MARKET COMPLETION USING OPTIONS

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Abstract. Mathematical models for financial asset prices which include, for example, stochastic volatility or jumps are incomplete in that derivative securities are generally not replicable by trading in the underlying. In earlier work [Proc. R. Soc. London, 2004], the first author provided a geometric condition under which trading in the underlying and a finite number of vanilla options completes the market. We complement this result in several ways. First, we show that the geometric condition is not necessary and a weaker, necessary and sufficient, condition is presented. While this condition is generally not directly verifiable, we show that it simplifies to matrix non-degeneracy in a single point when the coefficients are real analytic functions. In particular, any stochastic volatility model is then completed with an arbitrary European type option. Further, we show that adding path-dependent options such as a variance swap to the set of primary assets, instead of plain vanilla options, also completes the market.

1. Introduction. It is well known that the Black-Scholes financial market model, consisting of a log-normal asset price diffusion and a non-random money market account, is complete: every contingent claim is replicated by a portfolio formed by dynamic trading in the two assets. Ultimately this result rests on the martingale representation property of Brownian motion. As soon as we attempt to correct the empirical deficiencies of the asset model by including, say, stochastic volatility, completeness is lost if we continue to regard the original two assets as the only tradables: there are no longer enough assets to ‘span the market’. However there are traded options markets for many assets such as single stocks or stock indices, so it is a natural question to ask whether the market becomes complete when these are included. An early result in this direction was provided by Romano and Touzi [13] who showed that a single call option completes the market when there is stochastic volatility driven by one extra Brownian motion (under some additional assumptions; see Section 5 below). But providing a general theory has proved surprisingly problematic. There are two main approaches, succinctly labelled ‘martingale models’ and ‘market models’ by Schweizer and Wissel [15].

In the former—which is the approach taken in Davis [1] and in this paper—one starts with a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \mathbb{P})$. $\mathbb{P}$ is a risk-neutral measure, so all discounted asset prices are $\mathbb{P}$-martingales which can be constructed by conditional expectation: the price process for an asset that has the integrable $\mathcal{F}_T$-measurable value $H$ at some final time $T$ is $S_t^H = \mathbb{E}[e^{r(T-t)}H | \mathcal{F}_t]$ for $t < T$, where $r$ is the riskless interest rate. The distinction between an ‘underlying asset’ and a ‘contingent claim’ largely disappears in this approach. A specific model is obtained by specifying some process whose natural filtration is $(\mathcal{F}_t)$, for example a diffusion process as in Section 2 below. In a ‘market model’ one specifies directly the price processes of all traded assets, be they underlying assets or derivatives. For the latter, say a call option with strike $K$ and exercise time $T$ on an asset $S_t$, it is generally more convenient to model a proxy such as the implied volatility $\hat{\sigma}$ which is related in a one-to-one way to the price process $A_t$ of the call by $A_t = \text{BS}(S_t, K, r, \hat{\sigma}_t, T-t)$, where $\text{BS}(\cdot)$ is the Black-Scholes formula. This is the approach pursued by Schönbucher [14] and, in different variants, in recent papers by Schweizer and Wissel [15] and Jacod and Protter [8]. This is not the place to debate the merits of these approaches; suffice it to say that the problem with martingale models is that the modelling of asset volatilities is too indirect, while the problem with market models is the extremely awkward set of conditions required for absence of arbitrage.

The paper is organised as follows. We first describe our market, i.e. we model the factor process spanning the filtration and write assets prices as conditional expectations. Then in Section 3 we give a necessary and sufficient condition for completeness of our market. Section 4 explores the case when the factor process solves an SDE with real analytic coefficients: we show that the completeness questions reduces from non-degeneracy of a certain matrix in whole domain to its non-degeneracy in a single point. The result is then applied in Section 5 to show completeness of stochastic volatility
models. Section 6 explores the use of path dependent derivatives, in particular variance swaps, in place of European type options and Section 7 concludes.

