G-EQUIVARIANT l-SHEAVES

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1. Introduction

Prof. Bernstein and prof. Zelevinski ([1]) showed that the category of sheaves over an l-space X is equivalent to the category of non-degenerate modules over an algebra $S(X)$ of Schwarz functions on X and that the category of G-equivariant sheaves over a point is equivalent to the category of smooth representations and also to the category of modules over the Hecke algebra $\mathcal{H}(G)$. The main aim of this document is to combine these two facts and show that the category of G-equivariant l-sheaves over an l-space X is equivalent to the category of modules over a certain algebra generated by $S(X)$ and $\mathcal{H}(G)$. To prove that, we introduce more down-to-earth definition of G-equivariant l-sheaves and prove that it is equivalent to the standard definition. The document is based on "Representations of the group GL(n,F) where F is a non-archimedean local field" paper ([1]) and "Draft of: Representations of p-adic groups" ([2]).

2. Preliminaries

Definition 1. *We will call a topological space an l-space if it is Hausdorff, locally compact, countable at ∞ and totally disconnected (each point has a basis of open compact neighbourhoods).*

Definition 2. *We will call a topological group an l-group if it is an l-space.*

We will use the following well known fact.

Proposition 3. *Let G be a topological group and $H \leq G$ be an open subgroup. Then H is closed.*

Lemma 4. *Let an l-group G act continously on an l-space X. Then for each open-compact subset U in X there exists a neighbourhood V of $e \in G$ s.t. $V \cdot U = U$.*

Proof. As U is open and the action map $G \times X \to X$ is continuous, for every point $x \in U$ there exist neighbourhood V_x of $e \in G$ and neighbourhood U_x of x s.t. U_x stays in U under the action of V_x (i.e. $V_x \cdot U_x \subset U$). Due to the fact that U is compact, we can cover U by finite number of $U_{x_1}, \ldots, U_{x_m}$ and take the intersection of V_{x_i} as V. We see that $V \cdot U = U$, as $e \in V$. ∎

Proposition 5. *Let G be an l-group. Then $e \in G$ has a basis of neighbourhood which are open compact subgroups.*

Proof. We need to prove that every open compact neighbourhood of e contains an open compact subgroup. Let $U \subset G$ be an arbitrary open compact neighbourhood of e. As G acts on itself, the lemma 4 implies that there exists an open $V \subset G$ ($e \in V$) s.t. $V \cdot U = U$. Without loss of generality, we may assume that $V \subset U$ and $V = V^{-1}$. Take $H := \bigcup_{i=1}^{\infty} V^i$ - it is an open subgroup of G. H is closed from lemma 3.

We will show that $H \subset U$. It will imply that H is compact, because closed subsets of compact sets are compact .

We see that if $V^{i-1} \subset U$ then $V \cdot V^{i-1} \subset V \cdot U = U$, so by mathematical induction $H \subset U$. □

Proposition 6. *For every l-group G, there exists an unique G-left-invariant measure μ_G. It is called a Haar measure.*

Proof. It follows from [1, §1.18]. □

Definition 7. *Let $\mathcal{F}$ be a sheaf of $\mathbb{C}$-vector spaces on an l-space X. We will call $\mathcal{F}$ an l-sheaf. For $U \subset X$, $S(U, \mathcal{F})$ will mean compactly supported sections on U.*

Proposition 8. *With notation as in the previous definition, $S(X, \mathcal{F})$ is a natural non-degenerate $S(X)$-module and $\mathcal{F} \mapsto S(X, \mathcal{F})$ gives us an equivalence of categories between $Sh(X)$ (category of sheaves of $\mathbb{C}$-vector spaces on X) and $\mathcal{M}(S(X))$ (non-degenerate modules over $S(X)$). In particular $\mathcal{F}(U) = e_U S(X, \mathcal{F})$ for an open compact subset U (e_U is the characteristic function of U).*

Proof. See [2, §1.3, theorem 1]. □

Let $\mathcal{F}$ be an l-sheaf on an l-space X. Let $s \in \mathcal{F}(U)$, where U is open and compact. Using gluing axiom we can glue s with $0 \in \mathcal{F}(X \setminus U)$ to receive a global section. It gives as a morphism $\mathcal{F}(U) \hookrightarrow \mathcal{F}(X)$.

