Mathematical Option Pricing

Robustness of Black-Scholes Hedging

If we assume the Black-Scholes price model
\[ dS_t = \mu S_t dt + \sigma S_t dw_t \] (1)
then the price at time \( t \) of an option with exercise value \( h(S_T) \) is
\[ C_{h}(S_t, r, \sigma, t) = C(t, S_t) \]
where \( C(t, s) \) satisfies the Black-Scholes PDE
\[ \frac{\partial C}{\partial t} + rs \frac{\partial C}{\partial s} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 C}{\partial s^2} - rC = 0, \] (2)
with boundary condition
\[ C(T, s) = h(s). \]

Suppose we sell an option at implied volatility \( \hat{\sigma} \), i.e. we receive at time 0 the premium \( C_{h}(S_0, r, \hat{\sigma}, 0) \), and we hedge under the assumption that the model (1) is correct with \( \sigma = \hat{\sigma} \).
The hedging strategy is then ‘delta hedging’: the number of units of the risky asset held at time \( t \) is the so-called option ‘delta’ \( \partial C/\partial s \):
\[ \phi_t = \frac{\partial C}{\partial s}(t, S_t). \] (3)

Suppose now that the model (1) is not correct, but the ‘true’ price model is
\[ dS_t = \alpha(t, \omega) S_t dt + \beta(t, \omega) S_t dw_t, \] (4)
where \( w_t \) is an \( \mathcal{F}_t \)-Brownian motion for some filtration \( \mathcal{F}_t \) (not necessarily the natural filtration of \( w_t \)) and \( \alpha_t, \beta_t \) are \( \mathcal{F}_t \)-adapted, say bounded, processes. It is no loss of generality to write the drift and diffusion in (4) as \( \alpha S, \beta S \): since \( S_t > 0 \) a.s. we could always write a general diffusion coefficient \( \gamma \) as \( \gamma_t = (\gamma_t / S_t) S_t = \alpha_t S_t \). In fact the model (4) is saying little more than that \( S_t \) is a positive process with continuous sample paths.

Using strategy (3) the value \( X_t \) of the hedging portfolio is given by \( X_0 = C(0, S_0) \) and
\[ dX_t = \frac{\partial C}{\partial s} dS_t + \left( X_t - \frac{\partial C}{\partial s} S_t \right) r dt \]
where \( S_t \) satisfies (4). By the Ito formula, \( Y_t \equiv C(t, S_t) \) satisfies
\[ dY_t = \frac{\partial C}{\partial s} dS_t + \left( \frac{\partial C}{\partial t} + \frac{1}{2} \beta^2 S_t^2 \frac{\partial^2 C}{\partial s^2} \right) dt. \]
Thus the hedging error \( Z_t \equiv X_t - Y_t \) satisfies
\[ \frac{d}{dt} Z_t = rX_t - rS_t \frac{\partial C}{\partial s} - \frac{\partial C}{\partial t} - \frac{1}{2} \beta^2 S_t^2 \frac{\partial^2 C}{\partial s^2}. \]
Using (2) and denoting \( \Gamma_t = \frac{\partial^2 C(t, S_t)}{\partial s^2} \), we find that
\[ dZ_t = rZ_t dt + \frac{1}{2} S_t^2 \Gamma_t^2 (\hat{\sigma}^2 - \beta_t^2). \]
Since \( Z_0 = 0 \), the final hedging error is
\[ Z_T = X_T - h(S_T) = \int_0^T e^{r(T-s)} \frac{1}{2} S_t^2 \Gamma_t^2 (\hat{\sigma}^2 - \beta_t^2) dt. \]
Comments:

This is a key formula, as it shows that successful hedging is quite possible even under significant model error. It is hard to imagine that the derivatives industry could exist at all without some result of this kind. Notice that:

- Successful hedging depends entirely on the relationship between the Black-Scholes implied volatility $\hat{\sigma}$ and the true ‘local volatility’ $\beta_t$. For example, if we are lucky and $\hat{\sigma}^2 \geq \beta_t^2 \text{ a.s.}$ for all $t$ then the hedging strategy (3) makes a profit \textit{with probability one} even though the true price model is substantially different from the assumed model (1), as long as $\Gamma_t \geq 0$, which holds for standard puts and calls.

- The hedging error also depends on the option convexity $\Gamma$. If $\Gamma$ is small then hedging error is small even if the volatility has been underestimated.