $\Omega$ being $\omega_1 \in \Omega^1$ and $\omega_2 \in \Omega^2$.

We define a mapping $\Upsilon$ from $\Omega$ into $D([0, \infty), \mathbb{R}^m)$ by $(\omega_1, \omega_2) \mapsto q$, where $q^n$ is the process solution to the equation

$$q^n(t) = q^n(0) + \sum_{i \in I} N_i \left( \int_0^t \mu_{i,n}(s, q^n(s), \xi_s) ds \right) v_i \quad (2.5)$$

The law of $q^n(t)$ is $P_n (\cdot) = P(\Upsilon^{-1}(\cdot))$. The probability conditional on the environment process $\xi_t$ denoted by $P_{n,\omega_t}$ is then defined by the conditional probability under the inverse mapping. Such processes is well defined (has a unique solution) as shown in Theorem 2.3.1 in [5].

In the following we will assume the random environment process $\xi_t$ is a finite-state Markov chain. Then we can characterise $P^1$ by the generator matrix $Q$ that defines the process.

### 3 Fluid and Diffusion Limits

The model as enunciated above can become numerically cumbersome as the dimension of the portfolio increases. However, relatively accurate and tractable analysis of the model can be done by studying the asymptotic behaviour of $q^n(t)$ as $n \rightarrow \infty$ conditional on the realisation of the environment process $\xi_t$ and using an appropriate scaling.

The model can be considered as a stochastic network where obligors represent clients in a service network that transit across different service stations (ratings) with infinite servers. Once a client is served it transits to a different service station (credit migration) or leaves the system (default). In this context $q^n(t)$ represent the state of the stochastic network at time $t$.

There is a wide interest in the convergence properties of queues and stochastic networks (queueing systems) according to different scalings—see Whitt [21]. A Functional Strong Law of Large Numbers (FSLLN) is derived for sequences of the form $q^n(t)/n$, this is commonly referred as the fluid limit. This strong convergence result is usually expressed as the unique solution to a partial differential equation and it helps to study the stability of the system. With the fluid limit $q^{(0)}(t)$ a weak convergence result can also be derived for sequences of the form

$$\frac{1}{\sqrt{n}} \left( q^n(t) - q^{(0)}(t) \right)$$

and is known as Functional Central Limit Theorem (FCLT) or diffusion limit. The FCLT gives a diffusion process that refines the approximation (in distribution) that may be obtained by the fluid limit.
Studies of the fluid and diffusion limits for queue applications can be traced to early works such as Iglehart and Whitt [11] and is still an area of development. An important branch of the development is based on heavy traffic condition assumptions, meaning an equilibrium between arrivals and service rates, leading to a stationary equilibrium in the state of the system. A less restrictive approach can be used when the system is Markov where by the use of random times convergence results are obtained, these kind of results has been studied in Kurtz [13] and Ethier and Kurtz [6], more recently Mandelbaum et al. [15] present results applicable to Markovian stochastic networks. Specific applications to the credit model explained above are presented below.

Slowly changing environments are processes within a system that change at a slower rate than the stochastic network or queue influencing its performance, that is the case of the random environment process in our credit model. An example is when the rate of service depends on an independent process and is kept constant when the number of clients increases. Choudhury et al. [1] show that for an infinite capacity server under slowly changing environments is possible to derive fluid and diffusion limits conditional on the realisation of the environment process. In this case, the fluid limit is the solution to a PDE whose parameters depend on the environment process. The diffusion limit is a Brownian diffusion conditional on the environment process.

In [5] we show that some of the results presented by [15] hold under slowly changing environments without any heavy traffic assumption. In particular the diffusion limit is given by a diffusion with stepwise constant diffusion coefficient and therefore the process can be completely defined by its covariance matrix obtained as the solution to the Lyapunov equation.

4 Convergence Results for the rating distribution process

In the following we present the convergence results for the credit model described in Section 2. These are applications of results presented in [5] and detailed proofs can be found there. These results are modifications of results presented by Choudhury et al. [1] concerning queueing networks.

4.1 The fluid limit

**Theorem 1** Assume we have a sequence of rating processes \((\xi, q^n)\) \(n = 1, 2, \ldots\) according to the Markov environment process \(\xi_t\) and to the rating process (2.2). Suppose that

\[
\frac{q^n(0)}{n} \xrightarrow{n \to \infty} q^{(0)}(0) \quad (4.1)
\]

Then for each \(\omega_1 \in \Omega^1\)

\[
\frac{q^n(t)}{n} \xrightarrow{n \to \infty} q^{(0)}(t) \quad (4.2)
\]

a.s. in \(P^\omega_1\), where \(q^{(0)}(t)\) defined in \(\mathbb{R}^m\) is the solution to the PDE system

\[
\frac{d}{dt}q^{(0)}(t) = A(\xi_t)q^{(0)}(t) \quad (4.3)
\]

that is deterministic conditional on the realisation of \(\xi\). Here the matrix-valued function \(A\) is given by (2.4)
This theorem is a special case of Theorem 2.3.2 in [5]. Since $\xi$ has at most finitely many jumps in any bounded interval of time, we can define the countable set of jump times of $\xi$, $t_0 = 0, t_1 < \ldots$, and define $q^n(t)$ recursively as
\[
q^n(t) = e^{(t-t_i)A(t_i)}q^n(t_i)
\]
for $t_i < t \leq t_{i+1}$.

The function $q^{(0)}(t)$ is called the fluid limit of the sequence $\{q^n(t)\}_{n \geq 0}$.

### 4.2 The diffusion limit

**Theorem 2** Assume we have a sequence of rating processes $(\xi, q^n) n = 1, 2, \ldots$ according to the Markov environment process $\xi$, and to the rating process 2.2. For $P^1$ a.e. $\omega \in \Omega^1$, if
\[
\lim_{n \to \infty} \sqrt{n} \left( \frac{q^n(0)}{n} - q^{(0)}(0) \right) = d q^{(1)}(0)
\]

w.r.t $P^{\omega_1}_Q$, then
\[
\lim_{n \to \infty} \sqrt{n} \left( \frac{q^n(t)}{n} - q^{(0)}(t) \right) = d q^{(1)}(t)
\]

w.r.t $P^{\omega_1}_Q$, where the process $q^{(1)}(t)$ takes values in $\Omega^2$ and it is the solution to the stochastic integral equation
\[
q^{(1)}(t) = q^{(1)}(0) + \int_0^t A_t q^{(1)}(t) + \sum_{k=1}^m \int_0^t \left( q^{(0)}_k(t) \right)^{1/2} B_k d\tilde{W}^k_t
\]

where $\tilde{W}$ is a $m$-dimensional vector of independent standard Brownian motions for $k = 1, \ldots, m$, and $A_t = A(\xi_t)$. The matrices $B_k$ have components
\[
(B_k(t))_{ij} = \begin{cases} 
-\mu_{ij}^{1/2} & \text{if } i = j \\
\mu_{ij}^{1/2} & \text{if } k = i \neq j \\
0 & \text{otherwise}
\end{cases}
\]

This is, conditional on the realisation of $\xi$ the limit rating process $q^{(1)}(t)$ is normally distributed for any $t > 0$.

The theorem can be seen as a special case of theorem 2.3.3 in [5].