Definition 9. *We will call this morphism: extension by zero.*

We define l-germ-sheaf $\mathcal{F}$ on an l-space X as a family of vector spaces $\mathcal{F}(x)$ ($x \in X$) and family of sections (we will call them regular sections), $\phi : x \mapsto v_x \in \mathcal{F}_x$ satisfying following properties:

a A locally regular section is regular (s is locally regular if for every point in $x \in X$ there exists neighbourhood U of x and a regular section which is equal to s on U).
b Any vector in $\mathcal{F}(x)$ extends to a regular section.
c If a regular section is zero at a point, then it is zero in neighbourhood

To show that l-sheaves and l-germ-sheaves are "generally the same", we will need the following lemma:

Lemma 10. *Every cover of an l-space X has a countable refinement consisting of pairwise disjoint open compact sets*

Proof. Let $X = \bigcup_{i=1}^{\infty} K_i$ where K_i are compact subsets of X. As K_i are compact, each of them can be covered by a finite number of compact open sets. Let the union of sets covering K_i be denoted by $\tilde{U}_i$. Sets $\tilde{U}_i$ are open compact (as finite unions of open compact sets) and $X = \bigcup_{i=1}^{\infty} \tilde{U}_i$.

It holds that

$$U_{i+1} = \tilde{U}_{n+1} \setminus \bigcup_{i=1}^{n} \tilde{U}_i \tag{1}$$

is open compact, so X is a union of pairwise disjoint open compact subsets U_i. Since U_i are disjoint, the problem simplifies to the case of compact spaces (we restrict the cover W_α of X to the cover W'_α of U_i by intersecting with U_i). It is enough to find a required refinement for a cover W'_α of U_i - afterwards we take an union of refinements for every U_i.

As U_i is compact, every cover has a finite refinement consisting of open compact sets. By using procedure (1) we make the sets in the refinement pairwise disjoint. □

Proposition 11. *The category of l-germ-sheaves is equivalent to the category of l-sheaves.*

Proof. Firstly, we will prove that stalks $\mathcal{F}(x) = \mathcal{F}_x$ and global sections of an l-sheaf give us an l-germ-sheaf (i.e. they satisfy properties mentioned above).

Let us assume that a section s is locally regular. Lemma 10 implies that there exist pairwise disjoint open compact subsets $U_1, U_2, \dots$ and sections $s_i \in \mathcal{F}(U_i)$ s.t. $X = \bigcup_{i=1}^{\infty} U_i$ and $s_{|U_i} = s_i$. From gluing axiom $s \in \Gamma(\mathcal{F})$, which proves (a).

Let $v \in \mathcal{F}_x$ for $x \in X$. The definition of stalk implies that there exists an open compact subset $U \subset X$ and a section $s \in \mathcal{F}(U)$ s.t. $s_x = v$. We can extend s by zero (definition 9) to get a regular section required in (b).

Property (c) follows from the definition of a stalk.

Now, let $\mathcal{F}'$ be an l-germ-sheaf. Define a presheaf $\mathcal{F}$ by:

$$\mathcal{F}(U) = \{s_{|U} \mid \text{s is a regular section}\}$$

Let us prove that it is a sheaf. Property (a) implies gluing axiom. Uniqueness follows from the fact that $\mathcal{F}$ is described by values at each point.

