M3S11 / M4S11 GAMES, RISKS AND DECISIONS

EXERCISES

1. Show that for any real numbers a, b, c, d, e, the two-person zero-sum game with pay-off matrix

$$\begin{matrix} a & a & b & b \\ c & d & c & d \\ c & e & c & e \end{matrix}$$

always has a pure -strategy saddle-point.

If a finite two-person zero-sum game with pay-off matrix G has a pure-strategy saddle-point, does the game with pay-off matrix G^T (the transpose of G) necessarily have a pure-strategy saddle-point?

2. Consider the 2 x 2 two-person zero-sum game in which distinct payoffs a, b, c, d are allocated at random to the entries of the pay-off matrix G. What is the probability that the game has a pure-strategy saddle-point?

Show that the probability that a 3 x 3 game with distinct pay-offs $a, b, c, d, e, f, g, h, i$, allocated randomly, has a pure strategy saddle-point is 3/10.

3. Describe one player's strategy space for the game of GOPS with 3 cards. How many elements does it contain?

4. In the two-person zero-sum game $\mathcal{G}$ the pure strategy spaces for the two players are X_S and Y_S, where $X_S = Y_S = [0, 1]$. If the pay-off to player X is $g(x, y)$, find the upper and lower pure values of $\mathcal{G}$ from first principles when $g(x, y) = |x - y| - (x - y)^2$

5. In a two-person zero-sum game with pay-off g each of the pairs of randomised strategies (α, β) and (γ, δ) is in equilibrium. Show that each of the pairs (α, δ) and (γ, β) is also in equilibrium and that

$$g(\alpha, \beta) = g(\alpha, \delta) = g(\gamma, \beta) = g(\gamma, \delta).$$

6. In a two-person zero-sum game $\mathcal{G}$ the pay-offs to A (and losses to B) are

		B		
		b_1	b_2	b_3
	a_1	1	2	3
A	a_2	2	3	2
	a_3	3	2	x

where a_i and b_i $(i = 1, 2, 3)$ are A's and B's pure strategies, and x is a real number.

Show that when $1 \leq x \leq 3$, the game has a simple solution and that the value of the game is $(x+7)/4$.

Solve the game when $(a) x > 3$ and (b) $x < 1$.

7. Solve the game of "scissors - paper-stone".

8. An army unit (A) is trying to defend m targets against an enemy (B). The targets are arranged in a row and only one target and any targets immediately next to it can be defended. B only has sufficient resources to attack a single target. If B attacks a defended target there is no loss to either side but if B attacks an undefended target then A loses the target to B. All targets are equally valuable to both sides.

Find good strategies for A and B and the value of the game to A when (a) $m = 6$ (b) $m = 8$.

9. Find good strategies for A and B and the value of the game for the two-person zero-sum game with pay-off matrix $G = (g(a, b))$, where $g(a, b)$ is the gain to A when A plays a and B plays b :

$$\text{(i)} \begin{array}{ccc} 3 & -4 & 1 \\ 2 & 6 & 0 \\ 5 & 4 & 3 \end{array} \qquad \text{(ii)} \begin{array}{ccc} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{array} \qquad \text{(iii)} \begin{array}{ccc} x & -1 & -1 \\ -1 & y & -1 \\ -1 & -1 & z \end{array}$$

where x, y, z are real numbers.

10. Prove that in a two-person zero-sum game an admissible equaliser strategy is minimax (or maximin).

11. Consider the two-person zero-sum game with pay-off matrix $\begin{array}{ccc} & \mathbf{b}_1 & \mathbf{b}_2 \\ \mathbf{a}_1 & x & y \\ \mathbf{a}_2 & z & w \end{array}$

Show that if the game has no pure-strategy saddle-points then A should play a_1 and a_2 with probabilities

$$\frac{| z - w |}{| x - y | + | z - w |} \text{ and } \frac{| x - y |}{| x - y | + | z - w |}$$

respectively.