It is always possible rewrite (4.7) as
\[
q^{(1)}(t) = q^{(1)}(0) + \int_0^t A_t q^{(1)}(t) + \int_0^t B(s) d\tilde{W}^k_t
\]

where $\tilde{W}$ is a $m$-dimensional vector of independent standard Brownian motions. In the case $m = 2$ the SDE is
\[
dq^{(1)}_1(t) = -q^{(1)}_1(t)(\mu_1(t) + \mu_{12}(t))dt + q^{(1)}_2(t)\mu_{12}(t)dt - (q^{(0)}_1(t))^{1/2}(\mu_{11}(t)d\tilde{W}^{(1)}_{1,t} + \mu_{12}(t)d\tilde{W}^{(1)}_{2,t}) + (q^{(0)}_2(t))^{1/2}(\mu_{12}(t))d\tilde{W}^{(1)}_{2,t}
\]
\[
dq^{(1)}_2(t) = -q^{(1)}_2(t)(\mu_2(t) + \mu_{21}(t))dt + q^{(1)}_1(t)\mu_{21}(t)dt - (q^{(0)}_2(t))^{1/2}(\mu_{22}(t)d\tilde{W}^{(2)}_{2,t} + \mu_{21}(t)d\tilde{W}^{(2)}_{1,t}) + (q^{(0)}_1(t))^{1/2}(\mu_{12}(t))d\tilde{W}^{(1)}_{1,t}
\]
That is equivalent to the following SDE system

\[ dq^{(1)}(t) = A_t q^{(1)}(t) dt + B(t) d\hat{W}_t \]  

(4.10)

where

\[
B(t) = \begin{pmatrix} \sigma_1(t) & 0 \\ \rho(t) \sigma_2(t) & \sqrt{1 - \rho^2(t)} \sigma_2(t) \end{pmatrix}
\]

(4.11)

\[ \sigma_1^2(t) = q_1^{(0)}(t)(\mu_1(t) + \mu_{12}(t)) + q_2^{(0)}(t)\mu_{12}(t) \]

(4.12)

\[ \sigma_2^2(t) = q_2^{(0)}(t)(\mu_2(t) + \mu_{21}(t)) + q_1^{(0)}(t)\mu_{12}(t) \]

(4.13)

\[ \rho(t)\sigma_1(t)\sigma_2(t) = -q_1^{(0)}(t)\mu_{12}(t) - q_2^{(0)}(t)\mu_{21}(t) \]

(4.14)

and

\[ \hat{W}_t = (\hat{W}_t^{(1)}, \hat{W}_t^{(2)})^t \]

(4.15)

is a standard Brownian motion in \( \mathbb{R}^2 \).

In general we can derive that the matrix \( \hat{B} = B'B \) has components

\[
\hat{B}_{ij}(t) = \begin{cases} 
q_1^{(0)}(t) \sum_k \mu_{ik} + \sum_k q_k^{(0)} \mu_{ki} & \text{if } i = j \\
-q_i^{(0)} \mu_{ij} - q_j^{(0)} \mu_{ji} & \text{otherwise}
\end{cases}
\]

(4.16)

On the interval \( [t_i, t_{i+1}) \), \( A_t = A \) is a constant matrix equal to \( A(\xi_t) \) and hence the solution of the SDE is given by

\[ q^{(1)}(t) = e^{(t-t_i)A} q^{(1)}(t_i) + \int_{t_i}^t e^{(t-s)A} B(s) d\hat{W}_s \]

(4.17)

with \( q^{(1)}(0) = 0 \). The process is a stable Gaussian system with covariance matrix given by the integral

\[ C_t = \text{Cov}[q^{(1)}(t), q^{(1)}(t)] = e^{(t-t_i)A} C_{t_i} e^{(t-t_i)A'} + \int_{t_i}^t e^{(t-s)A} B(s) B(s)' e^{(t-s)A'} ds \]

(4.18)

that can be calculated numerically by solving the Lyapunov matrix ordinary differential equation

\[
\frac{d}{dt} C_t = A C_t + C_t A' + B(s) B'(s),
\]

(4.19)

with the known initial condition at \( t = t_i \). This has a unique solution equal to the expression in (4.18).

Similarly to the fluid limit, conditional on the realisation of \( \xi_t \) and applying (4.17) - (4.19) piecewise we can express the solution as

\[ q^{(1)}(t) = e^{(t-t_i)A_{t_i}} q^{(1)}(t_i) + \int_{t_i}^t e^{(t-s)A_{t_i}} B(s) d\hat{W}_s \]

(4.20)

for \( t_i < t < t_{i+1} \) where \( t_i \) is the time of the \( i \)-th jump of \( \xi_t \).

Conditional in the realisation of the random environment, the process \( q^n(t) \) may be approximated by the two processes

\[ q^n(t) \simeq n q^{(0)}(t) + \sqrt{n} q^{(1)}(t) \]

where \( q^{(0)}(t) \) is deterministic and \( q^{(1)}(t) \) is a diffusion called the “diffusion limit” of the sequence \( q^n(t) \). This is, conditional on the random environment, the distribution of the process \( q^n(t) \), at any time \( t > 0 \), may be approximated by a normal distribution.
4.3 The infinitesimal generator of the single-obligor process and the probability of default

In this section we derive a the probability of default for a single obligor at time \( t > 0 \). This is important for calibration purposes as it allows us to price CDS in a closed form.

To obtain the probability of default of a single bond consider \( q^1_t \) the rating distribution process of a single bond portfolio under the random environment process \( \xi_t \) and initial rating \( 1 \leq k \leq m \). The pair process \((\xi_t, q^1_t)\) is then a Markov chain in continuous time with space state

\[ \Xi \times \{\{e_i; i = 1, ..., m\} \cup 0\} \]

and transition rates defined as

\[
\lambda((\eta, e_i) \rightarrow (\eta, e_j)) = \mu^n_{ij} \quad \text{for credit rate transitions},
\]

\[
\lambda((\eta, e_i) \rightarrow (\zeta, e_i)) = Q_{\eta\zeta} \quad \text{for environment state transitions},
\]

\[
\lambda((\eta, e_i) \rightarrow (\eta, 0)) = \nu^\eta_i \quad \text{for transition to default, for for } \eta, \zeta \in \Xi \]

\[
\lambda((\xi, 0) \rightarrow (\zeta, 0)) = Q_{\xi\zeta} \quad \text{for environment state transitions once default has occurred.}
\]

\( i, j = 1, ..., m. \)

The default states \((\xi, 0)\) are absorbent. Alternatively, we can think of the process as a Markov chain in the space \( \Xi \times \{e_i; i = 1, ..., m\} \) killed at a rate \( \mu^\xi_i \) in state \((\xi, e_i)\) and defaults corresponding to the cemetery state of the process.

The default probability at time \( t \) is then given by

\[
\mathbb{E}\{\mathbb{1}_{t \geq t}\} = \int_0^t P_s(1 - \sum q^1_s(t))ds
\]

where \( P \) is the infinitesimal generator of the pair process \((\xi, q^1_t)\).

Considering the process \( q^1(t) \) under the environment state \( \xi_t = \xi \) we know that the generator matrix \( A_\xi \in \mathbb{R}^{m+1 \times m+1} \) is given by

\[
(A_\xi)_{ij} = \begin{cases} 
-\sum_{k \neq i} \mu^\xi_{ik} - \mu^\xi_i & i = j \leq m \\
\mu^\xi_{ij} & i \neq j, i, j \leq m \\
\mu^\xi_i & i = m+1, j \leq m+1 \\
0 & \text{otherwise}
\end{cases}
\]

and from there it is possible to derive the generator of the process \((\xi_t, q^1_t)\).