Proposition 8 implies that the constructions of sheaves mentioned above are inverse to each other, but we don't know yet whether the family of vector spaces (germs) is preserved. In order to finish the proof, we must show that for $\mathcal{F}$ constructed from regular sections, it holds: $\mathcal{F}_x \cong \mathcal{F}(x)$ (the family of vector spaces $\mathcal{F}(x)$ is really a family of stalks of our sheaf). We have a natural map $\mathcal{F}_x \to \mathcal{F}(x)$ which is the evaluation of functions at a point x. It is surjective due to the property (b) and it is injective due to the property (c). Therefore it is required isomorphism. □

Definition 12. *For and l-space X let $C^\infty(X)$ be the space of locally constant functions on l-space X with values in $\mathbb{C}$. Let $S(X) \subset C^\infty(X)$ denote those functions which are compactly supported.*

Definition 13. *Let G be an l-group. Let $\mathcal{H}(G) \subset S^*(G)$ denote those distributions ($\mathbb{C}$-functionals on $S(G)$) which are locally constant (under the action of G on $S^*(G)$) and compactly supported.*

There is given a natural structure of algebra under convolution on $S(G)$, where G is an l-group equipped with a Haar measure $d\mu$, i.e. for $f, h \in S(G)$ and $x \in G$

$$(f * h)(x) := \int_{g\in G} f(g)h(g^{-1}x)d\mu$$

Proposition 14. *Distributions in $\mathcal{H}(G)$ are exactly of the form $h \mapsto \int_{g\in G} f(g)h(g)d\mu$ for $f \in S(G)$ and a Haar measure μ.*

Proof. See [2, §2.1, proposition 5]. □

We will denote distributions in $\mathcal{H}(G)$ of the aforementioned form by $\widetilde{f}$. There is given a natural structur of an algebra under convolution on $\mathcal{H}(G)$, i.e. $\widetilde{h} * \widetilde{g} := \widetilde{h * g}$.

Definition 15. *We say that an algebra A is idempotented if for every finite set of elements $\{a_i\}$ of A there exists an indempotent element $e \in A$ preserving them, i.e. $a_i e = e a_i$ for every a_i.*

Proposition 16. *$\mathcal{H}(G)$ is an idempotented algebra.*

Proof. Let $\{\widetilde{h_i}\}$ be a finite set of elements of $\mathcal{H}(G)$ and let $\{h_i\}$ be a set of corresponding elements of $S(G)$. Let U be a sum of supports of h_i. Then $\widetilde{e_U}\widetilde{h_i} = \widetilde{h_i} = \widetilde{h_i}\widetilde{e_U}$. □

We will consider G-representations over $\mathbb{C}$. We do not demand that the space is finitely dimensional or has a topology.

Definition 17. *We call a G-representation M smooth if the action of G on M is locally constant, i.e. for all $v \in M$ it holds that $Stab_G(v)$ is open.*

Let M be a smooth G-module. We define the structure of $\mathcal{H}(G)$-module on M as follows: for $\mathcal{E} \in \mathcal{H}(G)$ given by $\mathcal{E} : h \mapsto \int_{g\in G} h(g)f(g)d\mu$ and $m \in M$

$$\mathcal{E}m := \int_{g\in G} f(g)(g \cdot m)d\mu$$

Proposition 18. *These constructions induce an equivalence of categories between smooth G-representations and non-degenerate $\mathcal{H}(G)$-modules.*

Proof. See [2, theorem 2]. □

Let G be an l-group acting on an l-space X. Let $\pi : G\times X \to X$ be the projection, $a : G \times X \to X$ be the action map of G.

Definition 19. *We say that an l-sheaf $\mathcal{F}$ together with an isomorphism $\rho : \pi^*(\mathcal{F}) \cong a^*(\mathcal{F})$ is a G-equivariant sheaf, if ρ is compatible with the group structure on G, i.e. the two morphisms of sheaves $\alpha, \beta : proj^*(\mathcal{F}) \to multi^*(\mathcal{F})$ induced by ρ are equal* *where $proj : G \times G \times X \to X$ is projection onto X, and $multi : G \times G \times X \to X$ is defined by $multi(g, g', x) = gg'x$.*

3. Equivalent definition of G-equivariant l-sheaf

Let us fix and l-group G and an l-space X.

Proposition 20. *Let $\mathcal{F}$ be an l-sheaf on X. A G-equivariant structure on $\mathcal{F}$ is equivalent to the following one, which we will call the action of G on $\mathcal{F}$. This action is precisely a family of morphisms $g : \mathcal{F}(U) \to \mathcal{F}(gU)$ for every $g \in G$ and open $U \subset X$ such that*

1. *they are compatible with restrictions of sheaf*
2. *$(g_2 \cdot g_1)s = g_2 \cdot (g_1 \cdot s)$ for every $U \subset X$ and $s \in \mathcal{F}(U)$*
3. *the action of G on $S(X, \mathcal{F})$ (induced by our map) is smooth*

Proof. At first we will construct the action of G on $\mathcal{F}$ from the structure of G-equivariant sheaf on $\mathcal{F}$.

