1. Exercise value and zero-vol valuation. The exercise value of the payer’s swaption exercised at $T_0$ with payment dates $T_1 \ldots T_n$ is

\[
\left[ 1 - K \sum_{i=1}^{n} \theta_{i-1} p(T_0, T_i) - p(T_0, T_n) \right]^{+} = A(T_0) \left[ \frac{1 - p(T_0, T_n)}{A(T_0)} - K \right]^{+},
\]

where $A$ is the annuity

\[
A(t) = \sum_{i=1}^{n} \theta_{i-1} p(t, T_i).
\]

In the zero-volatility case we have

\[
p(T_0, T_i) = \frac{p(0, T_i)}{p(0, T_0)} := D_i / D_0.
\]

Hence

\[
\left[ 1 - p(T_0, T_n) \right]^{+} = \left[ \frac{1 - D_n}{\sum_i \theta_{i-1} D_i / D_0} - K \right]^{+} = \left[ \frac{D_0 - D_n}{\sum_i \theta_{i-1} D_i} - K \right]^{+}
\]

and

\[
A(T_0) = \sum_i \theta_{i-1} D_i / D_0.
\]

The value at time 0 is

\[
D_0 \times A(T_0) \times [\cdots]^{+} = A(0)[S_{T_0, T_0}^{T_0} - K]^{+}
\]

where $A(0) = \sum_i \theta_{i-1} D_i$ and $S_{T_0, T_0}^{T_0}$ is the forward swap rate.

2. HW Zero-coupon Bond Option Volatility. Recalling that $B(t, \lambda) = \frac{1}{\lambda}(1 - e^{-\lambda t})$, this is given by

\[
\begin{align*}
\sigma^2 T_0 &= \sigma^2 \int_0^{T_0} (B(T_1 - t, \lambda) - B(T_0 - t, \lambda))^2 dt \\
&= \sigma^2 \int_0^{T_0} \frac{1}{\lambda^2} e^{2\lambda t} (e^{-\lambda T_0} - e^{-\lambda T_1})^2 dt \\
&= \frac{\sigma^2}{2\lambda^3} (e^{2\lambda T_0} - 1) (e^{-\lambda T_0} - e^{-\lambda T_1})^2 \\
&= \frac{\sigma^2}{2\lambda^3} (1 - e^{-2\lambda T_0}) \left( 1 - e^{-\lambda (T_1 - T_0)} \right)^2
\end{align*}
\]

so that

\[
\sigma = \sigma B(T_1 - T_0, \lambda) \sqrt{\frac{B(T_0, 2\lambda)}{T_0}}.
\]
3. The Jamshidian Decomposition. The ZC bond \( p(T_0, T_i) \) is expressed as \( p(T_0, T_i) = p_i(r) \) where \( r \) is the short rate at time \( T_0 \). Let \( r^* \) be the unique value of \( r \) such that

\[
K \sum_{i=1}^{n} \theta_{i-1} p_i(r^*) + p_n(r^*) = 1
\]

and denote

\[
\alpha_i = K \theta_{i-1} p_i(r^*), \quad i < n
\]

\[
\alpha_n = (1 + K \theta_{n-1}) p_i(r^*).
\]

Then \( \sum_1^n \alpha_i = 1 \) and the exercise value (1.1) can be expressed as

\[
\sum_{i=1}^{n} [\alpha_i - K \theta_{i-1} p_i(r)]^+ + [\alpha_n - (1 + K \theta_{n-1}) p_n(r)]^+ = \sum_{i=1}^{n} K \theta_{i-1} [p_i(r^*) - p_i(r)]^+ + (1 + K \theta_{n-1}) [p_n(r^*) - p_n(r)]^+,
\]

expressing the swaption as a linear combination of zero-coupon bond put options with strikes \( p_i(r^*) \).

(The point is that all the options on the right are in the money precisely when \( r > r^* \).)

It is shown in Hull’s book\(^1\) that in the HW model

\[
p(t, T)(r) = \frac{p(0, T)}{p(0, t)} \exp \left( -B(T - t, \lambda) \frac{\partial}{\partial t} \log p(0, t) - H(t, T) \right),
\]

where

\[
H(t, T) = \frac{\sigma^2}{4\lambda^2} (e^{-\lambda T} - e^{-\lambda t})^2 (e^{2\lambda t} - 1).
\]

Using this formula, we can find the value of \( r^* \), and hence the values of \( p_i(r^*) \), by binary search. Then the ZC bond option values are given by the Black formula with volatility \( \sigma \) given by (2.1).

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\(^1\)J. Hull, Options, Futures and Other Derivatives, formula (17.25), page 434 in the 3rd edition. We will see the proof when we study the HJM model.