12. A randomised strategy α^* for A in a two-person zero-sum game with pay-off g to A is called a Bayes strategy with respect to a randomised strategy β for B if

$$g(\alpha^*, \beta) = \sup_{a \in \mathcal{A}} g(a, \beta).$$

Show that if α^* and β^* are maximin and minimax strategies for A and B and if the game has a value, then α^* is a Bayes strategy with respect to β^*.

13. Consider the two-person zero-sum game with pay-off matrix $\begin{matrix} a & b \\ c & d \end{matrix}$.

Show that if $(a-b)(b-d)(d-c)(c-a) \neq 0$, then the game has a pair of equaliser strategies if and only if $(d-b)(a-c) > 0$ and $(d-c)(a-b) > 0$. Find the equaliser strategies and the value of the game if these conditions hold.

Solve the game when $(a-b)(b-d)(d-c)(c-a) = 0$.

14. Solve the two-person zero-sum games with the following pay-off matrices:

(i) $\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{matrix}$ (ii) $\begin{matrix} 1 & 2 & 3 \\ 6 & 1 & 2 \\ 2 & 3 & 2 \end{matrix}$

15. Two people together inherit a painting worth £400. They each make a sealed bid. When the envelopes are opened the higher bidder inherits the painting and pays the other person an amount equal to the higher bid. If both make an equal bid then a fair coin is tossed to determine who shall inherit the painting and this person pays the other the amount of the (common) bid.

Solve this game if it is assumed that the only bids allowed are multiples of £100 (up to £400).
Solve the game if any real-numbered bid from £0 to £400 is allowed.

16. Show that in a two-person zero-sum game, a unique minimax strategy is admissible.

17. Solve the games for which the pay-off matrices are

(i) $\begin{matrix} 1 & 3 & 2 & 7 \\ 8 & 2 & 4 & 1 \end{matrix}$ (ii) $\begin{matrix} 1 & 4 & 3 \\ 3 & 2 & 4 \\ 5 & 1 & 4 \end{matrix}$

18. The 3x3 game with pay-off matrix $\begin{matrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{matrix}$ has no pure strategy saddle-points. Prove that a, b and c are all non-zero, that a and c have the same sign and that a and b have opposite signs. Solve the game.

19. In a two-person zero-sum game the space of pure strategies for each player is [0,1] and the gain to X when X plays x and Y plays y is

$$g(x,y) = |x-y| - (x-y)^2.$$

Show that the randomised strategy which is uniformly distributed over [0,1] is an equaliser strategy for each player.

20. In an S-game show that
(i) if s is a unique Bayes strategy w.r.t the strategy α, then s is admissible.
(ii) if s is a Bayes strategy w.r.t. the strategy $\alpha = (p_1, ..., p_m)$, where $p_i > 0$ for $i = 1, 2, ..., m$, then s is admissible.

21. In a two-person zero-sum game, the strategy spaces for A and B are A_S and B_S, each of which consists of a finite set of non-zero real numbers. Each of the two strategy spaces, which may be distinct, contains at least one negative number and one positive number. Solve the game when the pay-off to A when A plays a and B plays b is

(i) $g(a, b) = ab$ (ii) $g(a, b) = sign(ab)\ c^{a+b}$ where $c > 0$ is a fixed real number.

22. Two companies, one large and one small, manufacture the same product. Each company wishes to build a new store in one of the 4 towns W, X, Y and Z which are located along a particular main road, in that order, with a distance of 5 miles between each consecutive pair of towns The total population of the 4 towns is distributed as follows:

Town W	20%
Town X	40%
Town Y	20%
Town Z	20%

If the large company's store is nearer to a town then it will capture 80% of the business; if both stores are equally distant from a town the large store will capture 60% of the busines; if the small store is nearer a town then the larger store only captures 40% of the business. By making reasonable assumptions, model this situation as a two-person zero-sum game and find good strategies for the two companies.