Let $j : G \times X \to G \times X$ be the map $(g, x) \mapsto (g, g^{-1}x)$ and let $i_g : X \to G \times X$ for $g \in G$ be an injection $x \mapsto (g, x)$. Obviously, $a \circ j = \pi$.

The proof will be based on the following diagram:

$$\begin{tikzcd} X & G \times X \arrow[l, "a"'] & G \times X \arrow[l, "j"'] \arrow[ll, bend right, "\pi"'] & X \arrow[l, "i_g"'] \end{tikzcd} \tag{2}$$

At first, we see that $j^*a^*\mathcal{F} \cong \pi^*\mathcal{F}$, because $a \circ j = \pi$. Therefore $a^*\mathcal{F}(O)$ (where $O \subset G \times X$) is isomorphic to $\pi^*\mathcal{F}(\{(g, gx) \mid (g, x) \in O\})$. Composing these isomorphisms with ρ, we receive isomorphisms $\varphi_O : \pi^*\mathcal{F}(O) \to \pi^*\mathcal{F}(\{(g, gx) \mid (g, x) \in O\})$.

Now, we define the action of G on $\mathcal{F}$ as follows: $g \cdot s = i_g^* \circ \varphi_{G\times U}(\pi^*(s))$ for $g \in G$, $U \subset X$ and $s \in \mathcal{F}(U)$. Let us notice that $\pi^*(s) \in \pi^*\mathcal{F}(G \times U)$, $\varphi_{G\times U}(\pi^*(s)) \in \pi^*\mathcal{F}(\bigcup_{g\in G}\{g\} \times gU)$ and $g \cdot s \in \mathcal{F}(gU)$. In informal words, the action of $g \in G$ on $\mathcal{F}$ is induced by morphism $\varphi_{G\times U}$ by "restricting" it to the subset $\{g\} \times U$.

The first required property of the action of G is satisfied from the definition. The second property is equivalent to the fact that ρ is compatible with the group structure on G.

Let us take an open compact subset $U \subset X$ and identity $e \in G$. Consider a small neighbourhood V of e s.t. $VU = U$ (it exists from lemma 4). Subset V acts on $\mathcal{F}(U)$ and sends it to $\mathcal{F}(U)$ (because of $VU = U$ and definition of φ_O). Consider an arbitrary section $s \in \mathcal{F}(U)$. We will find smaller neighbourhood of e s.t. all elements of it will preserve s (act constantly). As every element of $S(X, \mathcal{F})$ has support on some open compact set, it will prove the third property of the action.

From the definition of pull-back and sheafification, $\pi^*\mathcal{F}$ is locally constant in G-direction, i.e. the function $g \mapsto i_g^*(f) \in \mathcal{F}(U)$ is locally constant on G for every open compact $U \subset X$ and $f \in \pi^*\mathcal{F}(G \times U)$). Therefore $\varphi_{V\times U}(\pi^*(s)) \in \pi^*\mathcal{F}(V \times U)$ must be locally constant section, which exactly means that the action near e is identity:

map $g \mapsto i_g^*\big(\varphi_{V\times U}(\pi^*(s))\big) = g \cdot s$ for $g \in V$ is locally constant (and e acts as identity).

Now, we will prove the other direction of the proposition. Let us consider the action of G on $\mathcal{F}$. We will create the isomorphism ρ of $\pi^*(\mathcal{F})$ and $a^*(\mathcal{F})$.

Using the action of G on $\mathcal{F}$ we will construct the family of maps:

$$\varphi_O : \pi^*\mathcal{F}(O) \to \pi^*\mathcal{F}(\{(g, gx) \mid (g, x) \in O\})$$

where $O \subset G \times X$.