2. Market model. Consider a market in which investors can trade in $d$ assets $A = (A^1, \ldots, A^d)$. We assume there is no arbitrage in the market and we want to investigate market completeness on $[0, T]$. We therefore assume existence of an equivalent martingale measure and we chose to work under this measure, which we denote $\mathbb{P}$. The market is spanned by some factor process. More precisely, market factors are modeled with a $d$-dimensional diffusion process $(\xi_t)_{t \geq 0}$, solution to an SDE:

$$d\xi_t = m(t, \xi_t)dt + \sigma(t, \xi_t)dw_t, \quad \xi_0 = x_0 \in \mathcal{D},$$

(2.1)

where $w_t$ is a $d$-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, w.r.t. its natural filtration, and where $\mathcal{D} \subset \mathbb{R}^d$ is an open connected set. We assume

$$(A1) \quad (2.1) \text{ has a unique strong solution with } \mathbb{P}(\xi_t \in \mathcal{D}) = 1, \ t \geq 0, \ \sigma(t, x)\sigma(t, x)^T \text{ is strictly positive definite for a.e. } (t, x) \in (0, T) \times \mathcal{D}.$$  

The assumption of ellipticity above seems natural and corresponds to full factor representation. The semi-group of $(\xi_t)$ is denoted $(P_{u,t})$, i.e. $P_{u,t}h(x) = \mathbb{E}_{u,x}[h(\xi_t)]$, $u \leq t$, and $(\mathcal{F}_t)$ is the natural filtration of $(\xi_t)$ taken completed.

Traded assets are of European type, asset $A^i$ has a given payoff $h_i(\xi_T)$ at maturity $T_i$, larger than the time-horizon on which we investigate market completeness, $T \leq T_i$. We assume implicitly $\mathbb{E}[h_i(\xi_T)] < \infty$. As we choose to work under the risk neutral measure, the discounted price process of an asset is a martingale. More precisely,

$$A^i_t = \mathbb{E}\left[e^{-r(T_t-t)}h_i(\xi_{T_t}) \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T_i.$$  

(2.2)

The setup we have in mind in particular is: $A^1_t = S^1_t = \xi^1_t$ is the stock price itself, however for the questions considered here there is no benefit from making this particular assumption. We could also consider assets with path dependent payoffs, we will come back to this in Section 6.

The Markov property of $(\xi_t)$ implies that

$$A^i_t = v_i(t, \xi_t), \quad \text{where} \quad v_i(t, x) = e^{-r(T_t-t)}P_{t,T_i}h_i(x).$$  

(2.3)

We assume that

$$(A2) \quad v_i \text{ are of class } C^{1,2} \text{ on } (0, T) \times \mathcal{D}, 1 \leq i \leq d.$$  

Under suitable regularity conditions (cf. Friedman [2, Chp. 6]) we can apply the Feynman-Kac formula which immediately implies this property. Let $G(t, x)$ be the matrix of partial derivatives,

$$G(t, x) = \left(\frac{\partial v_i(t, x)}{\partial x_j}\right)_{1 \leq i, j \leq d} = \left(\nabla v_i(t, x)\right)_{1 \leq i \leq d}.$$  

(2.4)

Let $(M_i)$ be the martingale part of $(\xi_t)$, so that $dM_i = \sigma(t, \xi_t)dw_t$. Using the Itô formula together with the fact that discounted prices are martingales, we see that

$$d\tilde{A}_t = d(e^{-rt}A_t) = G(t, \xi_t)dw_t = G(t, \xi_t)\sigma(t, \xi_t)dw_t, \quad t \leq T.$$  

(2.5)

In what follows, we refer to the above setup simply as the market.

3. Market completeness: stochastic criterion. A predictable process $(\alpha_t)$ in $\mathbb{R}^d$ with

$$\mathbb{E} \int_0^T (\alpha_t)^2 dt < \infty, \quad 1 \leq i \leq d$$

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is a valid trading strategy, where $\alpha^i_t$ represents the number of units of $i^{th}$ asset invested at time $t$. It induces a self-financing portfolio $(X^\alpha_t)$ whose value behaves according to

$$dX^\alpha_t = \sum_{i=1}^{d} \alpha^i_t dA^i_t + \left( X^\alpha_t - \sum_{i=1}^{d} \alpha^i_t \bar{A}^i_t \right) r dt,$$

$$d\bar{X}^\alpha_t = d(e^{-rt}X_t) = \sum_{i=1}^{d} \alpha^i_t d\bar{A}^i_t.$$ 

An $\mathcal{F}_T$-measurable claim can be hedged if it is a final value of a self-financing portfolio with some starting capital, that is if there exists $(\alpha_t)$ with

$$H = \mathbb{E} H + X^\alpha_T = \mathbb{E} H + e^{rt} \sum_{i=1}^{d} \int_0^T \alpha^i_t d\bar{A}^i_t, \ a.s. \tag{3.2}$$