Before committing themselves to these good strategies, each company considers the effect of a change in the population distribution. There is some evidence that although the total population is remaining constant, there is a drift from town X to the other towns, with each of these 3 towns absorbing equal numbers of people from X. If p is the probability that a person leaves X, show that the good strategies found above remain good strategies provided $p \leq 3/8$. Solve the game if $p > 3/8$.

23. Show that if x, y, z, w are real numbers then the two-person non-cooperative game with the following pay-offs to A and B

$$\begin{array}{ccll} & & B & \\ & & b_1 & b_2 \\ A & a_1 & (x,z) & (y,z) \\ & a_2 & (x,w) & (y,w) \end{array}$$

is Nash solvable.

24. Show that in a Nash solvable non-cooperative two-person game the set of equilibrium pairs of strategies is a convex set.

25. In a two-person non-cooperative game the pay-offs to A and B are given by

$$\begin{array}{ccll} & & B & \\ & & b_1 & b_2 \\ A & a_1 & (1,4) & (9,0) \\ & a_2 & (7,1) & (3,3) \end{array}$$

Show that there are no pairs of pure strategies in equilibrium. Find a pair of randomised strategies in equilibrium.

26. Prove that in the non-cooperative Prisoners' Dilemma problem, there is just one equilibrium pair of strategies.

27. Sketch the region of the (x, y)- plane corresponding to pay-offs for randomised strategies for the non-cooperative two-person game

$$\begin{array}{ccll} & & B & \\ & & b_1 & b_2 \\ A & a_1 & (2,1) & (3,4) \\ & a_2 & (4,3) & (1,2) \end{array}$$

Is this game solvable in the strict sense?

28 . Two airlines, Pigeonair and Flapalot, are anxious to secure custom on cheap flights to and from Vilepest. Each airline can charge either a low fare for all of their return flights, or a high fare. If Pigeonair and Flapalot both charge a low fare then each secures half of the potential custom. If each charges a high fare some customers will decide not to fly because it has become too expensive, and each airline secures only 30% of the potential custom. If one charges a low fare and the other a high fare the former gets 80% of the potential custom and the latter only 10%. It is well known that staff at Pigeonair and staff at Flapalot never talk to one another. Describe the pay-off set. You should justify your answer.

Suppose now that if Pigeonair and Flapalot both charge a low fare, then each secures only 40% of the potential custom, all other percentages remaining the same. For what values of x is the point (x, x) in the pay-off set? Sketch the pay-off set (without further calculation).

29. In a two-person non-cooperative game the pay-offs to A and B are given by

$$\begin{array}{ccccc} & & & B & \\ & & b_1 & & b_2 \\ A & a_1 & (1,2) & & (4,3) \\ & a_2 & (3,1) & & (4,2) \end{array}$$

Find all the equilibrium pairs of randomised strategies and all the jointly admissible pairs of randomised strategies.

Is this game Nash solvable?

If A and B now cooperate,what is the Shapley solution?

30. For each non-zero sum two-person game below

(a) determine the pay-off set

$$S = \{(x, y) : x = g_A(\alpha, \beta), y = g_B(\alpha, \beta) \text{ for some } \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$$

when the game is played non-cooperatively.

Find the set of jointly admissible strategies and the equilibrium pairs of strategies. Is the game Nash solvable?

(b) determine the negotiation set and the Shapley solution when the game is played cooperatively.

(i) $\begin{array}{cccc} & & B & \\ & & b_1 & b_2 \\ A & a_1 & (2,4) & (4,2) \\ & a_2 & (3,1) & (1,3) \end{array}$ (ii) $\begin{array}{cccc} & & B & \\ & & b_1 & b_2 \\ A & a_1 & (2,1) & (3,0) \\ & a_2 & (0,4) & (2,5) \end{array}$

31. Find the Shapley solution to the non-zero cooperative game

$$\begin{array}{cccc} & & B & \\ & b_1 & & b_2 \\ & & & \\ a_1 & (a-b, a+b) & & (-b, a) \\ A & & & \\ a_2 & (a, b) & & (0, 0) \end{array}$$

where a and b are positive integers and $a > b$.