**Proposition 3** The infinitesimal generator of the single obligor process \((\xi_t, q^1_t)\) is the linear operator \( A \in \mathbb{R}^{n_\xi(m+1) \times n_\xi(m+1)} \) defined with the matrices \( A_\xi, \xi \in \Xi \), as

\[
A = \begin{pmatrix}
A_0 - \lambda_0 I & \lambda_1 Q_{10}I & \ldots & \lambda_{n_\xi-1} Q_{n_\xi-1,0}I \\
\lambda_1 Q_{01}I & A_1 - \lambda_1 I & \ldots & \lambda_{n_\xi-1} Q_{n_\xi-1,1}I \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_0 Q_{0n_\xi}I & \lambda_0 Q_{1n_\xi}I & \ldots & A_{n_\xi} - \lambda_{n_\xi} I
\end{pmatrix}.
\]

Here \( n_\xi \) is the cardinality of \( \Xi \) and \( \lambda_i = \sum_j Q_{ij} \).

With the infinitesimal generator the probability transition matrix can be calculated by simple integration and the probabilities of default can be calculated.

**Proposition 4** The probability of default in a horizon time \( t \) for an obligor with initial rate \( i \) is given by

\[
\mathbb{E}\{\mathbb{1}_{t \geq t}\} = \sum_{k=1}^{n_\xi} (e^{tA})_{k(m+1),i}
\]
where $A$ is the infinitesimal generator of the single obligor process.

**Proof:** The probability of the time to default $\tau$ being less than $t$ is the probability that the process is in a state $(\xi, 0)$ at time $t$. Then the probability is given by the sum of the entries of the transition matrix $e^{tA}$. \qed

## 5 Computational Aspects: Quadratures

Some of the computational tools available for pricing general claims, where analytical formulas are not straightforward, are the use of a numerical solutions to the backward or forward equations and procedures based on sampling the environment process such as Monte Carlo. When solving the Kolmogorov equations numerically, the problem can be reduced to handling a PDP system, however the solution can be computationally expensive.

We will consider a general European claim on $q^n_T$, the rating/default distribution of the portfolio with $n$ elements exercised at time $T$. Let us denote the exercise value of this claim as $f(q^n_T)$. In the case of the losses in the portfolio, for example, the form of the claim is $f(x) = n - \sum x_i$. Since we assume that the associated diffusion limit $q_t = \sqrt{nq^{(0)}_t} + \sqrt{nq^{(1)}_t}$ approximates $q^n_T$ “reasonably well”, we will assume as well that $f$ is “smooth enough” to allow for the approximation of $f(q^n_T)$ by $f(q_t)$.

A natural approach to calculate expectations is by conditioning on the environmental process path $\{\xi_t : 0 \leq t \leq T\}$, then

$$
\mathbb{E}\{f(q_T)\} = \mathbb{E}\left[\mathbb{E}\{f(q_T) | \xi_t : 0 \leq t \leq T\}\right].
$$

(5.1)

As we are concerned with pricing, in the following we will assume $\mathbb{E}$ is the expectation under a risk neutral measure $Q$ unless stated differently.

The advantage of this idea is that $q_t$ conditional on the environment path is jointly normal distributed and expectations are straightforward to calculate, depending solely in the expectation (fluid limit) and variance of the process (given by the Lyapunov equation in Section 4). The remaining choice is whether to apply Monte Carlo methods to generate paths $\{\xi_t|0 \leq t \leq T\}$ or to approximate the distribution of the jump times $t_i$ and apply numerical integration. While the former method requires little programming effort and gives good results, a refined version of sampling can give even better results.

We will consider the process $\{\xi_t : 0 \leq t \leq T\}$ and $N_T$, the associated counting process of the number of jumps in $\xi_t$ in interval $[0,T]$. Assuming the environment process takes only two values, the process $\xi_t$ can be characterized by the initial state $\xi_0$ and the vector of jump times $\mathbf{t} = (t_1, ..., t_N) \in [0,T]^n$ where $t_N < T$ and $t_{N+1} > T$, corresponding to $N_T = N$. Alternatively it can be defined in terms of the vector of inter-jump times $\mathbf{\tau} = (\tau_1, ..., \tau_n) \in S^n_T$ where $\tau_i = t_i - t_{i-1}$ with $t_0 = 0$. We denote by $S^n_T$ the $n$-dimensional simplex set defined by the points $\mathbf{\tau}$ such that $\sum \tau_i \leq T$. In the general case we must consider all possible paths as we explain later, but to simplify the exposition we will assume a two-state environment process in the remainder of this section unless otherwise stated.

In the case of European claims of the form $f(q_T)$, the conditional variable $f(q_T)|\{\xi_t; 0 \leq t \leq T\}$ can be expressed in terms of the vector $\mathbf{t}$ and will be denoted as $f(q_T | \mathbf{t})$. Alternatively, can be expressed in terms of the time between jumps, $\tau_i = t_i - t_{i-1}$ for $i = 1, ..., N$ and $t_0 = 0$, denoted as $f(q_T | \tau)$. The vectors $\mathbf{t}$ and $\tau$ are random vectors which we will characterize explicitly later.

Using a naive Monte Carlo approach, a first version of our pricing algorithm is:
1. For \( i = 1 \) to \( M \) the number of iterations.

(a) Simulate a sequence of jumps \( t_1, ..., t_{N_i} \) and the process \( \xi_t \), such that \( t_{N_i} < T \) and \( t_{N_i+1} \geq T \), according to the rate \( \lambda(t_{j-1}) \), \( j = 1, ..., N_i + 1 \) and the transition matrix \( Q \).

(b) Based on the simulated process \( \xi_t \) generate the parameters \( \mu(\xi) \) and \( C_t \) of the diffusion process \( q_t \).

(c) Calculate \( f(T) = \mathbb{E} \{ f(q_T) | \xi_t : 0 \leq t \leq T \} \)

2. Average the results to obtain \( \mathbb{E} \{ f(q_T) \} \)

This algorithm has some problems, the most obvious is that if the jump rates of the environment process \( \xi_t \) are low, a large proportion of samples will show no jumps at all, so that the number of samples required to obtain convergence will be large.

An alternative is to use Importance Sampling techniques where a measure, say \( \tilde{Q} \), is chosen to increase the frequency of jumps in the environment process. Hopefully this will improve the convergence in the pricing, particularly for senior tranches. Under the Importance Sampling scheme, the basic algorithm is modified as follows:

1. For \( i = 1 \) to \( M \) the number of iterations.

(a) Simulate a sequence of jumps \( t_1, ..., t_{N_i} \) and the process \( \xi_t \), such that \( t_{N_i} < T \) and \( t_{N_i+1} \geq T \), according to the rate \( \tilde{\lambda}(t_{j-1}) \), \( j = 1, ..., N_i + 1 \) and the transition matrix \( \tilde{Q} \) consistent with the measure \( \tilde{Q} \).

(b) Based on the simulated process \( \xi_t \) generate the parameters \( \mu(\xi) \) and \( C_t \) of the diffusion process \( q_t \) and the Radon-Nikodym derivative.

