

Different viewpoints on multiplier ideal sheaves and singularities of theta divisors

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Abstract

In those notes, we give an overview of various approaches to multiplier ideal sheaves. Further, we discuss about singularities of the theta-null divisor on a moduli space of principally polarized abelian varieties.

1 Introduction

Multiplier ideal sheaves play a crucial role in modern algebraic geometry. Being developed as a tool to study solutions of certain partial differential equation, they turned out to be a significant invariant of singularities of complex and algebraic singularities.

There are two main aims of those notes. Firstly, we show how to understand multiplier ideal sheaves from perspectives of geometry, analysis and arithmetic. We present the proof of the fact that the reduction mod big enough p of a multiplier ideal sheaf coincide with so called big test ideal – an ideal coming from Frobenius actions. Moreover, we briefly explain, how the main invariant of multiplier ideal sheaves, the log canonical threshold, is related to differential operators and jet spaces.

Secondly, we discuss singularities of theta divisors – symmetric ample divisors on abelian varieties, whose associated line bundles have only one section. Those divisors are of surprising importance in algebraic geometry.

The story starts with a discovery that the theta divisor on the Jacobian of a curve can be used to study the geometry of the curve itself. Moreover, the celebrated Torelli theorem says the Jacobian with the theta divisor distinguishes the curve uniquely. It raised the following natural question, known as the Schottky problem: which abelian varieties are Jacobians of curves. It is believed that this problem is strongly related to understanding singularities of theta divisors. The study of singularities of theta divisors were further motivated by an astonishingly beautiful result of Clemens and Griffiths. Comparing singularities of the theta divisors of intermediate ja-

cobians of cubic threefolds and of rational threefolds, they showed that the former are not rational.

Having explained the importance of the singularities of theta divisors, let us briefly sketch the history of the research about them. First, George Kempf proved that the theta divisors of Jacobians are irreducible, normal and have rational singularities, and conjectured that in the general case of an irreducible theta divisor on an abelian variety, the singularities should be always rational. Later on, Kollar proved that normal theta divisors are log canonical. But it was not before Ein and Lazarsfeld ([5]), that the conjecture of Kempf has been cracked. The key point in the proof was their celebrated generic vanishing theorem.

Given that singularities of theta divisors on abelian varieties have been better understood, it is natural to ask about singularities of special divisors on moduli spaces of abelian varieties. In those notes, we will discuss a question raised by Prof. Shepherd-Barron – what can we say about singularities of the so-called theta-null divisor. We show that standard techniques of dealing with the problem does not lead to a success. We hope that our overview would be helpful for somebody who would try to undertake this question again.

The notes are organised in the following way. In Section 2 we discuss a general theory of multiplier ideal sheaves. In Section 3, we present the proof of the fact that singularities of irreducible theta divisors are log canonical and rational. Further, we give a sketch of a proof of the generic vanishing theorem. In Section 4 we give an overview of the theory of moduli spaces of principally polarized abelian varieties, and discuss singularities of the theta-null divisor.

The readers wanting to enhance their knowledge about algebro-geometric theory of multiplier ideal sheaves are recommended to consult an excellent book [9]. The analytic theory of multiplier ideal sheaves is well described in [3]. For the Frobenius-related theory, we refer to unpublished notes of Karl Schwede [6] or an article [4]. An overview of various approaches to multiplier ideal sheaves may be found in [1]. As for the theory of moduli spaces of principally polarized abelian varieties and singularities of special divisors on them, we recommend the survey [10] or preliminaries in the article [12].

2 Multiplier ideal sheaves and the log canonical threshold

The goal of this section is to present different ways of approaching singularities related to birational geometry, with a view toward the problem of understanding singularities of the theta-null divisor. We believe that it may serve as a brief but comprehensible independent survey of the theory.

In contrast to the excellent article [1] from an International Congress of Mathematicians, the following presentation is more down-to-earth, and elaborates more on foundational results.

2.1 The geometric definition and properties of a multiplier ideal sheaf

In this subsection we define multiplier ideal sheaves and present their basic properties.

