1. The binomial tree. The tree has $N$ time steps corresponding to times $k = 0, 1, \ldots, N$, and models an asset price $S_k$. The price is normalized to $S_0 = 1$, and at each branch in the tree the price moves up to $S_{k+1} = uS_k$ or down to $S_{k+1} = dS_k$ where $u > 1$ and $d = 1/u$ so that the tree is re-combining; see Figure 1.1. If we define the ‘up’ probability $p$ (the same at every node) then $S_k$ is a discrete-time stochastic process. We let $\mathcal{F}_k$ be the $\sigma$-field generated by $\{S_0, S_1, \ldots, S_k\}$. At time $k$ the possible price values are specified by a vector $s_k = s_k[0], \ldots, s_k[k]$ with

$$s_k[j] = u^k d^{2j} = d^{2j-k}$$

(1.1)

(so that $s_k[0] = u^k$, $s_k[k] = d^k$, i.e. the prices are listed in decreasing order.) There is a riskless savings account such that $\$1$ invested at time $k$ is worth $\$R$ at time $k+1$. The condition for no arbitrage is

$$d < R < u.$$  (1.2)

The up probability $p$ is risk neutral if the discounted price $S_k/R^k$ is a martingale. At time 0 this requires that

$$S_0 = 1 = \frac{1}{R} (pu + (1-p)d) = E\left[\frac{S_1}{R}\right]$$

and it is easily seen that this is the same condition as that required at any node in the tree. Hence

$$p = \frac{R-d}{u-d}$$

and we note that $0 < p < 1$ if and only if (1.2) holds.

Most of the time, as we will see below, we are more interested in identifying the node the price process is at, rather than the actual value $S_k$ of the price. If we define

$$J_k = \frac{1}{2} \left( k + \frac{\log S_k}{\log d} \right)$$

(1.3)

then from (1.1) we see that if $S_k = d^{2j-k}$ then $J_k = j$, so $J_k$ is the process followed by the index of the price. Since there is (at each $k$) a 1-1 correspondence between $S_k$ and $J_k$, the $\sigma$-fields these processes generate are the same: $\mathcal{F}_k = \sigma\{S_0, \ldots, S_k\} = \sigma\{J_0, \ldots, J_k\}$.

In the classic American put option with strike $K$, the holder has the right to exercise at any time $k = 1, \ldots, N$ and receives at that time the exercise value $[K - S_k]^+$. If $S_k = s_k[j]$ then the exercise value will be $[K - u^k d^{2j}]^+$. The questions we wish to answer are (i) what is the price of this option at time 0?, and (ii) what is the holder’s optimal exercise strategy?

![Fig. 1.1. Binomial tree with $N = 4$ time steps.](image-url)
2. Optimal stopping. We will generalize slightly and consider a general exercise value $\hat{Y}_{kj} \geq 0$.

Define a function $\hat{V}$ by backwards recursion as follows:

$$
\hat{V}_{Nj} = \hat{Y}_{Nj}, \quad j = 0, \ldots, N \tag{2.1}
$$

$$
\hat{V}_{(k-1)j} = \max \left\{ \hat{Y}_{(k-1)j}, \frac{1}{R} \left[ p \hat{V}_{kj} + (1-p)\hat{V}_{k(j+1)} \right] \right\}, \quad j = 0, \ldots, k-1, \quad k = N, \ldots, 1 \tag{2.2}
$$

We denote by $S$ the stopping region $S = \{(k,j) : \hat{V}_{kj} = \hat{Y}_{kj}\}$ and by $C$ the complementary continuation region $C = \{(k,j) : \hat{V}_{kj} > \hat{Y}_{kj}\}$ Now let $\tau^*$ be the random stopping time\footnote{We write $V_{kj} = \hat{V}(k,j)$ as convenient.}

$$
\tau^* = \min \{ k : \hat{V}(k,j_k) = \hat{Y}(k,j_k) \} = \min \{ k : (k,j_k) \in S \}.
$$

Note that $\hat{Y} = \hat{Y}$ at time $N$ if not before, so $\tau^*$ is well defined. We claim

**Theorem 2.1.** (i) $\hat{V}_{00} = \max_{\tau \in \Sigma} \mathbb{E}[R^{-\tau} \hat{Y}(\tau, J_\tau)]$, where $\Sigma$ is the set of all $\mathcal{F}_k$-stopping times.