We say that the market is complete if any claim can be hedged. More precisely, using the above, we make the following definition

**Definition 3.1.** We say that the market on $[0, T]$ is complete if for any $\mathcal{F}_T$-measurable random variable $H$, $\mathbb{E} H^2 < \infty$, there exists a predictable process $(\alpha_t)$ such that $\forall 1 \leq i \leq d \ \mathbb{E} \int_0^T (\alpha^i_t)^2 d\langle \bar{A}^i \rangle_t < \infty$ and $H = \mathbb{E} H + \sum_{i=1}^{d} \int_0^T \alpha^i_t d\bar{A}^i_t$.

The assumption of integrability on $H$ is natural. General $H$ can still be represented but we need to authorize trading strategies $(\alpha_t)$ such that $\int_0^T \alpha_t dA_t$ is well defined while $\int_0^T \alpha^i_t d\bar{A}^i_t$ are not well defined (see Jacod and Shiryaev [9, Ex. III.4.10]) which makes little sense in market terms.

Let $\mathcal{S}$ be the set of zeros of the determinant of $G$ on $(0, T) \times \mathcal{D}$

$$\mathcal{S} := \left( \det G \right)^{-1}(\{0\}) \subset (0, T) \times \mathcal{D}, \tag{3.3}$$

which is a well defined Borel set. We can now state the characterization of market completeness.

**Theorem 3.2.** Under the assumptions (A1) and (A2), the market is complete, in the sense of Definition 3.1, if and only if $\int_0^T 1_{(t, \xi_t) \in \mathcal{S}} dt = 0 \ a.s.$

We can rephrase the above criterion by saying that $G(t, \xi_t)$ is non-singular dt-a.e. on $(0, T)\ a.s.$ In particular, if the law of $\xi_t$ admits a density, the market is complete if and only if $\mathcal{S}$ is of $(d+1)$-dimensional Lebesgue measure zero.

**Proof.** “$\Rightarrow$”

Let $G^{-1}(t, x) = G^{-1}(t, x) 1_{(t, x) \notin \mathcal{S}}$ and let $H$ be any $\mathcal{F}_T$-measurable random variable with $\mathbb{E} H^2 < \infty$. Using the representation theorem (cf. Rogers and Williams [12, V.25.1]) for $(\xi_t)$ we know there exists a predictable process $(\chi_t)$, $\mathbb{E} \int_0^T |\chi_t \sigma(t, \xi_t)|^2 dt < \infty$, where $|x|^2 = xx^T$, with $H = \mathbb{E} H + \int_0^T \chi_t \sigma(t, \xi_t) dw_t$.

Put $\alpha_t := \chi_t G^{-1}(t, \xi_t)$ which is a predictable process with

$$\mathbb{E} \sum_{i=1}^{d} \int_0^T (\alpha^i_t)^2 d\langle \bar{A}^i \rangle_t \leq \mathbb{E} \int_0^T |\alpha_t G(t, \xi_t) \sigma(t, \xi_t)|^2 dt \leq \mathbb{E} \int_0^T |\chi_t \sigma(t, \xi_t)|^2 dt < \infty.$$

We have

$$\int_0^T \alpha_t d\bar{A}_t = \int_0^T \chi_t \sigma(t, \xi_t) dw_t - \int_0^T \chi_t \sigma(t, \xi_t) 1_{(t, \xi_t) \in \mathcal{S}} dw_t = H - \mathbb{E} H,$$

where we used the assumption of the theorem (and Fatou lemma) to deduce that $\int_0^T |\chi_t \sigma(t, \xi_t)|^2 1_{(t, \xi_t) \in \mathcal{S}} dt = 0 \ a.s.$ and thus $\int_0^T \chi_t \sigma(t, \xi_t) 1_{(t, \xi_t) \in \mathcal{S}} dw_t = 0 \ a.s.$