Definition 2.1. We say that (X, D) is a *log pair* if $K_X + D$ is \mathbb{Q} -Cartier, where X is a projective variety over an algebraically closed field and D is an effective \mathbb{Q} -divisor on X .

A multiplier ideal sheaf measures how singular a variety is.

Definition 2.2. Let (X, D) be a log pair, and let $\mu: \bar{X} \rightarrow X$ be a resolution of singularities. We define the *multiplier ideal sheaf* of (X, D) to be

$$\mathcal{I}(X, D) := \mu_* \mathcal{O}_{\bar{X}}([K_{\bar{X}} - \mu^*(K_X + D)]).$$

We can show that a multiplier ideal sheaf does not depend on the choice of a resolution. Using a multiplier ideal sheaf we can define:

Definition 2.3. Let (X, D) be a log pair. We say that it is *kawamata log terminal* (klt in short) if

$$\mathcal{I}(X, D) = \mathcal{O}_X.$$

We say that it is *log canonical* (lc in short) if

$$\mathcal{I}(X, (1 - \epsilon)D) = \mathcal{O}_X$$

for all $0 < \epsilon \ll 1$.

Equivalently, one can define klt (respectively lc) singularities as those for which

$$\text{ord}_E(K_{\bar{X}/X} - \mu^*D)$$

is greater (respectively greater or equal) than -1 for any divisor $E \subseteq \bar{X}$.

Now, we can define an important invariant of singularities.

Definition 2.4. We define the *log canonical threshold* (LCT in short) of a log pair (X, D) , such that $(X, 0)$ is klt, to be

$$\text{lct}(X, D) = \sup\{\lambda \in \mathbb{R}_{>0} \mid \mathcal{I}(X, \lambda D) = \mathcal{O}_X\}.$$

Further, if $\lambda \in \mathbb{R}_{>0}$ is such that

$$\mathcal{I}(X, \lambda D) \neq \mathcal{I}(X, (\lambda - \epsilon)D),$$

for any $\epsilon > 0$, then we call λ a *jumping number*.

Note that the log canonical threshold is the greatest jumping number.

The importance of multiplier ideal sheaves is justified by the following vanishing theorem.

Theorem 2.5 (Nadel vanishing [9, Theorem 9.4.17]). *Let (X, Δ) be a log pair in characteristic 0 and let D be a big and nef \mathbb{Q} -Cartier divisor such that $K_X + \Delta + D$ is Cartier. Then*

$$H^i(X, \mathcal{O}_X(K_X + \Delta + D) \otimes \mathcal{I}(X, \Delta)) = 0$$

for $i > 0$.

If (X, Δ) is klt, then it recovers Kawamata-Viehweg vanishing theorem.

For further applications, let us note the following lemma, which says that that by restricting to a subvariety, the singularities may only get worse.

Lemma 2.6 ([9, Theorem 9.5.1]). *Let (X, Δ) be a log pair and let $H \subseteq X$ be a normal variety such that $H \not\subseteq \text{Supp } \Delta$. Let $\Delta_H \subseteq H$ be the restriction of Δ to H . Then $\mathcal{I}(H, \Delta_H) \subseteq \mathcal{I}(X, \Delta)_H$, where the latter is the restriction of $\mathcal{I}(X, \Delta)$ to H .*

Moreover, note the following fact, which we will need later.

Proposition 2.7 (cf. [9, Theorem 9.3.37]). *Consider a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ and assume that coefficients of monomials occurring in it are sufficiently general. Let D be the divisor $\{f(x) = 0\}$. Then*

$$\mathcal{I}(X, D)$$

does not depend on the coefficients of monomials in f .

For further usage, we remark the following: a log pair (X, D) such that D is an integral divisor, is log canonical if and only if $(D, 0)$ is log canonical (cf. [9, Corollary 9.5.11]).

