Multi-Asset Options

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February 27, 2003

This note covers pricing of options where the exercise value depends on more than one risky asset. Section 1 describes a very useful formula for pricing exchange options, while section 2 gives a model for the FX market, where the option could be directly an FX option or an option on an asset denominated in a foreign currency.

1 The Margrabe Formula

This is an expression, originally derived by Margrabe [1], for the value

\[ C = E[e^{-rT} \max(S_1(T) - S_2(T), 0)] \]

of the option to exchange asset 2 for asset 1 at time \( T \). It is assumed that under the risk-neutral measure \( P \), \( S_1(t) \) and \( S_2(t) \) satisfy

\[ dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dw_1, S_1(0) = s_1 \]  
\[ dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dw_2, S_2(0) = s_2, \]

where \( w_1, w_2 \) are Brownian motions with \( E[dw_1dw_2] = \rho dt \). The riskless rate is \( r \). The Margrabe formula is

\[ C(s_1,s_2) = s_1 N(d_1) - s_2 N(d_2) \]  

where \( N(\cdot) \) is the normal distribution function,

\[ d_1 = \frac{\ln(s_1/s_2) + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \]
\[ d_2 = d_1 - \sigma \sqrt{T} \]
\[ \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2} \]  

1.1 The Probabilistic Method

First, \( C \) does not depend on the riskless rate \( r \) since \( S_i(t) = e^{rt} \tilde{S}_i(t), i = 1, 2 \), where \( \tilde{S}_i(t) \) is the solution to (1),(2) with \( r = 0 \), and hence

\[ C = E[\max(\tilde{S}_1(T) - \tilde{S}_2(T), 0)] \]
\[ = E[\tilde{S}_2(T) \max(\tilde{S}_1(T)/\tilde{S}_2(T) - 1, 0)] \]  

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Henceforth, take $r = 0$ so that $\tilde{S}_i(t) = S_i(t)$. By the Ito formula, $Y(t) = S_1(t)/S_2(t)$ satisfies

$$dY = Y(\sigma_2^2 - \sigma_1\sigma_2\rho)dt + Y(\sigma_1dw_1 - \sigma_2dw_2)$$

and

$$\frac{1}{s_2}S_2(T) = \exp(\sigma_2w_2(T) - \frac{1}{2}\sigma_2^2T)$$

is a Girsanov exponential defining a measure change

$$\frac{d\tilde{P}}{dP} = \frac{1}{s_2}S_2(T).$$

Thus from (5)

$$C = s_2\tilde{E}[\max(Y(T) - 1, 0)].$$

By the Girsanov theorem, under measure $\tilde{P}$ the process

$$d\tilde{w}_2 = dw_2 - \sigma_2dt$$

is a Brownian motion. We can write $w_1$ as $w_1(t) = \rho w_2(t) + \sqrt{1-\rho^2}w'(t)$ where $w'(t)$ is a Brownian motion independent of $w_2(t)$ (under measure $P$). You can check that with $\tilde{P}$ defined by (7), $w'$ remains a Brownian motion under $\tilde{P}$, independent of $\tilde{w}_2$. Hence $d\tilde{w}_1$ defined by

$$d\tilde{w}_1 = \rho dw_2(t) + \sqrt{1-\rho^2}dw'(t)$$

$$= dw_1(t) - \rho\sigma_2dt$$

is a $\tilde{P}$-Brownian motion. The equation for $Y$ under $\tilde{P}$ turns out—miraculously—to be

$$dY = Y(\sigma_1d\tilde{w}_1 - \sigma_2d\tilde{w}_2)$$

which we can write

$$dY = Y\sigma dw,$$

where $w$ is a standard Brownian motion and $\sigma$ is given by (4). In view of (8), (9) the exchange option is equivalent to a call option on asset $Y$ with volatility $\sigma$, strike 1 and riskless rate 0. By the Black-Scholes formula, this is (3).

1.2 The PDE Method

This follows the original Black-Scholes “perfect hedging” argument. This time we work under the “objective” probability measure, under which $S_1$ and $S_2$ have drifts $\mu_1, \mu_2$ (rather than the riskless drift $r$) in (1),(2). Form a portfolio

$$X_t = C - \alpha_1S_1 - \alpha_2S_2 - \alpha_3P,$$

where $P(t) = \exp(-r(T - t))$ is the zero-coupon bond and the hedging component is self-financing, i.e. satisfies

$$d(\alpha_1S_1 + \alpha_2S_2 + \alpha_3P) = \alpha_1dS_1 + \alpha_2dS_2 + \alpha_3dP.$$
Assuming \( C(t, s_1, s_2) \) is a smooth function and writing \( C_1 = \partial C / \partial S_1 \) etc we have by the Ito formula,

\[
\begin{align*}
\text{d}X_t &= C_t + C_1 dS_1 + \frac{1}{2} C_{11} \sigma_1^2 s_1^2 dt + C_2 dS_2 + \frac{1}{2} C_{22} \sigma_2^2 s_2^2 dt \\
&\quad + C_{12} \rho \sigma_1 \sigma_2 S_1 S_2 dt - \alpha_1 dS_1 - \alpha_2 dS_2 - \alpha_3 r P dt.
\end{align*}
\]