(c) Calculate

\[
    f(T) = \mathbb{E} \{ f(q_T) | \xi_t : 0 \leq t \leq T \}
\]

and the Radon-Nikodym derivative \( \frac{dQ}{d\tilde{Q}} \), see Rogers and Williams [18] for a study of change of measures in Markov Chains.

2. Average the results to obtain

\[
    \mathbb{E}_{\tilde{Q}} \left\{ f(q_T) \frac{dQ}{d\tilde{Q}} \right\} = \mathbb{E} \{ f(q_T) \}
\]

We propose an alternative technique that combines this conditional analysis with the use of Gaussian Quadratures.

5.1 CDO pricing

In principle we are concerned with the loss process at time \( T \) up to the attachment point of the tranche \( K \) (we consider base tranches as building blocks [16]). In this section we will consider the limit process \( q(t) = q_t^{(0)} + \frac{1}{\sqrt{n}} q_t^{(1)} \), this is, instead of absolute losses we will consider losses as the proportion of the original notional. In the CDO case the claims can be expressed as

\[
    f(q_T) = \min \left\{ \left( 1 - \sum_i q_i(T) \right) (1 - r), K \right\} = L_t \wedge K
\]
where $r$ is the deterministic recovery rate. We will denote the loss process $L_T = (1 - \sum_i q_i(T))(1 - r)$.

Assuming a deterministic discount function $D_i = D(t_i)$ and that the payment of the losses occurs at the end of the coupon period, the spread $u$ of the CDO is the solution to the equation

$$u \tau \sum_{i=0}^{N-1} D_i \mathbb{E}(K - f(q_{t_i}))^+ = \sum_{i=1}^N D_i \mathbb{E}(f(q_{t_i}) - f(q_{t_{i-1}}))$$

(5.2)

where $\tau$ is the accrual period between coupons. The left hand side of the equation is the spread leg of the CDO, represents the coupon paid on the non-defaulted proportion of the underlying portfolio. The right hand side is the sum of the compensation for the losses occurring between $t_{i-1}$ and $t_i$, assuming that these losses are paid at the end of each accrual period. From the above we shall then focus on calculating the expected value of $f(q_T)$ under the pricing risk neutral measure.

Since conditional on the vector $\tau$ the process $q_t$ is normally distributed with parameters $\mu(\tau)$ and $C(\tau)$, the conditional loss process $L_T|\tau = 1 - \sum_i q_i(T|\tau)$ and therefore is normal distributed with mean $1 - \sum_i \mu_i(\tau)$ and variance $\sum_{i,j} C_{ij}(\tau)$. However, the following result shows that to estimate the parameters of the distribution of $L(t|\tau)$ is not required to calculate $C(\tau)$ explicitly.

**Proposition 5** If conditional on the vector $\tau$ all obligors are independent with probability of default $p_i$ given by the initial rate $i$ at time 0, then the random variable $L(t|\tau)$ is normal with the following parameters

$$\tilde{p}(\tau) = \sum_i q_i(0)p_i(\tau)$$

(5.3)

$$\tilde{\sigma}^2(\tau) = q_i(0)p_i(\tau)(1 - p_i(\tau))$$

(5.4)

and the price of the equity tranche with attachment point $K$ is given by

$$\mathbb{E}_Q\{f(t, \tau)\} = \mathbb{E}_Q\{f(t|\tau)|L(t|\tau) \leq K/(1 - R)\} + KP(L(t|\tau)(1 - r) > K)$$

$$= \int_0^{K'} z(1 - R)\phi [x; \tilde{p}, \tilde{\sigma}/\sqrt{\tilde{n}}] dz + K\Phi \left( -\frac{K' - \tilde{p}}{\tilde{\sigma}/\sqrt{\tilde{n}}} \right)$$

(5.5)

where $K' = K/(1 - r)$.

**Proof:** Since for $n > 0$ we can express

$$\sum_i q^n_i(t) = \sum_i \sum_j q^{n,i}_j(t)$$

where $q^{n,i}_j(t)$ represents a portfolio with initial position of $q^n_i(0)$ units in rate $i$. Since conditional on the environment process $\xi$, the elements in the portfolio are independent and homogeneous up to the credit rating, it is true that

$$\text{var}\left(\sum_i q^n_i(t) \bigg| \tau(t)\right) = \sum_i (q^n_i(0))^2 \text{var}\left(\sum_j q^{n,i}_j(t) \bigg| \tau(t)\right)$$

$$= \sum_i (q^n_i(0))^2 p_i(1 - p_i)$$

11
since \( \sum_j \hat{q}_j^{1,i}(t) = 0 \) if and only if default has occurred by \( t \) and is equal 1 otherwise, therefore is Bernoulli \( p_i \). Now by Theorem 2 we know that conditional on \( \tau(t) \)

\[
\frac{1}{\sqrt{n}} \left( \hat{q}^{n,i}(t) - nq^{(0,i)}(t) \right) \rightarrow^d q^{(1)}(t)
\]

where \( q^{(0,i)}(t) \) denotes the fluid limit of \( \hat{q}^{n,i}(t) \). By the Helly-Bray lemma ([2], pp. 251) we know that

\[
\text{var} \left( \sum_j \hat{q}_j^{n,i}(t) \mid \tau(t) \right) \rightarrow \text{var} \left( q^{(1,i)}(t) \mid \tau(t) \right)
\]

(5.6)

where \( q^{(1,i)}(t) \) denotes the diffusion limit associated to \( \hat{q}^{n,i}(t) \). Since the fluid limit is deterministic and \( \sum_j \hat{q}_j^{n,1} \) is a sum of identical Bernoulli random variables

\[
\text{var} \left( \frac{1}{\sqrt{n}} \left( \hat{q}^{n,i}(t) - nq^{(0)}(t) \right) \right) = p_i(\tau)(1 - p_i(\tau))
\]

(5.7)

and then (5.4) follows. Since \( L_t \) is normally distributed conditional the rest of the results is trivial.

Therefore, when interested exclusively in the default process of the portfolio, rather than the rating distribution, it is possible to obtain the conditional distribution of the defaults from the fluid limit without solving the Lyapunov equation (4.19).

5.2 Changes of measure, the Poisson space and Quadrature formulas

By conditioning on the actual number of jumps in the environment process we can rewrite the pricing formula as

\[
\mathbb{E}(f(T) \mid \tau) = \sum_{i=0}^{\infty} P(N_T = i) \mathbb{E}(f(T) \mid \tau; N_T = i)
\]

\[
= \sum_{i=0}^{\infty} P(N_T = i) \mathbb{E}(f(T) \mid (\tau_1, \ldots, \tau_i))
\]

\[
\approx \sum_{i=0}^{N^\ast} P(N_T = i) \mathbb{E}(f(T) \mid (\tau_1, \ldots, \tau_i))
\]

(5.8)

where \( N^\ast \) is set such that the rest of the series is small enough. The process \( N_t \) is a Poisson process when \( \lambda_\xi = \lambda \) for all \( \xi \in \Xi \) and this conditioning and integrating approach to the process is related to the canonical space of a Poisson process [17], we discuss this below. To derive the quadrature approximation, first we have to study the distribution of the jump times in the environment process. Each of the conditional expectations \( F(T, \tau_1, \ldots, \tau_N) = \mathbb{E}_Q(f(T) \mid (\tau_1, \ldots, \tau_N)) \) may be calculated using Gaussian Quadratures. For simplicity of notation we will assume \( T = 1 \) in the following development.