32 . Consider:

Lottery	*Payoff*	*probability*
1	£5,000	1
2	£25,000	0.1
	£5,000	0.89
	£0	0.01
3	£5,000	0.11
	£0	0.89
4	£25,000	0.1
	£0	0.9

Show that a person who prefers Lottery 1 to 2 and Lottery 4 to 3 is not obeying the utility axioms. This example is often quoted as an argument against the reasonableness of the axioms.

33. Consider the function

$$u(z) = 2\beta z - \beta^2 z^2 \qquad \text{for } 0 \leq z \leq 1/\beta$$
$$u(z) = 1 \qquad \text{for } z > 1/\beta$$

where β is a small positive constant. Explain the features of this function which might make it suitable to represent the utility of money, £z. How would you interpret the constant β?

A person has this utility function for extra money where $\beta < 0.1$, and they are presented with the two lotteries:

Lottery	*Payoff*	*probability*
1	£10	0.4
	£5	0.4
	£0	0.2
2	£10	0.1
	£5	0.9

Show that Lottery 1 is preferred to Lottery 2 if $\beta < 2/35$.

Suppose now that $\beta = 0.01$ and that all prizes in Lottery 1 and Lottery 2 are multiplied by a factor α, where $10 < \alpha < 20$. Show that Lottery 2 is preferred to Lottery 1 and that the utility of Lottery 2 is at least 0.775

34. A decision maker has the following utilities over the relevant portion of her overall assets scale, her current total assets being £7,000:

Assets ($£1,000s$)	5	6	7	8	9
utility	0.32	0.46	0.59	0.67	0.72

(i) She is offered a place on a lottery which gives a reward of £1000 with probability p and which loses her £1,000 with probability 1-p. For what values of p should she accept?

(ii) She is now offered a place on a lottery which consists of two independent plays of the lottery in (i). For what values of p should she accept?

35. A lottery gives a reward of £10,000 with probability 2/3 and a loss of £10,000 with probability 1/3. Five people with the same utility function (part of which is shown below) and each with assets of £10,000, each assigns the same probabilities to these consequences and they agree that they will share any profit or loss equally if each (independently) takes part in the lottery. Is this an attractive deal for each individual?

Assets($£1,000s$)	0	2.5	5	10	15	20
Utility	0	0.45	0.65	0.85	0.95	1.00

36. Describe the features of the function $u(z) = 1 - e^{-\lambda z}$ $(z \geq 0)$, where λ is a positive constant, which might make it suitable to represent the utility of money ($£z$). How would you interpret the constant λ?

An investor whose utility for extra money is $u(z)$ wishes to invest a total of $£M$ split between two independent companies, A and B. Any money, $£m$, invested in A brings a return of $£am$ with probability p and is lost with probability $1 - p$, where $a > 0$. Any money $£n$ invested in B brings a return of $£an$ with probability q and is lost with probability $1 - q$ where $0 < q < p < 1$. Assuming that the investor wants to maximise their expected utility of return, find the sum they should invest with company A.

37. In each case below find the posterior distribution of θ and the posterior mean and variance of θ :

(a) $X_i (i = 1, 2, \ldots, n)$ are independent negative binomial random variables with known index m and unknown parameter θ, and θ has a prior beta(α,β) distribution where α and β are known.

(b) $X_i (i = 1, 2, \ldots, n)$ are independent exponential random variables with unknown parameter θ where θ has a gamma prior distribution with index a and mean b where a and b are known.

38. The random variables $X_i (i = 1, 2, \ldots, n)$ are independent and each has a uniform distribution on $(0, \theta)$. A priori θ has a Pareto density:

$$\pi(\theta) = ba^b/\theta^{b+1} \qquad \text{for } \theta > a,$$

where a and b, positive constants, are the parameters of the Pareto distribution. Show that the posterior distribution of θ is also Pareto, with parameters $\max(a, x_{(n)})$ and $b + n$, where $x_{(n)} = \max(x_i)$.