If we choose \( \alpha_1 = C_1, \ \alpha_2 = C_2 \) this reduces to

\[
\text{d}X_t = (C_t + \frac{1}{2} C_{11} \sigma_1^2 s_1^2 + \frac{1}{2} C_{22} \sigma_2^2 s_2^2 + C_{12} \rho \sigma_1 \sigma_2 S_1 S_2 - \alpha_3 r P) dt \tag{11}
\]

and by standard “no-arbitrage” arguments \( X_t \) must grow at the riskless rate, i.e. satisfy

\[
\text{d}X_t = r X_t dt. \tag{12}
\]

We see that (10), (11), (12), are satisfied if \( C \) satisfies the PDE

\[
C_t + \frac{1}{2} C_{11} \sigma_1^2 s_1^2 + \frac{1}{2} C_{22} \sigma_2^2 s_2^2 + C_{12} \rho \sigma_1 \sigma_2 S_1 S_2 = r C - r C_1 s_1 - r C_2 s_2 \tag{13}
\]

(The terms involving \( \alpha_3 \) cancel.) Now from the definition of \( C \) and the price processes (1), (2) it is clear that \( C \) satisfies

\[
\lambda C(s_1, s_2) = C(\lambda s_1, \lambda s_2),
\]

for any \( \lambda > 0 \), so taking \( \lambda = 1/s_2 \) we have

\[
C(t, s_1, s_2) = s_2 f(t, s_1/s_2) \tag{14}
\]

where \( f(t, y) = C(t, y, 1) \). We can now calculate derivatives of \( C \) in terms of those of \( f \); for example

\[
C_1 = \frac{\partial f}{\partial y}, \quad C_2 = f - \frac{s_1}{s_2} \frac{\partial^2 f}{\partial y^2}.
\]

From these expressions we see that \( s_1 C_1 + s_2 C_2 = C \), so that the right-hand side of (13) is equal to zero for any \( C \) satisfying (14). This removes the dependence on \( r \) in (13). Substituting the remaining expressions into (13) we find that (13) is equivalent to the following PDE for \( f \):

\[
\frac{\partial f}{\partial t} + \frac{1}{2} y^2 \sigma^2 \frac{\partial^2 f}{\partial y^2} = 0, \tag{15}
\]

where \( \sigma \) is as defined above. The boundary condition is

\[
f(T, y) = C(T, y, 1) = \max(y - 1, 0). \tag{16}
\]

But (15) (16) is just the Black-Scholes PDE whose solution is (3).

Since the left-hand side of (13) is equal to zero, we see from (11) that \( X_t \equiv 0 \) if and only if \( X_0 = C(0, S_0) \) and \( \alpha_3 \equiv 0 \). The hedging strategy places no funds in the riskless asset.
1.3 Exercise Probability

As in Section 1.1, define $Y(t) = S_1(t)/S_2(t)$. We see from (5) that exercise takes place when $Y(T) > 1$, and under the risk-neutral measure $Y(t)$ satisfies (6). Hence the forward is $F = (s_1/s_0) \exp((\sigma_2^2 - \sigma_1 \sigma_2 \rho)T)$, and by standard calculations

Risk-neutral probability of exercise $= P[Y(T) > 1] = N(\hat{d}_2)$,

where

$$\hat{d}_2 = \frac{\ln(F) - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}}$$

$$= \frac{\ln(s_1/s_2) + \frac{1}{2}(\sigma_2^2 - \sigma_1^2)T}{\sigma \sqrt{T}}.$$

Note that the exercise probability is not $N(d_2)$, which is the exercise probability under the transformed measure $\tilde{P}$.