2.2 The analytic definition of a multiplier ideal sheaf

The following section is based on [9] and [3]. Surprisingly, the singularities of a divisor can be measured using purely analytic methods. A hypersurfaces in \mathbb{C}^n defined by a power series $f \in \mathbb{C}[[z_1, \dots, z_n]]$ is kawamata log terminal if and only if $\frac{1}{|f|^2}$ is locally integrable!

In general, we have.

Proposition 2.8 ([3, Remark 5.9]). *Let (X, D) be a log pair such that X is a smooth complex projective variety. Let*

$$D = \sum a_i D_i,$$

and let D_i be defined by a holomorphic function $g_i \in \mathcal{O}_X(U)$ for some open set $U \subseteq X$ in the standard topology. Then.

$$\mathcal{I}(X, D)^{an}(U) = \bar{\mathcal{I}}(X, D)(U) := \left\{ f \in \mathcal{O}_X(U) \mid \frac{|f|^2}{\prod |g_i|^{2a_i}} \in L_{loc}^1(U) \right\},$$

where $\mathcal{I}(X, D)^{an}$ is the analytification of the multiplier ideal sheaf.

Proof. First, we show it in the case, when D is a simple normal crossing divisor with a nonsingular point at some $x \in U$. In this case $\mathcal{I}(X, D)(U) = \mathcal{O}_X(-[D])(U)$, and so we are left to show that

$$\frac{|f|^2}{\prod |z_i|^{2a_i}}$$

is locally integrable at x if and only if $|z_1|^{[a_1]} \cdot \dots \cdot |z_n|^{[a_n]} \mid f$. Looking at monomial ideals of f one by one, we see that the result follows from the standard fact in analysis:

The function $\frac{1}{|z|^{2a}}$ in a complex variable z is locally integrable if and only if $a < 1$.

The case when D is general follows from the lemma below and a standard change of coordinates formula

$$\mu_*(\mathcal{O}_Y(K_Y) \otimes \mathcal{I}(Y, \mu^* D)) = \mathcal{O}_X(K_X) \otimes \mathcal{I}(X, D),$$

where $\mu: Y \rightarrow X$ is a birational map. □

Lemma 2.9 ([3, Proposition 5.8]). *We keep the notation as above. Let $\mu: Y \rightarrow X$ be a birational map, where Y is smooth. Then*

$$\mu_*(\mathcal{O}_Y(K_Y) \otimes \bar{\mathcal{I}}(Y, \mu^* D)) = \mathcal{O}_X(K_X) \otimes \bar{\mathcal{I}}(X, D).$$

Proof. By definition, for an open subset $U \subseteq X$, we have that $(\mathcal{O}_X(K_X) \otimes \bar{\mathcal{I}}(X, D))(U)$ consists of holomorphic forms $f \in \mathcal{O}_X(K_X)(U)$, such that

$$\frac{f \wedge \bar{f}}{\prod |g_i|^{2a_i}} \in L_{loc}^1.$$

Let $S \subseteq X$ and $\mu^* S \subseteq Y$ be the subsets where μ is not an isomorphism. The key point is that

$$(\mathcal{O}_X(K_X) \otimes \bar{\mathcal{I}}(X, D))(X \setminus S) = (\mathcal{O}_X(K_X) \otimes \bar{\mathcal{I}}(X, D))(X),$$

that is, if you take a form $f \in (\mathcal{O}_X(K_X) \otimes \bar{\mathcal{I}}(X, D))(X \setminus S)$, then $f \wedge \bar{f} \in L_{loc}^2(X \setminus S)$, and so it must extend to X .

Hence, the lemma follows by noting that

$$\int_{X \setminus S} \frac{f \wedge \bar{f}}{\prod |g_i|^{2a_i}} = \int_{Y \setminus \mu^*(S)} \frac{\mu^*(f) \wedge \mu^*(\bar{f})}{\prod |\mu^*(g_i)|^{2a_i}}$$

implies that

$$\mu_*(\mathcal{O}_Y(K_Y) \otimes \bar{\mathcal{I}}(Y, \mu^* D))(X \setminus S) = (\mathcal{O}_X(K_X) \otimes \bar{\mathcal{I}}(X, D))(X \setminus S).$$

□

2.3 An arithmetic definition of the multiplier ideal sheaf

All the rings in this sections are assumed to be geometric and of positive characteristic, that is finitely generated over an algebraically closed field of characteristic $p > 0$. The section is based on [6] and [4].