We are interested in applying Gaussian Quadrature methods in the simplex set \( S^N \) to estimate integrals of the form

\[
\int_{S^N} f(\tau) \cdot w(\tau) d\tau
\]

where \( w \) is the weight function (density) and \( f \) is the payoff of the claim to be priced.
The concept of change of measure is useful to calculate \( W_N(A) \), the conditional probability of the set \( A \) conditional on \( N_T = N \), and the probability \( P(N_T = N) \). In the following consider \( \mathbf{K} \) the index set for the set of all arrays of possible paths for \( \xi \) characterized by the sequences of states \((i_0^k, \ldots, i_N^k) \in \Xi^{N+1} \) and the number of transitions \( \{n_k^j\} \), \( k \in \mathbf{K} \) and \( \sum_{i,j} n_{ij} = N \). By assuming a change to a measure \( \tilde{\mathbf{P}} \) where \( \xi \) follows a Markov chain with generator matrix \( \Upsilon \) such that \( v_{ij} = \overline{v} \) for \( i \neq j \) and \( v_{ii} = n_x v \), applying the Girsanov formula (see section IV.22 in [18]) we obtain

\[
P(N_T = N) = \mathbb{E}\left\{ \mathbb{I}_{N_T = N} \frac{d\mathbf{P}}{d\tilde{\mathbf{P}}} \right\}
\]

\[
= \frac{1}{N!} \sum_{k \in \mathbf{K}} \prod_{i \neq j} v_{ij}^{n_{ij}^k} \int_{S^N} \exp \left( -\sum_{j=0}^{n_{ij}^k} \tau_j^k \right) d\tilde{\mathbb{P}}
\]

where \( \tau_j^k = 1 - \sum_{j=0}^{n_{ij}^k} \tau_j^k \). Using the same argument, for \( A \in S^N \) measurable

\[
W_N(A) = \mathbb{E}\left\{ \mathbb{I}_A \frac{d\mathbf{P}}{d\tilde{\mathbf{P}}} \right\}
\]

\[
= \frac{1}{P(N_T = N)} \sum_{k \in \mathbf{K}} \prod_{i \neq j} v_{ij}^{n_{ij}^k} \int_A \exp \left( -\sum_{j=0}^{n_{ij}^k} \tau_j^k \right) d\tilde{\mathbb{P}}
\]

from where it is possible to derive the density function

\[
w_N(\Sigma) = \frac{1}{P(N_T = N)} \sum_{k \in \mathbf{K}} \prod_{i \neq j} \left( \frac{v_{ij}}{q} \right)^{n_{ij}^k} \sum_{j=0}^{n_{ij}^k} \tau_j^k \exp \left( -\sum_{j=0}^{n_{ij}^k} \tau_j^k \right)
\]

5.2.1 The canonical space of a Poisson Process

The decomposition of the sample space of the environment process in a series of simplex sets is similar to the analysis of a Poisson process by Neveu [17] in what is known as the canonical space of a Poisson process.

It is well known that conditional in the number of jumps in a given interval of time, the inter arrival times of a Poisson process are uniformly distributed. Neveu defines a Poisson process in terms of the inter arrival times as points uniformly distributed in \( n \)-dimensional hypercubes \( n = 1, \ldots \). Moreover, he defines a point process in terms of the distribution of these points in the hypercubes. By using the pricing approach we have developed a framework similar to the one defined by Neveu taking simplex sets instead of hypercubes. Both approaches are equivalent in the case of a Poisson process, as we now explain this.

The canonical space of a Poisson process in the interval \([0, T]\) is the probability space \((\hat{\Omega}, \hat{\mathfrak{B}}, \hat{\mathbf{P}})\) defined as follows: The sample space \( \hat{\Omega} \) is formed for all vectors \( \hat{t} \in \mathbb{R}^n \) for \( n \in \mathbb{N} \) and a “distinguished” element \([0, T]^0 = \{a\} \), i.e. \( \hat{\Omega} = \bigcup_{n=0}^{\infty} [0, T]^n \). The \( \sigma \)-field is \( \hat{\mathfrak{B}} = \{G : G \cap [0, T]^n \in \mathfrak{B}([0, T]^n)\} \) where \( \mathfrak{B}([0, T]^n) \) denotes the Borel algebra of \([0, T]^n\) for \( n \geq 1 \). Finally, if \( \mu^n \) is the Lebesgue measure in \([0, T]^n\) and \( \mu^0 = \delta_a \), then for \( A \in \hat{\mathfrak{B}} \),

\[
\hat{\mathbf{P}}(A) = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \mu^n(A \cap [0, T]^n)
\]
On this probability space the random variable
\[
\hat{N}_t\{a\} = 0 
\]
\[
\hat{N}_t(t_1, \ldots, t_n) = \sum_{i=1}^{n} \mathbb{I}_{[0,t]}(t_i) 
\]
is a Poisson process with intensity rate \(\lambda\). Notice that in the vector \(t\) the order is not important, if \(\Pi(t)\) is a permutation of the elements of \(t\), \(\hat{N}_t(\Pi(t)) = \hat{N}_t(t)\). Under this approach, the Poisson process is a particular case of point process. Point processes can be thought as a population permutations of the vector \(t\) represents the elements of \(\mathbb{R}\) in the current population, where \(a\) represents the empty population. From the characteristics of the Poisson space we know that the jump times in process are uniformly distributed conditional on the number of jumps.

In the exposition of the previous sections we assume the probability space \((\Omega, \mathcal{B}, P)\) as follows. The sample space \(\Omega = \bigcup_{n=0}^{\infty} S^n\), where \(S^n = \{t \in \mathbb{R}^n : \sum_{i=0}^{n} t_i \leq T\}\) is a simplex in positive part of \(\mathbb{R}^n\), and \(S^0 = \{a\}\). The \(\sigma\)-algebra is \(\mathcal{B} = \{G : G \cap [0,T]^n \in \mathcal{B}(S^n)\}\). With the measure \(W_n(\cdot)\) in \(S^n\) defined as in (5.10) for \(n \geq 1\), and \(W_0 = \delta_a\). We define for \(A \in \mathcal{S}\),
\[
P(A) = \sum_{n=0}^{\infty} P(N_T = n)W_n(A \cap S^n). 
\]  
In this form the probability space \((\Omega, \mathcal{B}, P)\) defines a point process as described by [17]. We then define the counting process
\[
N_t\{a\} = 0 
\]
\[
N_t(\tau_1, \ldots, \tau_n) = \sum_{i=1}^{n} \mathbb{I}_{[0,t]}(\tau_1 + \ldots + \tau_i) 
\]
in the case that \(\lambda_i = \lambda\) for all \(i\), the measure \(W_n = n!\mu^n\) where \(\mu^n\) is the Lebesgue measure in the simplex \(S^n\) and \(\mu^0\) is as above. In this case the probability measure (5.15) is
\[
P(A) = \sum_{i=0}^{\infty} e^{-\lambda t} \frac{\lambda^n}{n!} n!\mu^n(A \cap S^n) 
\]
and \(N_t\) is an standard Poisson process. For some \(n\) consider \(A \in S^n\) a measurable set (with respect to \(\mathcal{B}\)), then the set \(A = \{\Pi(t) : \tau(t) \in A\}\) is a measurable set with respect to \(\mathcal{B}\). Notice that for almost all points (with respect to the Lebesgue measure) in \(S^n\) the number of permutations of the vector \(\tau(t)\) is \(n!\), therefore \(W_n(A) = \hat{\mu}(A)\). Finally notice that \(\hat{N}_t(t) = N_t(\tau)\) for all \(t \in [0, T]\) if and only if \(t = \Pi(t(\tau))\) for some permutation, from this we conclude that \(N_t\) is a Poisson process with intensity \(\lambda\).