39. Let $Y_i \ (i = 1, 2, \ldots, n)$ be independent normal random variables with

$$E(Y_i) = \theta x_i \text{ and } var(Y_i) = 1.$$

where $x_i \ (i = 1, 2, \ldots, n)$ are known constants, at least one of which is non-zero and θ is an unknown parameter. If the prior distribution of θ is uniform over the real line, determine the posterior distribution of θ.

40. Two envelopes contain £m and £2m respectively, where m is unknown to both you and your opponent. You are each handed an envelope at random and you are allowed to look at the contents of your own envelope. If $\pi(m)$ is your prior density for m, show that your expected monetary reward if you swap envelopes is

$$\frac{x[\pi(x/2) + 4\pi(x)]}{2[\pi(x) + \pi(x/2)]},$$

where x is the amount of money in the envelope you have been given. Hence show that you should swap envelopes if and only if

$$2\pi(x) > \pi(x/2).$$

41. An observable random variable X is Poisson with unknown mean θ. Under squared error loss:

(a) find the risk function of the decision rule $d_c(x) = cx$, where c is a constant, and show that $d_c(x)$ is inadmissible for $c > 1$.

(b) find the Bayes rule for estimating θ, when θ has a prior gamma density π with known index α and parameter λ $(\alpha > 0, \lambda > 0)$. Find the risk function of the Bayes rule and the Bayes risk of the prior π.

42. A device has been created which can supposedly classify blood as type A, B, AB or O. The device measures a quantity X which has density $f(x|\theta) = \exp(-(x - \theta))$ for $x \geq \theta$. If $0 < \theta < 1$, the blood is of type AB, if $1 < \theta < 2$, the blood is of type A; if $2 < \theta < 3$, the blood is of type B; and if $\theta > 3$, the blood is of type O. In the population as a whole θ has a unit exponential density. The loss in misclassifying the blood is:

		classified as			
		AB	A	B	O
true	AB	0	1	1	2
blood	A	1	0	2	2
type	B	1	2	0	2
	O	3	3	3	0

If $x = 4$ what is the Bayes action?

43. Let $\Theta = (\theta_1, \theta_2)$ and $A = [0, \pi/2]$, with the loss function: $L(\theta_1, a) = -\cos a$, $L(\theta_2, a) = -\sin a$. A coin is tossed and comes up heads with probability $p(\theta)$, where $p(\theta_1) = 1/3$ and $p(\theta_2) = 2/3$. Find $R(\theta_1, d)$ and $R(\theta_2, d)$ for all (non-randomised) decision rules d. Find the Bayes rule δ_0 with respect to the prior distribution π which gives prior probability one-half to each θ_i.
Show that δ_0 is minimax and that π is least favourable.

44. An observation X is from the geometric distribution with unknown **mean** $\theta > 1$. The decision rule d_c, for estimating θ is given by

$$d_c(x) = cx, \text{ where } c > 0.$$

Find the risk function for d_c under squared error loss.
Show that when $c > 1$, d_c is inadmissible.
If θ has a prior distribution which has density proportional to $1/\theta^2$ (for $\theta > 1$) find the Bayes Rule for estimating θ under squared error loss.

45. Let $\Theta = [0, 1] = A$ and $L(\theta, a) = (1-\theta)a + \theta(1-a)$. Show that $d(x) = 1/2$ is minimax.

46.The observable random variables $X_1, \ldots, X_n$ form a random sample from the normal distribution with mean zero and unknown variance $\theta > 0$. Consider the decision rule

$$d_c(\underline{x}) = c\sum^n x_i^2,$$

for estimating θ, where $\underline{x} = (x_1, \ldots, x_n)$ and c is a constant. Find the risk function of d_c under squared error loss, and show that for $c \neq \frac{1}{n+2}$, d_c is inadmissible.
Suppose now that $\phi = \theta^{-1}$ has a prior density function $\alpha(\alpha\phi)^{m-1}e^{-\alpha\phi}/\Gamma(m)$, where $\alpha > 0$ and $m > 0$ are known. Find the posterior desnity of ϕ and hence show that the Bayes Rule, d_B, for estimating θ under squared error loss is

$$d_B(\underline{x}) = \frac{2\alpha + \sum^n x_i^2}{2m + n - 2}.$$

Show that as $\alpha \to 0, d_B$ becomes inadmissible, provided $m \neq 2$.