One of the most amazing discoveries of singularity theory, is that properties of the Frobenius map may reflect how singular a variety is. This observation is based on the fact that for a smooth local ring R , the e -times iterated Frobenius map $R \rightarrow F_*^e R$ splits. Further, the splitting does not need to hold when R is singular.

This lead to a definition of *F-split rings*, rings R such that for big enough $e > 0$ the e -times iterated Frobenius map $F^e: R \rightarrow F_*^e R$ splits. The problem with this property is that, in general, it does not behave well. For example, it is neither an open nor a close condition on fibers of a family.

Note, that $F^e: R \rightarrow F_*^e R$ splits if and only if there exists a map $\phi: F_*^e R \rightarrow R$ such that $1 \in \phi(F_*^e R)$. This suggests to consider rings which are *F-split* “under all small perturbations”. More formally we say that R is *strongly F-regular* if for every $c \in R \setminus 0$, there exists $e > 0$ a map $\phi: F_*^e R \rightarrow R$ such that $1 \in \phi(F_*^e cR)$.

For log pairs, we have the following definition.

Definition 2.10. We say that a log pair (X, Δ) , where $X = \text{Spec } R$ for an affine local ring R , is *strongly F-regular* if for every principal divisor $D \subseteq X$ there exists $e > 0$ and a map $\phi \in \text{Hom}_X(F_*^e \mathcal{O}_X([(p^e - 1)\Delta + D]), \mathcal{O}_X)$ such that $1 \in \phi(F_*^e \mathcal{O}_X)$.

For general log pairs, we require those conditions to hold at every point.

This definition may seem a bit mysterious in the first glance. Let us untangle it. Firstly, Grothendieck duality gives

$$\text{Hom}(F_*^e \mathcal{O}_X, \mathcal{O}_X) \simeq H^0(X, \omega_X^{1-p^e}).$$

This explains the following.

Proposition 2.11 ([11, Theorem 3.11, 3.13]). *There is a natural bijection.*

$$\left\{ \begin{array}{l} \text{Non-zero } R\text{-linear maps} \\ \phi: F_*^e R \rightarrow R \text{ up to} \\ \text{pre-multiplication by units.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Effective } \mathbb{Q}\text{-divisors } \Delta \text{ on} \\ X = \text{Spec } R \text{ such that} \\ (1 - p^e)\Delta \sim -(1 - p^e)K_X \end{array} \right\}$$

The \mathbb{Q} -divisor corresponding to $\phi: F_*^e R \rightarrow R$ will be denoted by Δ_ϕ .

Now, it is easy to see, that a log pair (X, Δ) , where X is affine local, is strongly F -regular if and only if for every principal divisor $D \subseteq X$ there exists a splitting $\phi: F_*^e \mathcal{O}_X \rightarrow \mathcal{O}_X$ such that $\Delta_\phi \geq \Delta + \frac{1}{p^e - 1}D$.

The same way klt singularities are detected by triviality of the corresponding multiplier ideal sheaves, the strongly F -regular singularities are related to the so-called big test ideal.

Definition 2.12. We define the big test ideal of an affine, local (X, Δ) to be the unique smallest non-zero ideal $J \subseteq \mathcal{O}_X$ such that $\phi(F_*^e J) \subseteq J$ for every $\phi \in \text{Hom}_X(F_*^e \mathcal{O}_X((p^e - 1)\Delta), \mathcal{O}_X)$. We denote it by $\tau(X, \Delta)$.