In the following section we will present numerical integration techniques and we will discuss its implementation in the space \((\Omega, \mathcal{B}, P)\) with a sample space as the union of simplex sets. In the case of the environment process being a Poisson process with intensity \(\lambda\) it does not make any difference the Poisson space we use and we have no special reason to prefer the use of our proposed space (of simplex sets).

It is always possible to translate the problem to a Poisson type by using the Girsanov theorem for Markov chains. In this case we have a constant weight function (if the Radon-Nikodym is not accounted in the weight function) and then the use of the canonical space may be preferred.
5.2.2 Gaussian Quadratures

Quadrature methods for numerical integration have been widely used as an alternative to the traditional Newton-Cotes approach. The appealing feature of quadratures is the lower number of points of evaluation required to obtain a desired degree of accuracy defined in terms of the maximum degree of the polynomials for which the formula is exact.

For example, on the real line a Gaussian quadrature of degree \( m \), with respect to a defined measure \( \mu \), provides an exact formula for polynomials up to degree \( 2m - 1 \), this is

\[
\int_{\mathbb{R}^n} x_i^k d\mu = \sum_{j=1}^{m} w_j x_{j,i}^k \tag{5.19}
\]

where \( x_i, i = 1, \ldots, n \), is the \( i \)-th component of \( x \) and \( k \leq 2m - 1 \). This result is the fundamental theorem of Gaussian Quadratures (see [19]). While the property of the formula being exact up to double degree polynomials is appealing, the selection of the abscissas \( x_j \) is not always trivial. The most common Gaussian Quadrature is the one related to the Lebesgue measure (known as Gaussian-Lebesgue). Since the density function involved in our calculations has the exponential form we will prefer the use of the Gaussian-Lebesgue quadratures over other less stable and more expensive methods as proposed in [14].

For the multiple dimensional integration problem, we can express formulas as a “product” of one dimensional quadratures, therefore these cases are referred as product formulas. A survey of these methods can be found in [20]. We will just explain briefly the case of a simplex set since it will be useful to our pricing technique.

Consider an integral of the form

\[
I = \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\cdots-x_n} f(x_1, \ldots, x_n) dx_n \cdots dx_1, \tag{5.20}
\]

setting the change of variable

\[
x_1 = y_1 \tag{5.21a}
\]
\[
x_2 = y_2(1 - y_1) \tag{5.21b}
\]
\[\vdots\]
\[
x_n = y_n(1 - y_1) \cdots (1 - y_{n-1}) \tag{5.21c}
\]

we obtain

\[
I = \int_{[0,1]^n} f(x_1(y_1), \ldots, x_n(y_n))(1 - y_1)^{n-1}(1 - y_2)^{n-2} \cdots (1 - y_{n-1})dy_n \cdots dy_1
\]

If a quadrature formula \( \{w_i, x_i\}_{i \in I} \) of degree \( m \) can be derived in the real interval \([0,1]\), then \( \{\prod_{j=1}^{n} w_{i_j}, (x_{i_1}, \ldots, x_{i_n})\} \) defines a quadrature formula in \([0,1]^n\) that is exact for all polynomials with coefficients less or equal to \( m \). Here \( \{(i_1, \ldots, i_j)\} \) represents the set of all possible combinations of indices \( i \in I \). From the change of variable above, the integral in the simplex is taken to the Cartesian product of the interval \([0,1]\) and product formulas can be applied.

It is evident that product formulas lose their appeal as the dimension increases. As an alternative there are available in the literature some other quadrature formulas. In particular we use the method proposed by Grundmann and Möller [9] that claims to use the minimum number of integration nodes.
5.3 Some comparisons

We present an example to illustrate the advantages of Gaussian quadratures and the use of change of measures. We set a two state model with two rates, the following transition rates (corresponding to each environment state)

\[ T_0 = \begin{pmatrix} 0.0125 & 0.0200 \\ 0.0100 & 0.0375 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1.500 & 0.030 \\ 0.005 & 3.000 \end{pmatrix} \]

and environment process jump rates

\[ \lambda = \begin{pmatrix} 0.05 \\ 1.0 \end{pmatrix} \]

We evaluate the expected loss in different equity tranches using different methods. Our aim is to achieve 6-digit accurate estimation, this is particularly difficult under Monte Carlo methods, particularly for large tranches. To complete our initial information, we assume a portfolio with initial distribution 50/50, 100 elements and a recovery rate \( r = 40\% \).

First, we implement Monte Carlo simulation with 1,000,000 simulations, and estimate the mean and variance of the loss in each tranche. The results are shown in Table 1. By (4.22) in Proposition 4 we find that the exact probability of default is 0.047207. From the results obtained it can be seen that in order to achieve a 6-digit (95% confidence) estimation in the 22% tranche we need over 1,199 million simulations.

<table>
<thead>
<tr>
<th>Attachment Point</th>
<th>Expectation (%)</th>
<th>Std Dev (bps)</th>
<th>95% Confidence (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.5343</td>
<td>32</td>
<td>6.40E-04</td>
</tr>
<tr>
<td>6</td>
<td>1.6919</td>
<td>94</td>
<td>1.85E-03</td>
</tr>
<tr>
<td>9</td>
<td>1.8240</td>
<td>155</td>
<td>3.04E-03</td>
</tr>
<tr>
<td>12</td>
<td>1.9493</td>
<td>213</td>
<td>4.18E-03</td>
</tr>
<tr>
<td>22</td>
<td>2.3145</td>
<td>390</td>
<td>7.64E-03</td>
</tr>
<tr>
<td>prob. Def.</td>
<td>4.7240</td>
<td>1131</td>
<td>2.22E-02</td>
</tr>
</tbody>
</table>

We must consider that the exponential form of the Radon-Nikodym derivative causes high variance when the jump rates are increased dramatically. Therefore while it would seem ideal to increase the rate of occurrence of jumps to eliminate the variance, the Radon-Nikodym derivative form makes this not practical.

While the reduction in the senior tranches is outstanding when we increase the jump rate to the stressed economic scenario, the number of simulations required still being relatively high.
In order to implement the Gauss quadratures we analyse the probability distribution of the number of jumps in the environment process. From Table 3 it is clear that by conditioning in the number of jumps, just a small number of conditional expectations will be enough for an accurate valuation. We will use 6 jumps, however it can be easily work with just four (the probability of a 5 jumps is lower than the order of accuracy and the expected loss is less than 1). Furthermore, the error tolerance required decreases while the dimension (number of jumps) increases since the probability of higher number of jumps is smaller.

