THE HULL-WHITE SWAPTION FORMULA

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1. Exercise value and zero-vol valuation. The exercise value of the payer’s swaption exercised at $T_0$ with payment dates $T_1 \ldots T_n$ is

\[
(1.1) \quad \left[1 - K \sum_{i=1}^{n} \theta_{i-1} p(T_0, T_i) - p(T_0, T_n)\right]^+ = A(T_0) \left[1 - p(T_0, T_n) \frac{A(T_0)}{A(T_0)} - K\right]^+, 
\]

where $A$ is the annuity

\[
A(t) = \sum_{i=1}^{n} \theta_{i-1} p(t, T_i). 
\]

In the zero-volatility case we have

\[
p(T_0, T_i) = \frac{p(0, T_i)}{p(0, T_0)} := \frac{D_i}{D_0}.
\]

Hence

\[
\left[1 - p(T_0, T_n) \frac{A(T_0)}{A(T_0)} - K\right]^+ = \left[1 - \frac{D_n}{\sum_i \theta_{i-1} D_i} - \frac{D_0}{D_0} - K\right]^+ 
\]

and

\[
A(T_0) = \sum_i \theta_{i-1} \frac{D_i}{D_0}. 
\]

The value at time 0 is

\[
D_0 \times A(T_0) \times [\ldots]^+ = A(0)[S_{T_0}^{d} - K]^+ 
\]

where $A(0) = \sum_i \theta_{i-1} D_i$ and $S_{T_0}^{d}$ is the forward swap rate.

2. HW Zero-coupon Bond Option Volatility. Recalling that $B(t, \lambda) = \lambda^{-1}(1 - e^{-\lambda t})$, this is given by

\[
\sigma^2 T_0 = \sigma^2 \int_0^{T_0} (B(T_1 - t, \lambda) - B(T_0 - t, \lambda))^2 dt \\
= \sigma^2 \int_0^{T_0} \frac{1}{\lambda^2} e^{2\lambda t} \left(e^{-\lambda T_0} - e^{-\lambda T_1}\right)^2 dt \\
= \frac{\sigma^2}{2\lambda^3} \left(e^{2\lambda T_0} - 1\right) \left(e^{-\lambda T_0} - e^{-\lambda T_1}\right)^2 \\
= \frac{\sigma^2}{2\lambda^3} \left(1 - e^{-2\lambda T_0}\right) \left(1 - e^{-\lambda(T_1 - T_0)}\right)^2
\]

so that

\[
(2.1) \quad \sigma = \sigma B(T_1 - T_0, \lambda) \sqrt{\frac{B(T_0, 2\lambda)}{T_0}}. 
\]
3. The Jamshidian Decomposition. The ZC bond $p(T_0, T_i)$ is expressed as $p(T_0, T_i) = p_i(r)$ where $r$ is the short rate at time $T_0$. Let $r^*$ be the unique value of $r$ such that
\[
K \sum_{i=1}^{n} \theta_{i-1} p_i(r^*) + p_n(r^*) = 1
\]
and denote
\[
\alpha_i = K \theta_{i-1} p_i(r^*), \quad i < n
\]
\[
\alpha_n = (1 + K \theta_{n-1}) p_i(r^*).
\]
Then $\sum_1^n \alpha_i = 1$ and the exercise value (1.1) can be expressed as
\[
\sum_{i=1}^{n} [\alpha_i - K \theta_{i-1} p_i(r)]^+ + [\alpha_n - (1 + K \theta_{n-1}) p_n(r)]^+
\]
\[
= \sum_{i=1}^{n} K \theta_{i-1} [p_i(r^*) - p_i(r)]^+ + (1 + K \theta_{n-1}) [p_n(r^*) - p_n(r)]^+,
\]
expressing the swaption as a linear combination of zero-coupon bond put options with strikes $p_i(r^*)$. (The point is that all the options on the right are in the money precisely when $r > r^*$.)

It is shown in Hull’s book\(^1\) that in the HW model
\[
p(t, T)(r) = \frac{p(0, T)}{p(0, t)} \exp \left( -B(T - t, \lambda) \frac{\partial}{\partial t} \log p(0, t) - H(t, T) - B(T - t, \lambda) r \right),
\]
where
\[
H(t, T) = \frac{\sigma^2}{4\lambda^2} (e^{-\lambda T} - e^{-\lambda t})^2 (e^{2\lambda t} - 1).
\]
Using this formula, we can find the value of $r^*$, and hence the values of $p_i(r^*)$, by binary search. Then the ZC bond option values are given by the Black formula with volatility $\sigma$ given by (2.1).

\(^1\)J. Hull, *Options, Futures and Other Derivatives*, formula (17.25), page 434 in the 3rd edition. We will see the proof when we study the HJM model.