Note that to evaluate ϕ on $F_*^e J$, we first embed ϕ into $\text{Hom}_X(F_*^e \mathcal{O}_X, \mathcal{O}_X)$. It is easy to see, that (X, Δ) is strongly F -regular if and only if $\tau(X, \Delta) = \mathcal{O}_X$. The big test ideal behaves well under taking localization, and it extends to non-affine varieties.

The aim of this subsection is to show that a reduction modulo big enough p of the multiplier ideal sheaf is equal to the big test ideal. In particular, for such reductions klt singularities and F -regular singularities coincide.

In order to do so, we need to first understand how our theory behaves under taking resolutions of singularities.

Proposition 2.13 ([13, Proof of Theorem 6.7], [4, Exercise 4.17]). *Suppose that $\pi: \tilde{X} \rightarrow X$ is a proper birational map of varieties, where X is normal, and take $\phi \in \text{Hom}_X(F_*^e \mathcal{O}_X, \mathcal{O}_X)$. Then ϕ induces a map*

$$\tilde{\phi}: F_*^e \mathcal{O}_{\tilde{X}}((1 - p^e)(K_{\tilde{X}} - f^*(K_X + \Delta_\phi))) \rightarrow \mathcal{O}_{\tilde{X}}$$

which agrees with ϕ , where π is an isomorphism. Further $\tilde{\phi}$ is a generator of the space of maps between those two sheaves, and it induces

$$\tilde{\phi}: F_*^e \mathcal{O}_{\tilde{X}}([K_{\tilde{X}} - f^*(K_X + \Delta_\phi)]) \rightarrow \mathcal{O}_{\tilde{X}}([K_{\tilde{X}} - f^*(K_X + \Delta_\phi)]).$$

By taking π_* , we get a map

$$\pi_* \tilde{\phi}: F_*^e \mathcal{I}(X, \Delta_\phi) \rightarrow \mathcal{I}(X, \Delta_\phi).$$

In particular it shows that $\tau(X, \Delta_\phi) \subseteq \mathcal{I}(X, \Delta_\phi)$. One can easily prove that this, further, implies the following.

Corollary 2.14. *For a log pair (X, Δ) we have*

$$\tau(X, \Delta) \subseteq \mathcal{I}(X, \Delta).$$

In particular, if (X, Δ) is strongly F -regular, then it is klt.

In order, to achieve the aim of this section, we need two more technical tools: test elements and Hara's surjectivity lemma.

Definition 2.15. Let M be a torsion-free rank one R -module with a non-zero map $\phi: F_*^e M \rightarrow M$. We say that c is a test element for (M, ϕ) if for every $N \subseteq M$ such that $\phi(N) \subseteq N$, we have that

$$cM \subseteq N.$$

By [6, Theorem 12.14] we have the following. If $c \in R$ is such that R_c is regular and $M_c \simeq R_c$, then c has some power which is a test element.

In particular, we get from this:

Proposition 2.16 ([4, Lemma 3.6], [6, Theorem 12.14]). *Let (X, Δ) be a log pair such that X is affine. Consider a map $\phi \in \text{Hom}(F_*^e \mathcal{O}_X, \mathcal{O}_X)$. Then, a test element for (\mathcal{O}_X, ϕ) exists.*

The last piece is the following vanishing theorem.

Theorem 2.17 (Hara's surjectivity lemma [6, Lemma 23.1]). *Suppose that R is a ring of characteristic zero, $\pi: \tilde{X} \rightarrow \text{Spec } R$ is a log resolution of singularities, D is a π -ample \mathbb{Q} -divisor with simple normal crossing support. We reduce this setup to characteristic $p \gg 0$. Then, the natural map*

$$(F^e)^\wedge : H^0(\tilde{X}, F_*^e \omega_{\tilde{X}}([p^e D])) = \text{Hom}_{\mathcal{O}_{\tilde{X}}}(F_*^e \mathcal{O}_{\tilde{X}}([-p^e D]), \omega_{\tilde{X}}) \longrightarrow \\ \text{Hom}_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{X}}([-D]), \omega_{\tilde{X}}) = H^0(\tilde{X}, \omega_{\tilde{X}}([D]))$$

surjects.