“$\Longleftarrow$”

Suppose that $\mathbb{P} \left( \int_0^T 1_{(t, \xi_t) \in \mathcal{S}} dt > 0 \right) > 0$. Using Lemma 8.1 chose a measurable function $\beta : (0, T) \times \mathcal{D} \to \mathbb{R}^d$ such that

$$\begin{cases}
\beta(t, x) = 0 \\
G(t, x) \sigma(t, x)^T \beta(t, x)^T = 0, \ |\beta(t, x)\sigma(t, x)|^2 = 1
\end{cases} \text{ for } (t, x) \notin \mathcal{S}$$

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\end{cases} \text{ for } (t, x) \in \mathcal{S} \tag{3.4}$$
and let \( H = \int_0^T \beta(t, \xi_t) \sigma(t, \xi_t) dw_t \). Naturally \( H \) is \( \mathcal{F}_T \)-measurable and

\[
E H^2 = E \int_0^T |\beta(t, \xi_t) \sigma(t, \xi_t)|^2 dt = E \left[ \int_0^T 1_{(t, \xi_t) \in S} dt \right] \in (0, T].
\]

For any predictable process \( (\alpha_t) \) with \( E \int_0^T |\alpha_t G(t, \xi_t)\sigma(t, \xi_t)|^2 dt < \infty \) we have

\[
E \left[ H \cdot \int_0^T \alpha_t d\tilde{A}_t \right] = E \left[ \int_0^T \alpha_t G(t, \xi_t) \sigma(t, \xi_t) \sigma(t, \xi_t) \beta(t, \xi_t) dt \right] = 0,
\]

which proves that \( H \) is orthogonal to the space generated by the stochastic integrals w.r.t. \( \tilde{A} \) and the market is incomplete. \( \square \)

4. Market completeness: PDE approach. In Theorem 3.2 we stated a general necessary and sufficient condition for our market to be complete. So far however, we did not provide any easy means to verify the condition holds. This is the purpose of this section. We exploit the Feynman-Kac formula to rephrase our condition in terms of PDEs and then use classical results on the interior regularity of solutions of PDEs.

Let \( \mathcal{G}_t \) be the generator of \((P_{t,t+u})\) acting on regular functions \( f: \mathbb{R}^d \rightarrow \mathbb{R} \) via

\[
\mathcal{G}_t f(x) = \nabla f(x) m(t, x) + \frac{1}{2} \sum_{i,j=1}^d (\sigma(t, x) \sigma(t, x)^T)_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x).
\]

The Feynman-Kac formula, provided we can justify its application, shows that the functions \( v_i \) in (2.3) satisfy

\[
\mathcal{G} v_i := \frac{\partial v_i}{\partial t} + \mathcal{G}_t v_i - rv_i = 0, \quad (t, x) \in (0, T_i) \times \mathcal{D}
\]

\[
v_i(T_i, x) = h_i(x), \quad x \in \mathcal{D}.
\]

We need the above for our next result, so we assume explicitly that it holds:

\[
m, \sigma \text{ are such that } \xi_t \text{ admits a positive density on } \mathcal{D}, t \leq T,
\]

\[
(A3) \quad h_1 \text{ have at most polynomial growth,}
\]

\[
v_i \text{ are the unique solutions of (4.2) with at most polynomial growth}.
\]

A particular choice of conditions on \( m \) and \( \sigma \) which grants (A2) is not important, we refer the reader to Friedman [2, Thm. 6.5.3] for an example.

Theorem 3.2, in the above setting, states that the market is complete if and if only \( S \) is of zero Lebesgue measure. Thus, we can rephrase the question of market completeness into an equivalent question about proprieties of a solution to a system of PDEs. This seems to be a hard question in general, but we can solve it completely in the case of analytic coefficients, which covers a vast majority of studied models such as, for example, the Hull and White [7] or the Heston [4] stochastic volatility models.

