\[ \nu = \frac{1}{\sigma} \left( r - \frac{1}{2} \sigma^2 \right). \]

The price \( S_T \) is in the money but below the barrier level when \( x_T \in (a_1, a_2) \) where

\[ a_1 = \frac{1}{\sigma} \log \left( \frac{K}{S_0} \right), \quad a_2 = \frac{1}{\sigma} \log \left( \frac{B}{S_0} \right). \]

Denoting \( g(y,x) = \partial E_\nu(y,x) / \partial x \), the option value can now be expressed as

\[ E_\nu \left[ e^{-rT} [S_T - K]_+ 1_{M_T < B} \right] = e^{-rT} \int_{a_1}^{a_2} (S_0 e^{x \sigma} - K) g(y,x) \, dx. \]

Doing the calculations we obtain the option value given in [5] as a sum of four terms of the form \( c_1 N(c_2) \), as in the Black-Scholes formula. The up-and-out option price is

\[ S_0 \left( N(d_1) - N(x_1) + \left( \frac{B}{S_0} \right)^{2\lambda} (N(-y) - N(-y_1)) \right) \]

\[ + Ke^{-rT} \left( -N(d_2) + N(x_1 - \sigma \sqrt{T}) - \left( \frac{B}{S_0} \right)^{2\lambda - 2} (N(-y + \sigma \sqrt{T}) - N(-y_1 + \sigma \sqrt{T})) \right) \]

where \( d_1, d_2 \) are the usual coefficients and

\[ x_1 = \frac{\log(S_0/B)}{\sigma \sqrt{T}} + \lambda \sigma \sqrt{T} \]
\[ y_1 = \frac{\log(B/S_0)}{\sigma \sqrt{T}} + \lambda \sigma \sqrt{T} \]
\[ y = \frac{\log(B^2/(S_0 K))}{\sigma \sqrt{T}} + \lambda \sigma \sqrt{T} \]
\[ \lambda = \frac{r + \sigma^2 / 2}{\sigma^2} \]

Figures 2.2,2.3,2.4 show the value, delta and gamma of an up-and-out call option with strike \( K = 100 \), barrier level \( B = 120 \) and volatility 25%. The option matures at time \( T = 1 \). One can clearly see the “black hole” of barrier options: the region where the time-to-go is short and the priced is close to the barrier. In this region there is high negative delta, and there comes a point where hedging is essentially impossible because of the large gamma (i.e. unrealistically frequent rehedging is called for by the theory.)
Fig. 2.2. Barrier option value

Fig. 2.3. Barrier option delta
Fig. 2.4. Barrier option gamma