Now, we can prove the main theorem of this section. The following has been developed by many authors, including Smith, Takagi, Hara and Yoshida. We refer to Schwede's sketch of a proof, which does not use the language of tight closure theory. We explain details of his proof.

Theorem 2.18 ([6, Theorem 17.8]). *Let (X, Δ) be a klt pair in characteristic zero. Let (X_p, Δ_p) be its reduction modulo $p \gg 0$. Then*

$$\tau(X_p, \Delta_p) = \mathcal{I}(X_p, \Delta_p).$$

Proof. The inclusion in one direction follows from Corollary 2.14. Hence, we are left to show that $\mathcal{I}(X_p, \Delta_p) \subseteq \tau(X_p, \Delta_p)$. Without loss of generality we may assume that X is a spectrum of an affine local ring.

Let $\pi: \tilde{X} \rightarrow X$ be a log resolution of singularities. Assume we found an element $d \in H^0(X, \mathcal{O}_X)$, a π -ample snc \mathbb{Q} -divisor D , a prime number p , a number $m \in \mathbb{N}$, and a number $e \in \mathbb{N}$, such that

1. $[K_{\tilde{X}} - \pi^*(K_X + \Delta) + D] = [K_{\tilde{X}} - \pi^*(K_X + \Delta)]$,
2. the element d_p^m is a test element for $(\mathcal{O}_{X_p}, \phi_{X_p})$, where $d_p \in H^0(X_p, \mathcal{O}_{X_p})$ is the reduction modulo p of d ,
3. $\pi_* \mathcal{O}_{\tilde{X}_p}([K_{\tilde{X}_p} - \pi^*(K_{X_p} + \Delta_p) + p^e D_p]) \subseteq d_p^m \mathcal{O}_X$,
4. $\phi_{\tilde{X}_p}$ is surjective, where

$\phi_{X_p}: F_*^e \mathcal{O}_X \rightarrow \mathcal{O}_X$ is a map induced by Δ_p , and where, obtained from Proposition 2.13 by tensoring with D_p and taking round-up, we have a map

$$\phi_{\tilde{X}_p}: \pi_* F_*^e \mathcal{O}_{\tilde{X}_p}([K_{\tilde{X}_p} - \pi^*(K_{X_p} + \Delta_p) + p^e D_p]) \rightarrow \pi_* \mathcal{O}_{\tilde{X}_p}([K_{\tilde{X}_p} - \pi^*(K_{X_p} + \Delta_p) + D_p]).$$

We assume that p is big enough so that the index of $K_X + \Delta$ is not divisible by p , and hence ϕ_X inducing Δ_p exists by Proposition 2.11.

We finish the proof assuming the above conditions. We have the following commutative diagram. Note that the first assumption implies that the object in the upper right corner is exactly $\mathcal{I}(X_p, \Delta_p)$.

$$\begin{array}{ccc} \pi_* F_*^e \mathcal{O}_{\tilde{X}_p}([K_{\tilde{X}_p} - \pi^*(K_{X_p} + \Delta_p) + p^e D_p]) & \xrightarrow{\phi_{\tilde{X}_p}} & \pi_* \mathcal{O}_{\tilde{X}_p}([K_{\tilde{X}_p} - \pi^*(K_{X_p} + \Delta_p) + D_p]) \\ \downarrow \subseteq & & \downarrow \subseteq \\ F_*^e(d_p^m \mathcal{O}_X) & \xrightarrow{\phi_{X_p}} & \mathcal{O}_X \end{array}$$

Since $\phi_{\tilde{X}_p}$ is surjective, we have that

$$\phi_{X_p}(F_*^e(d_p^m \mathcal{O}_X)) \supseteq \mathcal{I}(X_p, \Delta_p).$$

But, by definition of the test element, $\tau(X_p, \Delta_p) \supseteq \phi_{X_p}(F_*^e(d_p^m \mathcal{O}_X))$, which concludes the proof.