2 Cross-Currency Options

This section concerns valuation of an option on an asset $S_t$ denominated in currency F (for “foreign”) which pays off in currency D (for “domestic”). We denote by $f_t$ the exchange rate at time $t$, interpreted as the domestic currency price of one unit of foreign currency. Thus the currency D value of the asset $S_t$ is $f_t S_t$ at time $t$. In this note we ignore interest-rate volatility and take the foreign and domestic interest rates as constants $r_F, r_D$ respectively, so that the corresponding zero-coupon bonds have values

$$P_F(t, T) = e^{-r_F(T-t)}$$

$$P_D(t, T) = e^{-r_D(T-t)}.$$

2.1 Forward FX rates

To deliver one unit of currency F at time $T$, we can borrow $f_0 P_F(0, T)$ units of domestic currency at time 0 and buy a foreign zero-coupon bond maturing at time $T$. At that time the value of our short position in domestic currency is $-f_0 P_F(0, T)/P_D(0, T)$. By standard arguments, an agreement to exchange $K$ units of domestic currency for one unit of currency F at time $T$ is arbitrage-free if and only if $K = f_0 P_F(0, T)/P_D(0, T)$. In summary:

Forward price $= f_0 e^{(r_D-r_F)T}$.

This coincides with the formula for the forward price of a domestic asset with dividend yield $r_F$.

2.2 The domestic risk-neutral measure

The traded assets in the domestic economy are the domestic zero-coupon bond, value $Z_t = P_D(t, T)$, the foreign zero-coupon bond, value $Y_t = f_t P_F(t, T)$, and the foreign asset, value $X_t = f_t S_t$. An analogous set of assets is traded in the foreign economy. It is important to realize that there are two risk-neutral measures, depending on which economy we regard as “home”.
We will assume that in the domestic risk-neutral (DRN) measure the asset $S_t$ is log-normal, i.e. satisfies
\[ dS_t = S_t\mu dt + S_t\sigma_S dw^S(t) \] (17)
for some drift $\mu$ and volatility $\sigma_S$. The asset is assumed to have a dividend yield $q$. Similarly the FX rate $f_t$ is log-normal:
\[ df_t = f_t\gamma dt + f_t\sigma_f dw^f(t), \] (18)
with drift $\gamma$ and volatility $\sigma_f$. $w^S$ and $w^f$ are Brownian motions with $E dw^S dw^f = \rho dt$.

The discounted domestic value of the foreign zero-coupon bond is
\[ e^{-r_D t} f_t P_F(t,T) = e^{-r_F T} f_t e^{-(r_D - r_F)t}. \] (19)
This is a martingale in the DRN measure, which is true if and only if
\[ \gamma = r_D - r_F. \] (19)

Now consider a self-financing portfolio of foreign assets in which we hold $\phi_t$ units of asset $S_t$ and keep the remaining value in foreign zero-coupon bonds. The portfolio value process $V_t$ then satisfies
\[ dV_t = \phi_t dS_t + q\phi_t S_t dt + (V_t - \phi_t S_t) r_F dt. \]
Using (18),(19) and the Ito formula we find that the domestic value $U_t = f_t V_t$ of this portfolio satisfies
\[ dU_t = r_D U_t dt + \sigma_f U_t dw^f_t + \phi_t f_t S_t \sigma_S dw^S_t + \phi_t f_t S_t (\mu + q - r_F + \rho \sigma_S \sigma_f) dt. \]
Again, the discounted value $e^{-r_D t} U_t$ is a martingale in the DRN measure, and this holds if and only if
\[ \mu = r_F - q - \rho \sigma_S \sigma_f. \] (20)

In summary, under the DRN measure the FX rate and asset value satisfy the following equations
\[ df_t = f_t(r_D - r_F) dt + f_t \sigma_f dw^f(t), \] (21)
\[ dS_t = S_t(r_F - q - \rho \sigma_S \sigma_f) dt + S_t \sigma_S dw^S(t) \] (22)
By applying the Ito formula to (21),(22) we find that $X_t := S_t f_t$, the asset price expressed in domestic currency, satisfies
\[ dX_t = X_t (r_D - q) dt + X_t (\sigma_S dw^S(t) + \sigma_f dw^f(t)). \] (23)
By computing variances we find that
\[ \sigma_S w^S(t) + \sigma_f w^f(t) = \tilde{\sigma} w(t), \] (24)
where $w(t)$ is a standard Brownian motion and
\[ \tilde{\sigma} = \sqrt{\sigma_S^2 + \sigma_f^2 + 2 \rho \sigma_S \sigma_f} \] (25)
\[ E[ dw^S dw^f ] = \frac{1}{\tilde{\sigma}}(\sigma_f + \rho \sigma_S). \] (26)
Thus (23) becomes
\[ dX_t = X_t (r_D - q) dt + X_t \tilde{\sigma} dw(t). \] (27)
2.3 Option Valuation

2.3.1 Options on Foreign Assets

This refers to, for example, a call option with value at maturity time $T$

$$\max[X_T - K, 0],$$

i.e. the foreign asset value is converted to domestic currency at the spot FX rate $f_T$ and compared to a domestically-quoted strike $K$. Since $X_t$ satisfies (27) we see that the option value is just the Black-Scholes value for a domestic asset with volatility $\tilde{\sigma}$ given by (26).