(ii) The time $\tau^*$ defined above is optimal: $\mathbb{E}[R^{-\tau^*} \hat{Y}(\tau^*, J_{\tau^*})] = \hat{V}_{00}$.

*Proof.* It is convenient to move to normalized units, defining $Y_{kj} = \hat{Y}_{kj}/R^k, V_{kj} = \hat{V}_{kj}/R^k$. Then (2.1), (2.2) become

$$
V_{Nj} = Y_{nj}, \quad j = 0, \ldots, N \tag{2.3}
$$

$$
V_{(k-1)j} = \max \left\{ Y_{(k-1)j}, pV_{kj} + (1-p)V_{k(j+1)} \right\}, \quad j = 0, \ldots, k-1, \quad k = N, \ldots, 1 \tag{2.4}
$$

From (2.4) we have $V(k-1, j_{k-1}) \geq \mathbb{E}[V(k, j_k)]|\mathcal{F}_{k-1}$, so that the process $k \mapsto V(k, j_k)$ is a supermartingale. In particular, for any stopping time $\tau \in \Sigma$ we have\footnote{Note that the Optional Sampling Theorem is valid here because all $\tau \in \Sigma$ are bounded stopping times.}$\hat{V}_{00} = \mathbb{E}[V(\tau, J_\tau)]$. On the other hand, for $\tau^*$ we have $\hat{V}_{00} = \mathbb{E}[V(\tau^*, J_{\tau^*})]$ since $j_k \in C$ for all $k < \tau^*$ and hence $V(k-1)j = pV_{kj} + (1-p)V_{k(j+1)}$ for $k \leq \tau^*$. \hfill $\square$

3. Option pricing and Hedging. To proceed further, define $X_k = V(k, j_k)$ which, as we saw above, is a supermartingale.

**Proposition 3.1.** $X_k$ can be expressed as

$$
X_k = M_k - A_k \tag{3.1}
$$

where $M_k$ is an $\mathcal{F}_k$-martingale and $A_k$ is an increasing process such that $A_0 = 0$ and $A_k$ is $\mathcal{F}_{k-1}$-measurable for $k = 1, \ldots, N$.

*Proof.* We have only to define $M_0 = X_0, A_0 = 0$ and

$$
M_n = \sum_{k=0}^{n-1} \Delta M_k, \quad A_n = \sum_{k=0}^{n-1} \Delta A_k, \tag{3.2}
$$

where

$$
\Delta M_k = X_{k+1} - \mathbb{E}[X_{k+1}|\mathcal{F}_k]
$$

$$
\Delta A_k = X_k - \mathbb{E}[X_{k+1}|\mathcal{F}_k].
$$

Clearly $M_k$ is a martingale, and $\Delta A_k \geq 0$ by the supermartingale property, so $A_k$ is increasing. (3.1) follows, since $\Delta M_k - \Delta A_k = \Delta X_k$. \hfill $\square$

The representation (3.1) is known as the *Doob decomposition* of $X_k$. The key point is the $\mathcal{F}_{k-1}$-measurability of $A_k$, i.e. $A_k$ is ‘known’ one step in advance. We use this below.

In the following, we assume to avoid triviality that $(0, 0) \in C$, i.e. it is not optimal to stop at $\tau = 0$.

**Proposition 3.2.** We can construct a self-financing portfolio in the underlying asset and the riskless asset whose value $W_k$ at time $k$ is equal to $R^kM_k$. In particular the initial endowment required is $W_0 = M_0 = V_0$.