Theorem 4.1. Suppose that (A1) and (A3) hold and that further \( m_i, \sigma_i : (0, T) \times \mathcal{D} \rightarrow \mathbb{R}, \ 1 \leq i \leq d \), are real analytic functions. Then the market is complete if and only if there exists a point \((t_0, x_0) \in (0, T) \times \mathcal{D} \) such that \( G(t_0, x_0) \) and \( \sigma(t, x_0) \sigma(t, x_0)^T \) are non-singular.

Proof. We first argue that \( v_i \) are real analytic a.e. on \((0, T) \times \mathcal{D}\). If \( \sigma \sigma^T \) is strictly positive definite on all \((0, T) \times \mathcal{D}\) we can directly evoke Theorem 7.5.1 in Hörmander [5]. The general argument is as follows. By Theorem 8.4.5 in Hörmander [6], the set of points where \( v_i \) is not analytic is the projection of the analytic wave front set of \( v_i \) on \((0, T) \times \mathcal{D} \) (the first \((d+1)\) coordinates). The wave front set in turn is contained in the characteristic set of the operator \( \mathcal{G} \), by Theorem 8.6.1 [6] and (4.2). The characteristic set of \( \mathcal{G} \) is the set of zeros of the principal symbol of \( \mathcal{G} \) that is (the cotangent bundle at any point is simply equal to \( \mathbb{R}^{d+1} \))

\[
\text{Char} \mathcal{G} = \left\{ (t, x, z) \in (0, T) \times \mathcal{D} \times \mathbb{R}^{d+1} : \sum_{i,j=1}^d (\sigma(t, x) \sigma(t, x)^T)_{i,j} z_i z_j = 0 \right\}.
\]
The projection of $\text{Char } G$ on $(0, T) \times D$ is equal to
\[\left\{(t, x) \in (0, T) \times D : \sigma(t, x) \sigma(t, x)^T \text{ not positive definite}\right\},\]
which is negligible by (A1). In consequence, $v_i(t, x)$ are real analytic for a.e. $(t, x) \in (0, T) \times D$.

Where $v_i$ are analytic so are their partial derivatives, which form the entries of the matrix $G$. Products and sums of analytic functions are also analytic and thus $\det G : (0, T) \times D \to \mathbb{R}$ is a.e. real analytic. In consequence, it is either equal to zero on the whole domain of analyticity or its set of zeros $S$ is Lebesgue negligible (cf. Krantz and Parks [11, p.83]). By Theorem 3.2, and since $\xi_t$ admits a density, the market is complete if and only if we are in the latter case. This in turn is equivalent since then by continuity of $\det G$, it is non-degenerate in some neighbourhood of $(t_0, x_0)$ of positive measure.

4.1. Example: correlated Brownian motion. Consider $\xi_t = \sigma w_t$ a correlated Brownian motion. In this simple case $D = \mathbb{R}^d$ and we know the semi-group so that we can write function $v_i$ explicitly. Assume for simplicity that all options mature at time $T$ and set $r = 0$. We then have
\[
v_i(t, x) = \mathbb{E}_0[h_i(\xi_{T-t})] = \frac{\sqrt{\det(Q)}}{(2\pi(T-t))^{d/2}} \int_{\mathbb{R}^d} h_i(y)e^{-\frac{Q(y-x)\cdot(y-x)}{2(T-t)}} dy,
\]
where $Q = (\sigma^{-1})^T\sigma^{-1}$. Note that in this setup one can show directly that $v_i$ are analytic on $(0, T) \times \mathbb{R}^d$.

We want to stress that the condition of non-degeneracy in at least one point is important in Theorem 4.1. As a counterexample, consider the situation when our assets are call options with different strikes.

More generally, suppose now that the payoffs depend only on the stock, i.e. $h_i(x) = h_i(x_1)$. We can represent $\xi_{T-t} = \tilde{\xi}_q(t) + \tilde{\xi}_jN$, with $N$ independent of $\xi^1_t$ which gives
\[
\mathbb{E}_0[h_i(x + \xi_{T-t}^1(\xi_{T-t}^j - x_j))] = \mathbb{E}_0[h_i(x + \xi_{T-t}^1)\xi_{T-t}^j].
\]

Working out the derivatives matrix we get
\[
G(t, x) = \frac{1}{T-t} \left( \mathbb{E}_x[h_i(\xi_{T-t})(\xi_{T-t}^j - x_j)] \right)_{i,j \leq d} \cdot Q,
\]
and (4.4) readily implies that $\det G \equiv 0$. In fact, in this setup whenever the payoffs depend only on $(d-2)$ or fewer factors $G$ is degenerate and market is incomplete. This is still true even if we consider options with different maturities.