At first we construct this map only for sections $\pi^*(f) \in \pi^*\mathcal{F}(V \times U)$, where $f \in \mathcal{F}(U)$, $V{\cdot}U = vU$ (where v is some element of V) and all elements of V acts in the same way on f. For such sections $\pi^*(f)$ we put: $\varphi_{V\times U}(\pi^*(f)) = \pi^*(v{\cdot}f) \in \pi^*\mathcal{F}(vU)$. This construction extends to all sections, because of the smoothness and because each section with compact support in $\pi^*(\mathcal{F})$ is a finite sum of sections, for which we defined φ_O (the extension is unique, because for every two covers of a set with subsets on which a section is constant in G-direction we can find a common refinement).

We notice that $(j^{-1})^*\pi^*\mathcal{F} = a^*\mathcal{F}$ (because $\pi = a \circ j$, see (2)). It gives us isomorphism between $\pi^*\mathcal{F}(\{(g, gx) \mid (g, x) \in O\})$ and $a^*\mathcal{F}(O)$ for $O \subset G \times X$. If we compose it with φ_O, we receive isomorphism of sheaves $\rho : \pi^*\mathcal{F} \to a^*\mathcal{F}$. The morphism ρ is the isomorphism, we were looking for.

Compatibility of G action with ρ follows from the associativity of the action of G on $\mathcal{F}$. This construction is inverse to the construction from the first part of the proof. □

Corollary 21. *With notation as in the previous proposition, a G-equivariant structure on $\mathcal{F}$ is equivalent to the structure of smooth G-module on $S(X, \mathcal{F})$ s.t.:*

(3) $$supp(g \cdot s) \subset g \cdot supp(s), \text{ for any } s \in S(X, \mathcal{F}).$$

Proof. In the light of the previous proposition 20, the only thing we need to do is to construct a map $G \times \mathcal{F} \to \mathcal{F}$ from the structure of G-module on $S(X, \mathcal{F})$.

For compact open subset $U \subset X$, let $g : \mathcal{F}(U) \to \mathcal{F}(gU)$ be the composition of maps:

$$\mathcal{F}(U) \hookrightarrow \mathcal{F}(X) \xrightarrow{g\cdot} \mathcal{F}(X) \twoheadrightarrow \mathcal{F}(gU)$$

where $\mathcal{F}(U) \hookrightarrow \mathcal{F}(X)$ is extension by zero (definition - 9). These maps extends to all open subsets of X. In order to prove the associativity of this action, let us draw the following diagram for a compact open U:

$$\begin{array}{ccccccc}
\mathcal{F}(X) & \xrightarrow{g_1\cdot} & \mathcal{F}(X) & = & \mathcal{F}(X) & \xrightarrow{g_2\cdot} & \mathcal{F}(X) \\
\uparrow{\scriptstyle ext} & & \downarrow{\scriptstyle res} & & \uparrow{\scriptstyle ext} & & \downarrow{\scriptstyle res} \\
\mathcal{F}(U) & \xrightarrow{g_1\cdot} & \mathcal{F}(g_1U) & = & \mathcal{F}(g_1U) & \xrightarrow{g_2\cdot} & \mathcal{F}(g_2g_1U)
\end{array}$$

The middle diagram is commutative for sections with support in g_1U, because of the property of *supp* (3), so the big diagram is commutative. The associativity of our action (lower row) follows from the associativity of G action on $S(X,\mathcal{F})$ (upper row).

□

4. The structure of $\mathcal{HS}(G\times X)$ module on sections of G-equivariant l-sheaf

We will construct an algebra $\mathcal{HS}(G\times X)$ which as a vector space will be equal to $S(G\times X)\cong S(G)\otimes S(X)\cong \mathcal{H}(G)\otimes S(X)$ (see definition 12).

We define multiplication in $\mathcal{HS}(G\times X)$ as follows: for $F_1,F_2\in\mathcal{HS}(G\times X)$:

$$F_1\cdot F_2(\tilde{g},\tilde{x}) := \int\limits_{g\in G} F_1(\tilde{g}g^{-1},g\tilde{x})F_2(g,\tilde{x})d\mu \tag{4}$$

where μ is the Haar measure on G. This gives $\mathcal{HS}(G\times X)$ the structure of $\mathbb{C}$-algebra. It is easy to see that it is associative.