2.3.2 Currency-Protected (Quanto) Options

Here the option value at maturity is $A_0 \max[S_T - K, 0]$ units of domestic currency, where $A_0$ is an arbitrary exchange factor, for example the time-zero exchange rate. The option value at time zero is

$$A_0 e^{-r_D T} E(\max[S_T - K, 0]).$$

The expectation is taken under the DRN measure, in which $S_t$ satisfies (22). Note that the volatility is $\sigma_S$ and the drift is $r_F - q - \rho \sigma_S \sigma_f = r_D - (q + r_D - r_F + \rho \sigma_S \sigma_f)$. We can therefore calculate the option value in two equivalent ways:

(i) Use the “forward” form of the BS formula with forward $F_T = S_0 \exp((r_F - q - \rho \sigma_S \sigma_f) T)$ and discount factor $\exp(-r_D T)$.

(ii) Use the “stock” form of BS with riskless rate $r_D$ and dividend yield $q + r_D - r_F + \rho \sigma_S \sigma_f$.

2.4 Hedging Quanto Options

2.4.1 Deriving the Hedge

The value of the quanto option given above has the usual interpretation as the initial endowment of a perfect hedging portfolio, but the formula does not indicate how the hedging takes place. To discover this, we re-derive the formula using the traditional Black-Scholes perfect hedging argument. For this we use the “objective” probability measure - not the risk-neutral measure - under which $X_t = S_t f_t$ and $f_t$ are log-normal processes satisfying

$$dX_t = \lambda X_t dt + \tilde{\sigma} X_t d\tilde{w}(t)$$

$$df_t = \nu f_t dt + \sigma_f f_t dw^f(t)$$

for some drift coefficients $\lambda, \nu$ the value of which, it turns out, we do not need to know. The point about the hedging argument is that from the perspective of a domestic investor, $S_t$ itself is not a traded asset: the traded assets are $X_t$ (the domestic value of $S_t$) and the foreign and domestic bonds $Y_t, Z_t$. From (29), the equation satisfied by $Y_t$ is

$$dY_t = (\nu + r_F) Y_t dt + \sigma_f Y_t dw^f.$$

Note that the sign of $\rho$ would be reversed if we had written the FX model in terms of $1/f_t$ rather than $f_t$.\footnote{Note that the sign of $\rho$ would be reversed if we had written the FX model in terms of $1/f_t$ rather than $f_t$.}
We know from section 2.3.2 that the quanto call option value at time \( t \) is a function \( C(t, S_t) = C(t, X_t/f_t) \) but we need to regard it as a function of \( X_t, f_t \) separately for hedging purposes. Note that if we define \( g(t, x, f) := C(t, x/f) \) then with \( C' = \partial C/\partial S \) we have

\[
\begin{align*}
\frac{\partial g}{\partial t} &= \frac{\partial C}{\partial t}, \\
\frac{\partial g}{\partial x} &= \frac{1}{f} C', \\
\frac{\partial g}{\partial f} &= -\frac{x}{f^2} C', \\
\frac{\partial^2 g}{\partial x^2} &= \frac{1}{f^2} C'', \\
\frac{\partial^2 g}{\partial f^2} &= \frac{2x}{f^3} C' + \frac{x^2}{f^4} C'' \\
\frac{\partial^2 g}{\partial x \partial f} &= -\frac{1}{f^2} C' - \frac{x}{f^3} C''
\end{align*}
\]

Let \( C(t, S_t) \) be the call value and consider the portfolio

\[ V_t = C(t, X_t/f_t) - \phi_t X_t - \psi_t Y_t - \chi_t Z_t, \tag{37} \]

where \( \phi_t, \psi_t, \chi_t \) are the number of units of \( X_t, Y_t, Z_t \) respectively in the putative hedging portfolio. Recall that \( S \) (and hence \( X \)) pays dividends at rate \( q \). Applying the Ito formula using (31) - (36) and (28),(29) and then substituting \( S = X/f \) we eventually obtain

\[
\begin{align*}
dV_t &= \left( \frac{\partial C}{\partial t} - SC'(v + \rho \sigma_S \sigma_f) + \frac{1}{2} \sigma_S^2 S^2 C'' - \psi(v + r_F) Y_t - \chi r_D Z_t \right) dt \\
&\quad + \left( \frac{1}{f} C' - \phi \right) dX - \phi q X dt - (\psi Y_t + SC') \sigma_f dw_f
\end{align*}
\]