First, we implement Gaussian-Legendre quadratures extended to a simplex as explained in Section 5.2. The second method uses ad hoc quadratures extended to a simplex; for lower order degree we derive directly the orthogonal polynomials and coefficients for the Jacobi Matrix, larger order quadratures use the sequence approach of [14]. A third approach is based on a change of measure to a homogeneous rate Poisson and applying the Gaussian-Legendre quadrature that is suitable since the conditional probability is uniform. We try two changes of measure: one to a Poisson with rate 0.25 that keeps a stable Radon-Nikodym derivative, and a second to a Poisson rate 0.05 where the probability of higher number of jumps vanishes faster. Finally, we implement a 5-degree Grundmann-Möller quadrature for a simplex presented in [9].
In Table 4 we can observe the number of points required to match a 6-digit accurate estimation in the loss of the tranche. The first conclusion is that in all cases it is clear the advantage with respect to Monte Carlo methods. Even when producing quadrature points is more expensive than Monte Carlo points, the incredible reduction of points required justifies the use of quadratures.

Table 4: Points required for 6-digit accuracy.

<table>
<thead>
<tr>
<th>Method</th>
<th>Attachment Point</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3        6        9        12       22       total</td>
</tr>
<tr>
<td>Gauss-Legendre</td>
<td>232      332      445       657       633       467</td>
</tr>
<tr>
<td>Gauss-Exponential</td>
<td>232      361      417       837       662       1023</td>
</tr>
<tr>
<td>Change of Measure (1)</td>
<td>212      236      606       657       625       225</td>
</tr>
<tr>
<td>Change of Measure (2)</td>
<td>433      437      595       595       595       341</td>
</tr>
<tr>
<td>Grundmann Moller(*)</td>
<td>127      127      127       127       127       127</td>
</tr>
</tbody>
</table>

We compare in Table 5 the most accurate result (Obtained with a Gauss-Legendre quadrature using 224 983 points) with our Monte Carlo simulation and a 5 degree quadrature ad hoc for simplex domains developed in [9] that claims to have the minimum number of integration nodes.

Table 5: Expected Loss (%).

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Gauss Quadrature</th>
<th>Monte Carlo (1mm)</th>
<th>Grundmann Möller d=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.5344</td>
<td>1.5343</td>
<td>1.5351</td>
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<td>6</td>
<td>1.6921</td>
<td>1.6919</td>
<td>1.6965</td>
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<td>1.8243</td>
<td>1.8240</td>
<td>1.8356</td>
</tr>
<tr>
<td>12</td>
<td>1.9496</td>
<td>1.9493</td>
<td>1.9705</td>
</tr>
<tr>
<td>22</td>
<td>2.3145</td>
<td>2.3145</td>
<td>2.2919</td>
</tr>
<tr>
<td>def prob</td>
<td>4.7207</td>
<td>4.7240</td>
<td>4.7212</td>
</tr>
</tbody>
</table>

6 Calibration

We consider the pool of CDS in the iTraxx Europe index for quotes from 11 April to 15 April 2005 and from 26 February to 5 March 2007. This data is presented in Tables 7 and 8, and gives the par spreads for 5-year protection on the tranches specified in the left-hand column. As is customary, the premium payment for the equity (0-3%) tranche is an up-front payment of a percentage (this is the number given in the table) of the notional amount, plus a running spread of 500bp. For example, the figures given for April 11 in Table 7 mean that equity tranche investors pay 25% up-front plus 500bp, while investors in the 3-6% tranche pay nothing up-front but a running spread of 161bp (and similarly for all other tranches). The calibration is performed by minimizing the square error of the pricing error (spreads) for the CDO iTraxx quotes and the CDS quote. The minimization routine is performed combining a gradient based method and a pattern search method from Matlab. The reason to alternate methods is to avoid
local minima (by using pattern search) while keeping speed (gradient method).

First, we performed the calibration assuming a 2-state environment process. As for the rating process, we assume both a single and a two-rate portfolio.

The calibration of the two-state model to the market data is poor. It is possible to improve the fitting of the senior tranche and represent a increasing behavior in the base correlation by increasing the default rate $\mu_1$ in the catastrophic state and diminishing its probability of occurrence by decreasing $\lambda_0$, however this increases the error in the mezzanine tranche prices.

Various examples show us that the tranche prices generated by a two-rate system can be accurately matched by a model using only a single rate. This may be explained by the fact that in a large portfolio the effect of idiosyncratic risk events is averaged inside the portfolio, by using the fluid and diffusion approximation we are then dampening this effect. On the other side, the systematic risk (represented by the random environment process) is the main driver of the model.

6.1 A 3-state environment process

The high level of the implied and base correlations associated to the senior tranches can be explained as the premium asked for the existence of a rare catastrophic event where a high proportion of default occur. We introduce a third environment state such that a two state switching regime provides the default structure of the equity and junior mezzanine tranches and the third state provides the “worst scenario” that adds value to the senior tranches.

6.1.1 Implementation

The valuation methodology for the 3-state model is the same as explained in the previous section. In addition to the transition matrices $A_1, A_2$ and $A_3$ a generator matrix $Q \in \mathbb{R}^{3 \times 3}$ for the environment process is assumed. Default probabilities can be calculated using Proposition 4.

To use a quadrature approach for general claims we must consider all possible paths in the environment process. In the general case of $n$-states we have $(n - 1)^k$ possible paths with $k - 1$ jumps. In the case of $Q$ as described above and considering up to 6 jumps, we must consider 127 paths. We decide to use product formulas from Lebesgue-Gaussian quadratures for the paths up to 3 jumps and the Grundmann-Möller quadrature formula of fifth degree thereafter in order to reduce the number of points valuated.

We assume a model with a single rating class, a 3-state environment process with transition matrix $Q$, and a 3-vector $\mu = (\mu_1, \mu_2, \mu_3)$ of default rates in the 3 environment states, giving a total of 9 parameters ($Q$ is specified by 6 parameters since the row sums are zero). We employ a combination of gradient based algorithm and pattern search algorithms provided by Matlab. In order to increase the speed we generate a common set of nodes for the quadratures according to a pre-specified distribution and update the Radon-Nikodym derivative as we modify the parameters. This method seems to improve the performance without losing accuracy. A more refined method must consider readjusting the measure of reference periodically. Applying this method to the 11 April 2005 data we obtain the following parameters:
\[ Q = \begin{pmatrix} -0.080931 & 0.076783 & 0.004148 \\ 3.204384 & -3.239922 & 0.035538 \\ 4.935346 & 5.732706 & -10.668052 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0.00365 \\ 0.09194 \\ 3.73706 \end{pmatrix} \]

By 2007 credit spreads had dropped by almost half, this could be explained by a change in the “steady” default rate or a change of appreciation of the catastrophic state. The new parameters for 27 February 2007 are

\[ Q = \begin{pmatrix} -0.244905 & 0.238673 & 0.006232 \\ 2.908366 & -2.959641 & 0.051275 \\ 2.227186 & 0.269876 & -2.497042 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0.00253 \\ 0.01174 \\ 0.19825 \end{pmatrix} \]

We can observe that while the model indicates a reduction in the prevalent instantaneous default rate, the default rates associated to the stressed scenarios have been brought down significantly. Table 6 shows the calibration error. Essentially, exact calibration is achieved across all tranches, supporting the argument of the existence of a third catastrophic state to explain the implied correlation structure observed in the market.