Let $\mathcal{F}$ be an l-sheaf over an l-space X. We define the action of $\mathcal{HS}(G\times X)$ on $S(X,\mathcal{F})$ as follows: for $F\in\mathcal{HS}(G\times X)$ and $s\in S(X,\mathcal{F})$

$$F\cdot s = \int\limits_{g\in G} \pi(g)\big(i_g^*(F)s\big)d\mu$$

Let s_x denote the image of $s\in S(X,\mathcal{F})$ in $\mathcal{F}_x$ for $x\in X$. For germs the formula for the action is as follows:

$$(F\cdot s)_x = \int\limits_{g\in G} \pi(g)\big(F(g,g^{-1}x)s_{g^{-1}x}\big)d\mu$$

Proposition 22. *The action of $\mathcal{HS}(G\times X)$ on $S(X,\mathcal{F})$ is compatible with multiplication, i.e. $F_1\cdot(F_2\cdot s)=(F_1\cdot F_2)\cdot s$ for every $F_1,F_2\in\mathcal{HS}(G\times X)$ and $s\in S(X,\mathcal{F})$.*

Proof. We only need to check it on germs. We have:

$$\begin{aligned} F_1\cdot(F_2\cdot s)_x &= \int\limits_{g_1\in G} \pi(g_1)\Big(F_1(g_1,g_1^{-1}x)\cdot\big(F_2\cdot s\big)_{g_1^{-1}x}\Big)d\mu = \\ &= \int\limits_{g_1\in G}\int\limits_{g_2\in G} \pi(g_1g_2)\Big(F_1(g_1,g_1^{-1}x)F_2(g_2,g_2^{-1}g_1^{-1}x)s_{g_2^{-1}g_1^{-1}x}\Big)d\mu d\mu = \\ &\quad (\text{ let } \tilde{g}=g_1g_2 \text{ and } g=g_2) \\ &= \int\limits_{\tilde{g}\in G}\pi(\tilde{g})\Big(\int\limits_{g\in G}F_1(\tilde{g}g^{-1},g\tilde{g}^{-1}x)F_2(g,\tilde{g}^{-1}x)d\mu\Big)s_{\tilde{g}^{-1}x}d\mu = \\ &= \int\limits_{\tilde{g}\in G}\pi(\tilde{g})\Big((F_1\cdot F_2)(\tilde{g},\tilde{g}^{-1}x)\Big)s_{\tilde{g}^{-1}x}d\mu = (F_1\cdot F_2)\cdot s \end{aligned}$$

□

Remark 23. *$\mathcal{HS}(G\times X)$ is an idempotented algebra.*

Proof. Every element $F\in\mathcal{HS}(G\times X)$ is preserved by some $\mathcal{E}_L\otimes e_K$ for "small enough" $L\subset G$ and "big enough" $K\subset X$, which are open subsets (see (4)). □

Remark 24. *$S(X,\mathcal{F})$ is a non-degenerate $\mathcal{HS}(G\times X)$ module.*

Proof. Let $s\in S(X,\mathcal{F})$. Then $\mathcal{E}_L\cdot s=s$ and $e_K\cdot s=s$ for some open compact subsets $L,K\subset X$. We see that: $(\mathcal{E}_L\otimes e_K)s=s$. □

Theorem 25. *The category of G-equivariant l-sheaves on X is equivalent to the category of non-degenerate $\mathcal{HS}(G\times X)$ modules.*

Proof. Let M be a non-degenerate $\mathcal{HS}(G\times X)$ module. As $\mathcal{HS}(G\times X)$ is an $S(X)$ module (i.e. for $f'\in S(X)$ and $h\otimes f\in\mathcal{HS}(G\times X)$, $f'\cdot(h\otimes f):=h\otimes f'f$), it induces the structure of $S(X)$ module on M (due to the non-degeneracy). Every element of $\mathcal{HS}(G\times X)$ is preserved by e_K (for some open compact $K\subset X$), where K is "big enough". Therefore M is non-degenerate $S(X)$-module and there exists an l-sheaf $\mathcal{F}_M$ s.t. $M=S(X,\mathcal{F}_M)$ (see proposition 8).