Taking

\[
\phi = \frac{1}{f} C', \quad \psi = -\frac{1}{Y} SC', \tag{38, 39}
\]

this becomes

\[ dV_t = \left( \frac{\partial C}{\partial t} + SC'(r_F - y - \rho \sigma_S \sigma_f) + \frac{1}{2} \sigma_S^2 S^2 C'' - \chi r_D Z_t \right) dt \tag{40} \]

The usual no-arbitrage argument implies that \( V_t \) must grow at the domestic riskless rate, i.e.

\[
\begin{align*}
dV_t &= V_t r_D dt \\
&= \left( C - \frac{C'}{f} X + \frac{SC'}{Y} Y - \chi Z \right) r_D dt \\
&= (C - \chi Z) r_D dt \tag{41}
\end{align*}
\]

and, from (40) and (41), this equality is satisfied if \( C \) satisfies

\[
\frac{\partial C}{\partial t} + SC'(r_F - q - \rho \sigma_S \sigma_f) + \frac{1}{2} \sigma_S^2 S^2 C'' - r_D C = 0. \tag{42}
\]
The boundary condition is

\[ C(T, s) = A_0 [s - K]^+ \]  

If we write

\[ r_F - q - \rho \sigma_S \sigma_f = r_D - (q + r_D - r_F + \rho \sigma_S \sigma_f), \]

we can see that (42),(43) is just \( A_0 \) times the Black-Scholes PDE with volatility \( \sigma_S \), riskless rate \( r_D \) and dividend yield \( q + r_D - r_F + \rho \sigma_S \sigma_f \). This agrees with the valuation obtained in Section 2.3.2.

We still have to check the two key properties of the hedging portfolio, namely perfect replication and self-financing. The former is obtained by suitably defining \( \chi_t \); from (41),

\[ V_t \equiv 0 \text{ if } \chi_t = C(t, S_t) Z_t. \]  

(44)

To check the latter, note that the hedging portfolio value is \( W = \phi X + \psi Y + \chi Z \) and we now know that this is equal to the option value \( C(t, S_t) \). Hence

\[ dW = dC \]

\[ = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma_S^2 S^2 C'' \right) dt + C' dS \]

\[ = r_D C dt - SC'(r_F - q - \rho \sigma_S \sigma_f) dt + C' dS. \]  

(45)

(The second line is just an application of the Ito formula and the third uses the Black-Scholes PDE (42).) Now \( S = X/f \), so using (28), (29) we get from the Ito formula

\[ dS = \frac{1}{f} dX - S \frac{df}{f} - \rho \sigma_f \sigma_S dt. \]

Thus

\[ dW = r_D C dt - SC'(r_F - q - \rho \sigma_S \sigma_f) dt + C' \frac{dX}{f} - \frac{C'}{f} df - \rho \sigma_S \sigma_f C' S dt \]

\[ = r_D C dt - SC' \left( r_F dt + \frac{df}{f} \right) + q SC' dt + C' \frac{dX}{f} \]  

(46)

Using the definitions of \( \phi, \psi \) and \( \chi \) at (38),(39),(44) we see that the first, third and fourth terms of (46) are equal to \( \chi dZ, \phi q X dt \) and \( \phi dX \) respectively. Now \( Y_t = e^{-r_F(T-t) f_t} \), so

\[ dY_t = r_F Y_t dt + e^{-r_F(T-t) f_t} df_t \]

\[ = r_F Y_t dt + Y_t \frac{df}{f}, \]

showing that the second term in (46) is equal to \( \psi dY \). Thus (46) is equivalent to

\[ dW = \phi dX + q \phi X dt + \psi dY + \chi dZ, \]

which is the self-financing property.
2.4.2 Interpretation of the Hedging Strategy

Recall that the hedging portfolio is

$$ \phi X + \psi Y + \chi Z $$

where

$$ \phi = \frac{1}{f}C' $$

$$ \psi = -\frac{SC'}{Y} $$

$$ \chi = \frac{C}{Z} $$

The net value of the first two terms is zero, and this is what eliminates the FX exposure: \( \phi \) represents a conventional delta-hedge in Currency F, financed by Currency F borrowing (this is \( \psi \)). All increments in the hedge value are immediately “repatriated” and deposited in the home currency riskless bond \( Z \). The value in this domestic account is \( \chi Z = C \) so that, in particular, the value at the exercise time \( T \) is (for a call option) \( A_0[S_T - K]^+ \), the exercise value of the option.

References