*Proof.* Suppose, to start with, that we take $V_{00}$ units of cash and buy a portfolio consisting of $U$ units of the risky asset and $B$ in cash where $U$ and $B$ are chosen such that

$$
u U + RB = RV_{10} \tag{3.2}
$$

$$
\nu U + RB = RV_{11} \tag{3.3}
$$
\[ U = \frac{R}{u \cdot d} (V_{10} - V_{11}) \]
\[ B = \frac{1}{u \cdot d} (-V_{10} d + V_{11} u). \]

We readily see that with these values \( U + B = V_0 \). Since \((0, 0) \in \mathcal{C}\) we have \( A_1 = X_1 = V(1, J_1) \). Hence (3.2), (3.2) imply that \( W_1 = R M_1 \).

Now suppose that for some \( k \) we have constructed a portfolio with initial capital \( V_0 \) such that
\[ W_n = R^k M_n \]
for all \( n \leq k \). The Doob decomposition gives \( M_{k+1} = X_{k+1} + A_{k+1} \), and \( A_{k+1} \) is known at time \( k \), so \( M_{k+1} = V(k+1, J_k) + A_{k+1} \equiv \alpha_0 \) if the price moves up after time \( k \) and \( M_{k+1} = V(k+1, J_k + 1) + A_{k+1} \equiv \alpha_1 \) if it moves down; \( \alpha_0 \) and \( \alpha_1 \) are \( \mathcal{F}_k \)-measurable. Exactly the same argument as above, replacing \( RV_{10}, RV_{11} \) in (3.2) and (3.3) by \( R^{k+1} \alpha_0, R^{k+1} \alpha_1 \), shows that the time-\( k \) capital \( W_k \) can be reinvested so as to replicate \( R^{k+1} M_{k+1} \) at time \( k+1 \).

We now arrive at the main result.

**Theorem 3.3.** \( V_0 = \hat{V}_0 \) defined by (2.1), (2.2) is the unique arbitrage-free value for the American option.

**Proof.** Suppose first that the option is traded at a price \( \pi > V_0 \). We sell the option, collecting the premium \( \pi \), of which \( V_0 \) is invested in the hedging portfolio of Proposition 3.2, and \( \pi - V_0 \) is invested in the riskless asset. The buyer has the right to exercise at any time, but \( W_k \geq \hat{Y}_k \) a.s. for each \( k \) since
\[ W_k = R^k M_k \geq R^k M_k - R^k A_k = \hat{V}(k, J_k) \geq \hat{Y}(k, J_k). \]

This shows that we are hedged against exercise by the buyer at any time of his/her choosing, leaving us with a positive balance of at least \( R^N (\pi - V_0) \) at time \( N \). This is an arbitrage.

In the converse case \( \pi < V_0 \) we short the hedge portfolio, i.e. raise cash \( V_0 \) in exchange for acquiring a portfolio whose value at time \( k \) is \( -W_k \), and buy the option. As the holder, we have the right to choose the exercise time, and we choose \( \tau = \tau^* \) at which time \( \hat{Y}_{\tau^*} = \hat{V}_{\tau^*} = W_{\tau^*} \) (note that \( A_{\tau^*} = 0 \)). Thus the exercise value is exactly enough to close out our short position in the hedge, leaving an arbitrage profit of \( R^N (V_0 - \pi) \) at the terminal time.

Finally, could there be an arbitrage if \( \pi = V_0 \)? We know that the strategy of selling the option and going long the hedging portfolio has value exactly zero if the buyer exercises optimally at \( \tau^* \) (the worst case for the seller). To get an arbitrage we would need to add a second portfolio \( \tilde{W}_k \) with zero initial capital and the property that
\[ \tilde{W}_k \geq 0 \text{ a.s. for each } k \text{ and } \mathbb{P}[\tilde{W}_{\tau^*} > 0] > 0. \]

However, this is impossible: Any portfolio process is a martingale under the risk-neutral measure and hence \( \mathbb{E}[\tilde{W}_{\tau^*}] = \mathbb{P}[\tilde{W}_0] = 0 \), whereas if (3.4) holds then necessarily \( \mathbb{E}[\tilde{W}_{\tau^*}] > 0 \).