[var(Z^2) $= 2$, where $Z\tilde{\ }N(0,1)$]

47. Show that if d^* is an equaliser decision rule, which is also a Bayes Rule with respect to a proper prior π_0,then $d^*$is a minimax rule and π_0 is least favourable.

48. Consider two boxes, A and B, each of which contains both red and black balls. One of the boxes contains equal numbers of red and black balls and the other contains a quarter red and three-quarters black balls, but you do not know which is which. Let W denote the box with equal numbers of red and black balls and suppose $P(W = A) = \theta, P(W = B) = 1 - \theta$.

You must select one of the boxes and then a ball (at random) from that box, and having noted its colour, you must then decide whether $W = A$ or $W = B$.

Show that if $1/2 < \theta < 2/3$ then in order to maximise your probability of a correct decision, you should select a ball from box B, but if $2/3 \leq \theta \leq 1$, then it does not matter which box you choose.

49. A customer insists that you give him a guarantee that a piece of machinery will not be faulty for one year. As the supplier you have the option of overhauling the machinery before delivering it to the customer, or of not overhauling it. The pay-offs you will receive by taking these actions are (in utility units):

	overhaul machinery	don't overhaul machinery
machine is faulty	700	0
machine is not faulty	800	1000

The machinery will work if a certain plate of metal is flat enough. You have scanning devices which sound an alarm if a small region on this plate is bumpy. The probabilities that any one of these scanning devices sounds an alarm given the plate is, or is not, faulty are respectively 0.9 and 0.4. You may decide to scan with n scanning devices (decision d_n), $n = 0, 1, 2\ldots,$ at a cost of $50n$ utility units and overhaul the machine, or not, depending on the number of alarms which ring. The scanning devices give independent readings conditional on whether the machine is faulty or not. Prior to scanning you believe that the probability that your machine is faulty is 0.2. Use the expected value of perfect information to eliminate certain non-optimal decisions. Draw a decision tree to find the optimal number of scanning devices and how to use them in order to maximise your expected utility.

50. A building has a structural fault which is either minor (with prior probability 0.9) or serious (with prior probability 0.1). It is not possible from

a superficial inspection to tell which kind of fault it is. If a correct diagnosis is made there are no extra costs incurred (above those for the appropriate repair of the building). If the fault is diagnosed as serious whereas in fact it is minor, an extra cost of £1 million is incurred. If the fault is diagnosed as minor whereas it is serious, an extra cost of £4 million is incurred.

A more detailed examination of the the building can be made at a cost of £c million, but this will only indicate the correct nature of the fault with probability 0.8.

The owner of the building must decide whether or not to have a detailed examination carried out, and must then make a diagnosis in the light of any examination results (or otherwise).

Assuming the owner wants to minimise the expected extra cost incurred, find the maximum amount he/she should be prepared to pay for the detailed examination.

51. A company is thinking of launching a new product which it has developed. The marketing executive estimates that the profit and loss resulting from different shares in the market are as follows:

		market share levels	
		10%	2%
prior	probabilities	0.7	0.3
DECISIONS			
launch		500	-250
scrap		0	0

The cost involved in scrapping the project can be regarded as zero since R&D costs are already sunk and are not relevant here.

The company is offered a market research proposal at a cost of 10 utility units.

The market research company will report either "the market share will be low" or "the market share will be high".

However, these do not measure up exactly to 10% and 2%. Here are the conditional probabilities for the possible market research reports given the market share:

		market share levels	
		10%	2%
Market	Research	report	
high		0.85	0.25
low		0.15	0.75

E.g. P (market research report says "high"| market share is 10%) = 0.85

Questions:

Would you accept the market research proposal?

How would you react to the findings of the market research company?