<table>
<thead>
<tr>
<th>tranche</th>
<th>27-Feb-07</th>
<th></th>
<th>11-Apr-05</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Market</td>
<td>Model</td>
<td>Error</td>
<td>Market</td>
</tr>
<tr>
<td>0-3% (%)</td>
<td>8.661</td>
<td>8.661</td>
<td>0.000</td>
<td>25.000</td>
</tr>
<tr>
<td>3-6% (bps)</td>
<td>44.025</td>
<td>44.032</td>
<td>0.007</td>
<td>161.000</td>
</tr>
<tr>
<td>6-9% (bps)</td>
<td>11.875</td>
<td>11.877</td>
<td>0.002</td>
<td>51.500</td>
</tr>
<tr>
<td>9-12% (bps)</td>
<td>5.691</td>
<td>5.695</td>
<td>0.003</td>
<td>24.000</td>
</tr>
<tr>
<td>12-22% (bps)</td>
<td>2.041</td>
<td>2.034</td>
<td>-0.007</td>
<td>14.000</td>
</tr>
<tr>
<td>CDS (bps)</td>
<td>22.100</td>
<td>22.098</td>
<td>-0.002</td>
<td>40.000</td>
</tr>
</tbody>
</table>

Finally, Tables 7 and 8 show the calibrated parameters for the weeks beginning 11-Apr-05 and 27-Feb-07 respectively. In both cases the model shows stability in the parameters despite the fact that the sample week from 2005 is particularly volatile.

### 7 Conclusions

We have presented a credit risk model whose interpretation and dynamic are intuitive. This model is intended for use in large portfolio modelling, CDO pricing and risk management. Based on well known results from stochastic networks theory, the model considers the systemic risk of default by postulating the existence of an exogenous environment state (that can be interpreted as the economic cycle). It is assumed that in a large portfolio the idiosyncratic risk (represented by the rating system) is averaged in the portfolio.

We provide a description of the asymptotic behaviour of the model and use it to approximate the value of claims written in a large portfolio such as CDOs.
Table 7: Calibration to iTraxx mid market quotes (April 2005).

<table>
<thead>
<tr>
<th></th>
<th>11-Apr</th>
<th>12-Apr</th>
<th>13-Apr</th>
<th>14-Apr</th>
<th>15-Apr</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-3%</td>
<td>25.00</td>
<td>23.95</td>
<td>23.95</td>
<td>26.75</td>
<td>30.63</td>
</tr>
<tr>
<td>3-6%</td>
<td>161.00</td>
<td>153.00</td>
<td>151.50</td>
<td>169.50</td>
<td>190.00</td>
</tr>
<tr>
<td>6-9%</td>
<td>51.50</td>
<td>47.75</td>
<td>47.00</td>
<td>53.50</td>
<td>61.00</td>
</tr>
<tr>
<td>9-12%</td>
<td>24.00</td>
<td>22.50</td>
<td>22.50</td>
<td>26.00</td>
<td>30.00</td>
</tr>
<tr>
<td>12-22%</td>
<td>14.00</td>
<td>13.00</td>
<td>13.00</td>
<td>14.50</td>
<td>16.25</td>
</tr>
<tr>
<td>CDS</td>
<td>40.00</td>
<td>39.00</td>
<td>39.00</td>
<td>42.00</td>
<td>45.00</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>0.3656</td>
<td>0.3539</td>
<td>0.3543</td>
<td>0.3853</td>
<td>0.4252</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>9.1943</td>
<td>8.9888</td>
<td>8.8852</td>
<td>8.9338</td>
<td>8.6264</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>373.7</td>
<td>360.1</td>
<td>359.8</td>
<td>351.6</td>
<td>279.2</td>
</tr>
<tr>
<td>$q_{12}$</td>
<td>7.6784</td>
<td>8.5438</td>
<td>8.7701</td>
<td>7.4639</td>
<td>7.7863</td>
</tr>
<tr>
<td>$q_{13}$</td>
<td>0.4148</td>
<td>0.4565</td>
<td>0.4722</td>
<td>0.3912</td>
<td>0.6284</td>
</tr>
<tr>
<td>$q_{21}$</td>
<td>320.4</td>
<td>356.3</td>
<td>363.9</td>
<td>290.1</td>
<td>296.1</td>
</tr>
<tr>
<td>$q_{23}$</td>
<td>3.5538</td>
<td>1.8432</td>
<td>1.2611</td>
<td>7.2539</td>
<td>2.4838</td>
</tr>
<tr>
<td>$q_{31}$</td>
<td>493.5</td>
<td>491.5</td>
<td>492.5</td>
<td>488.7</td>
<td>515.7</td>
</tr>
<tr>
<td>$q_{32}$</td>
<td>573.3</td>
<td>594.1</td>
<td>594.3</td>
<td>576.1</td>
<td>472.7</td>
</tr>
</tbody>
</table>

For pricing purposes we make use of Quadratures applied to a decomposition to the sample space similar to the characterisation of point processes in Neveu [17]. We show this technique gives us an algorithm that is fast and accurate to calculate the distribution of losses in the two-state model.

We present some numerical results that show the ability of the model to represent the skew present in the market in terms base correlation for different periods of time.

References


Table 8: Calibration to iTraxx mid market quotes (Feb–Mar 2007).

<table>
<thead>
<tr>
<th></th>
<th>27-Feb</th>
<th>28-Feb</th>
<th>1-Mar</th>
<th>2-Mar</th>
<th>5-Mar</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-3%</td>
<td>8.66</td>
<td>9.95</td>
<td>10.22</td>
<td>11.02</td>
<td>12.73</td>
</tr>
<tr>
<td>3-6%</td>
<td>44.03</td>
<td>45.26</td>
<td>47.81</td>
<td>49.16</td>
<td>53.08</td>
</tr>
<tr>
<td>6-9%</td>
<td>11.88</td>
<td>12.52</td>
<td>13.01</td>
<td>13.65</td>
<td>14.94</td>
</tr>
<tr>
<td>9-12%</td>
<td>5.69</td>
<td>5.45</td>
<td>5.73</td>
<td>6.10</td>
<td>6.95</td>
</tr>
<tr>
<td>12-22%</td>
<td>2.04</td>
<td>2.13</td>
<td>2.21</td>
<td>2.39</td>
<td>2.56</td>
</tr>
<tr>
<td>CDS</td>
<td>22.10</td>
<td>22.88</td>
<td>23.19</td>
<td>23.78</td>
<td>25.05</td>
</tr>
</tbody>
</table>

μ₁     | 0.2527 | 0.2927 | 0.2865 | 0.2814 | 0.2984 |
μ₂     | 1.1174 | 0.7726 | 0.8486 | 1.0725 | 1.1843 |
q₁₂    | 23.8673| 22.8874| 25.0613| 23.2297| 21.1600|
q₁₃    | 0.6232 | 0.7383 | 0.7352 | 0.6961 | 0.6864 |
q₂₁    | 290.8  | 283.0  | 280.6  | 282.3  | 294.4  |
q₂₃    | 5.1275 | 3.6429 | 4.0824 | 5.2553 | 8.1463 |
q₃₁    | 222.7  | 217.7  | 228.5  | 219.3  | 215.3  |
q₃₂    | 27.0   | 40.2   | 27.6   | 32.6   | 38.0   |