A simple example when the market is complete is obtained taking $\sigma = Id$ and $h_i(x) = x_i^2$. Then $G$ is a diagonal matrix with $G(t, x)_{ij} = 2x_i1_{i\neq j}$. The set of singularities $S = (0, T) \times \{x : x_1, \ldots, x_d = 0\}$ has $(d+1)$-dimensional Lebesgue measure zero and the market is complete by Theorem 3.2.

5. Complete stochastic volatility models. We specialize now to the case $d = 2$ which corresponds to stochastic volatility models. We use the conventional notation so that $A^1_t = \xi^1_t = S_t$ is the stock price process and $\xi^2_t = Y_t$ is the process driving the volatility. The process $(S_t, Y_t)$ under the risk-neutral measure $\mathbb{P}$ satisfies
\[
\begin{cases}
dS_t = rS_t dt + \sigma(t, S_t, Y_t) d\tilde{w}_t^1, & S_0 = s_0 > 0, \\
dY_t = \eta(t, S_t, Y_t) dt + \gamma(t, S_t, Y_t) d\tilde{w}_t, & Y_0 = y_0,
\end{cases}
\]
where $\tilde{w}_t = \rho(t, S_t, Y_t) w^1_t + \sqrt{1 - \rho(t, S_t, Y_t)^2} w^2_t$. Romano and Touzi [13] were able to show that the above market is completed with a European call under additional assumptions that $\sigma, \eta, \gamma, \rho$ do not depend on $S_t$. We replace these assumptions with analyticity assumption.

**Proposition 5.1.** Consider assets $A^1_t = S_t$ and $A^2_t$ European option with a payoff $h(S_T) \geq 0$, where $h$ is an arbitrary not-affine function. Under (A1) and (A3) and if $\sigma, \eta, \gamma, \rho$ are real analytic then the market is complete.

**Proof.** First note that if $\sigma$ does not depend on $Y$, i.e. $\forall t \leq T, s > 0, y_1, y_2, \sigma(t, s, y_1) = \sigma(t, s, y_2)$, than the market is complete just by trading in the stock (we have in fact a local volatility model driven
by a Brownian motion). We assume thus that $\sigma$ depends on $y$ and we fix a non-affine payoff function $h \geq 0$. As our first asset is the first factor, we have $v_1(t, s, y) = s$ and the first row of $G(t, s, y)$ is simply $(1, 0)$. By Theorem 4.1 the market is complete if and only if $G$ is non-degenerate in at least one point, which is thus in turn equivalent to showing that there exists $(t, s, y)$ such that $\frac{\partial}{\partial y} v_2(t, s, y) \neq 0$. Suppose to the contrary so that $v_2(t, s, y) = v_2(t, s)$ is independent of $y$. It follows from (4.2) that $v_2$ satisfies

$$\frac{\partial v_2}{\partial t} + rs \frac{\partial v_2}{\partial s} = \frac{\sigma^2(t, s, y) s^2}{2} \frac{\partial^2 v_2}{\partial s^2} - rv_2 = 0. \quad (5.2)$$

The only term in (5.2) which depends on $y$ is $\sigma$ which implies that $\frac{\partial^2 v_2}{\partial s^2} = 0$ so that $v_2(t, s)$ is linear in $s$. Writing $v_2(t, s) = \alpha(t) + s \beta(t)$ and plugging in (5.2) we see that $\beta'(t) = 0$ and $\alpha'(t) = r\alpha(t)$. It follows that $h$ is an affine function which gives the contradiction. \(\square\)