We will give M the structure of $\mathcal{H}(G)$ module as follows. For $h\in\mathcal{H}(G)$ and $m\in M$:

$$h\cdot m:=(h\otimes e_K)m \tag{5}$$

where an open compact set $K\subset X$ is chosen such that: $e_K\cdot m=m$ (existence of such K follows from the fact that M is a non-degenerate $S(X)$ module).

Let $K\subset K'$ and $e'_K\cdot m=m$. Then:

$$(h\otimes e'_K)m=(h\otimes e'_K)(e_Km)=(h\otimes e'_Ke_K)m=(h\otimes e_K)m$$

Therefore the definition doesn't depend on the choice of K.

Every element of $\mathcal{HS}(G\times X)$ is preserved by left multiplication by $\mathcal{E}_L\otimes e_K$ (for some L and "big enough" K). As M is a non-degenerate $\mathcal{HS}(G\times X)$ module, M is also a non-degenerate $\mathcal{H}(G)$ module. It gives us the structure of a smooth G-module on M (see proposition 18).

In light of corollary 21, in order to give $\mathcal{F}_M$ the structure of G-equivariant sheaf we should prove that if $s \in M$ then $g \cdot supp(s) \subset supp(g \cdot s)$.

In general, for $F \in \mathcal{HS}(G \times X)$ the following fact is satisfied:

$$\text{supp}(F(m)) \subset \left\{ x : \text{exists g s.t.: } g^{-1}x \in \text{supp}(m) \text{ and } F(g, g^{-1}x) \neq 0 \right\} \tag{6}$$

Consider arbitrary element $g \in G$ and $s \in M$. From the construction of G-module structure from $\mathcal{H}(G)$-module structure (see proposition 18), there exists a compact open subgroup $L < G$ and open compact subset $K \subset X$ s.t. $g \cdot s = (\mathcal{E}_{gL} \otimes e_K) \cdot s$. Lemma 4 implies that we can take L satisfying: $L \cdot supp(s) = supp(s)$. We observe, that:

$$\begin{aligned} \text{supp}(g \cdot s) &= \text{supp}\big((\mathcal{E}_{gL} \otimes e_K) \cdot s\big) \overset{(6)}{\subset} \\ &\subset \left\{ x : x \in g' \cdot \text{supp}(s) \text{ and } g' \in gL \right\} = \\ &= \left\{ x : x \in gL \cdot \text{supp}(s) \right\} = \\ &= \left\{ x : x \in g \cdot \text{supp}(s) \right\} = \\ &= g \cdot \text{supp}(s) \end{aligned}$$

Thus F_M is a G-equivariant sheaf.

We should show that this construction is inverse to the construction presented at the beginning of this section.

Let M be a $\mathcal{HS}(G \times X)$ module and let $\mathcal{F}_M$ be a sheaf constructed from this module. We see that $M \cong S(X, \mathcal{F}_M)$ as vector spaces. We need to prove that $\mathcal{HS}(G \times X)$ acts on both in the same way. The constructions were made in a such a way that the action of $S(X)$-module as well as $\mathcal{H}(G)$-module on M and $S(X, \mathcal{F}_M)$ are the same. The action of $\mathcal{HS}(G \times X)$ on M and $S(X, \mathcal{F}_M)$ is uniquely induced by the structure of $\mathcal{H}(G)$-module and $S(X)$-module, because for $h \in \mathcal{H}(G)$ open $L \subset X$ and $s \in M$:

$$(h \otimes e_L) \cdot s = (h \otimes e_K) \cdot (e_L s) = h \cdot (e_L s)$$

where K is some open compact subset containing L s.t. $e_K \cdot s = s$ (see 5) and every element of $\mathcal{HS}(G \times X)$ decomposes into a sum of elements of the form $h \otimes e_L$. .

The inverse in other direction is clear from the constructions.

□

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