6. On the choice of assets completing the market. We introduced in Section 2 the general setup of market driven by $d$-dimensional factor process $(\xi_t)$ in which we can trade in $d$ assets $(A_1, \ldots, A_d)$. As we work under risk-neutral measure, assets are specified uniquely via their maturities $T_i$ and payoffs $h_i(x)$, $A_i = E[e^{r(T_i-t)}h_i(\xi_T)|\mathcal{F}_t]$, where we assume $T_i \geq T$. In the basic setting $A_1 = S = \xi_1$, is the stock price itself and other options’ payoffs depend on the first coordinate only: $h_i(x) = h_i(x_1)$. More generally, we can think of having $n$ stocks, $A^i = S^i = \xi^i$, $1 \leq i \leq n$. Other assets then could include some basket options with payoffs $h(x) = h(x_1, \ldots, x_n)$. However so far we allowed only European style options. In various markets some path-dependent options, such as variance swaps, are very liquid and it may be natural to use them to complete the market. We show how now this can be incorporates in our setup.

Let $X_t = \log(S_t/S_0)$, where $S_t = A_1$ is the stock price process. The variance swap pays the quadratic variation $V_T = \langle X \rangle_T$ at maturity $T$ (cf. Gatheral [3, Chp. 11]). The process $\overline{X}_t = X_t - rt = \log(S_t/S_0)$ differs from $X$ by a finite variation term, so that $\langle X \rangle_T = \langle \overline{X} \rangle_T$. The price of a variance swap at time $t$ is given by

$$V_t = e^{-rt}E[e^{-rT}\langle \overline{X} \rangle_T|\mathcal{F}_t] = v_{V}(t, \xi_t).$$

The following derivation is well known:

$$\overline{X}_T = \int_0^T \frac{d\overline{S}_t}{\overline{S}_t} + \frac{1}{2} \int_0^T \frac{d\langle \overline{S} \rangle_t}{\overline{S}_t^2} = \int_0^T \frac{dS_t}{S_t} + \frac{1}{2} \langle \overline{X} \rangle_T,$$

and thus

$$V_t = 2e^{-r(T-t)} \int_0^t \frac{dS_u}{S_u} - 2E \left[ e^{-r(T-t)} \log(S_T/S_0) \big| \mathcal{F}_t \right]. \quad (6.1)$$

Suppose $d^{th}$ asset’s payoff is given as $h_d(x) = \log(x)$. It follows from (6.1) that

$$d\tilde{V}_t = d(e^{-rt}V_t) = 2\frac{dA^1}{A^1} - 2dA^d, \quad \text{or equivalently}$$

$$\nabla v_{V}(t, x) = \frac{2\nabla v_{V_1}(t, x)}{v_1(t, x)} - 2\nabla v_d(t, x). \quad (6.2)$$

In consequence, the rank of a matrix $G$, whose first row is $\nabla v_1$, remains unchanged when we replace the row $\nabla v_d$ by $\nabla v_V$. We state this as a proposition.

Proposition 6.1. Consider a market model of Section 2 with assets $A = (A_1, \ldots, A_d)$, where $A_1 = S$ is the stock price and $A^d$ has payoff $\log S_d$ at maturity $T$. Trading in $A$ completes the market if and only if trading in $\overline{A}$ completes the market, where $\overline{A} = A^i$, $i < d$ and $\overline{A}^d = V_t$.

The method presented above allows to investigate other path-dependent options, as long as we can write them as sum of trades in the remaining assets plus a different asset with a European payoff. We could for example consider an option paying $\langle S \rangle_T$.

7. Conclusions. To model realistically the dynamics of the stock price process one typically needs to consider models driven by more factors than just one Brownian motion. This naturally leads to market incompleteness when only trading in the stock is considered. Pricing of derivatives is no longer unique. It is in fact a challenging problem which has been extensively studied. However, in
present markets one does not need to price all derivatives. Indeed, some options are so liquid that they should be treated as inputs of the model. This was the starting point of our work.

The basic rule of the thumb is naturally: in order to have a complete market take as many assets (including the stock itself) as you have random processes spanning the filtration (see for example Karatzas and Shreve [10], Theorem 1.6.6). The question is then: when is this intuition actually correct? Theorem 4.1 shows that in the most regular case it is essentially always correct. More precisely, we consider market model written as an SDE with analytic coefficients. Then it suffices to check that the matrix governing the evolution of assets’ prices is non-singular in one point to deduce that the set of assets completes the market. In particular, in Proposition 5.1 we show that an arbitrary stochastic volatility model is always completed by a single European option.

It seems, there are two main open questions resulting from the present work. An analogue of Proposition 5.1 for higher-dimensional models would complement Theorem 4.1 and provide a full understanding of market completeness with options. Throughout, we considered the SDE (2.1) which is driven by a $d$-dimensional Brownian motion. The second remaining challenge is to extend this to the discontinuous setup. This is the subject of our further research.

8. Appendix. For completeness, we give the proof of the following measurable selection lemma used in the proof of Theorem 3.2.

**Lemma 8.1.** Let $G$ be defined via (2.4). There exists a measurable function $\beta : (0, T) \times \mathcal{D} \to \mathbb{R}^d$ satisfying (3.4).

**Proof.** Let $Z = (0, T) \times \mathcal{D}$ and define $\Gamma(z) = \Gamma(t, x) = G(t, x)\sigma(t, x)$, $z = (t, x)$. Matrix $\Gamma(z)$ induces a linear application on $\mathbb{R}^d$ and $\Gamma : Z \times \mathbb{R}^d \to \mathbb{R}^d$ given by $(z, y) \to \Gamma(z)y$ is a continuous function. For $z \in Z$ define

$$F(z) = \begin{cases} \{0\} & \text{if } \det \Gamma(z) \neq 0 \\ \text{otherwise} \end{cases} \quad (8.1)$$

$F(z)$ is a non-empty closed set in $\mathbb{R}^d$ for any $z \in Z$. Let $U \subset \mathbb{R}^d$ and put $F^-(U) = \{z \in Z : F(z) \cap U \neq \emptyset\}$. We will now argue that $F^-(U)$ is a measurable set for any closed $U$. Let $\tilde{U} = U \cap \{y : |y| = 1\}$ and observe that

$$F^-(U) = p_z \left( \Gamma^{-1}(\{0\}) \cap (Z \times \tilde{U}) \right), \quad (8.2)$$

where $p_z : Z \times \mathbb{R}^d \to Z$ is the projection, $p_z(z, u) = z$. Consider the set

$$\tilde{U}_n = \left\{ y \in \mathbb{R}^d : \inf_{u \in U} |y - u| < \frac{1}{n} \right\} \cap \left\{ y : ||y| - 1| < \frac{1}{n} \right\}. $$

Naturally, $\tilde{U}_n$ is open and $\bigcap_{n \geq 1} \tilde{U}_n = U$. Finally, let $B(0, 1/n) = \{y \in \mathbb{R}^d : |y| < 1/n\}$. The set $Y_n = \Gamma^{-1}(B(0, 1/n)) \cap (Z \times \tilde{U}_n)$ is open and thus its image by the projection $p_z$ is a measurable set. Furthermore, we have

$$\bigcap_{n \geq 1} p_z(T_n) = \left\{ z \in Z : \forall n \Gamma(z)^{-1}(B(0, 1/n)) \cap \tilde{U}_n \neq \emptyset \right\}$$

$$= \left\{ z \in Z : \forall n \exists \beta_n \in \Gamma(z)^{-1}(B(0, 1/n)) \cap \tilde{U}_n \ |\Gamma(z)\beta_n| < \frac{1}{n} \right\}$$

$$= \left\{ z \in Z : \exists \beta \in \tilde{U}_n \Gamma(z)\beta = 0 \right\} = p_z \left( \Gamma^{-1}(\{0\}) \cap (Z \times \tilde{U}) \right), \quad (8.3)$$

where the last equalities follow by choosing a converging subsequence $\beta_{n_k} \to \beta$ and observing that $\beta \in Ker(\Gamma(z)) \cap \tilde{U}$. In consequence, $F^-(U)$ is an intersection of measurable sets and $F$ is measurable.

As we work in metric spaces, $F$ is also weakly measurable and an application of the Measurable Selection Theorem of Kuratowski and Ryll-Nardzewski (cf. Wagner [16, Thm 4.1]) completes the proof. □
REFERENCES