Now, we need to describe d , D , p and e . Choose $d \in H^0(X, \mathcal{O}_X)$ such that $X \setminus \{d = 0\}$ is smooth and $\Delta \subseteq \{d = 0\}$. Without loss of generality, by, for example, changing the log resolution, we may assume that $\text{div}(d)$ and the exceptional locus are simple normal crossings. Take a relatively ample, anti-effective \mathbb{Q} -divisor $-F$ and $\epsilon > 0$ such that, for $D := -F - \epsilon \text{div}(d)$, the first property holds:

$$[K_{\tilde{X}} - \pi^*(K_X + \Delta) + D] = [K_{\tilde{X}} - \pi^*(K_X + \Delta)].$$

Now, for $p \gg 0$, we can find

- $m > 0$ such that d_p^m is a test element, and
- $e > 0$ such that the third condition holds.

We are left to show the fourth condition, that is the surjectivity of $\phi_{\tilde{X}}$. Since $(p^e - 1)(K_{X_p} + \Delta_p) \sim 0$ and $\phi_{\tilde{X}}$ comes from a generator (see Proposition 2.13), one can easily see that the surjectivity of $\phi_{\tilde{X}}$ is equivalent to the surjectivity of

$$(F^e)^\wedge : H^0(\tilde{X}, F_*^e \omega_{X_p}([p^e(-\pi^*(K_{X_p} + \Delta_p) + D_p)])) \rightarrow H^0(\tilde{X}, \omega_{X_p}([-\pi^*(K_{X_p} + \Delta_p) + D_p])).$$

Here, we used that F^e is a generator of a corresponding space of maps. Now, the surjectivity follows from Hara's lemma 2.17. \square

We finish by stating a very useful criterion for F-splitness of hypersurfaces.

Proposition 2.19 (Fedder's criterion, cf. [4, Theorem 2.14]). *Let k be an algebraically closed field of characteristic $p > 0$. Take a polynomial $f \in k[x_1, \dots, x_n]$. Then $k[x_1, \dots, x_n]/(f)$ is F-split if and only if*

$$f^{p-1} \notin (x_1^p, \dots, x_n^p).$$

2.4 The log canonical threshold via Bernstein-Sato polynomials

Here we present a relation between log canonical threshold of a hypersurface and a study of differential operators. The presentation is based on [1].

Take a nonzero polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$. We define the *Weyl algebra* to be

$$A_n := \mathbb{C}[x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}].$$

In the following, s denotes another variable.

Definition 2.20. We define *Bernstein-Sato* polynomial to be the monic polynomial of smallest degree for which there exists $P \in A_n[s]$ satisfying

$$b(s)f^s = P(s, x, \partial_x) \bullet f^{s+1},$$

where \bullet denotes the action of $A_n[s]$ on $\mathbb{C}[x_1, \dots, x_n, s]$.

The following theorem shows the magic of Bernstein-Sato polynomials.

Theorem 2.21 ([1, Theorem 5.2]). *Let $Y \subseteq \mathbb{A}^n$ be a scheme defined by $f \in \mathbb{C}[x_1, \dots, x_n]$ and let $\lambda \in (0, 1]$ be a jumping number of (\mathbb{A}^n, Y) . Then $-\lambda$ is a root of Bernstein-Sato polynomial of f . Further, the log canonical threshold is its largest root.*

The following application was probably known before.

Proposition 2.22. *Consider $f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$. Assume that*

$$\partial_{y_i} f = \sum_j a_{ij} \partial_{x_j} f$$

for some $a_{ij} \in \mathbb{C}$. Define $\bar{f}(x_1, \dots, x_n) := f(x_1, \dots, x_n, 0, \dots, 0)$. Then

$$\text{LCT}(\bar{f}) = \text{LCT}(f).$$

Proof. First, note that

$$\partial_{y_i} f^{s+1} = \sum_j a_{i,j} \partial_{x_j} f^{s+1}$$

Take the Bernstein-Sato polynomial of $f(x_1, \dots, x_n, y_1, \dots, y_m)$ together with a corresponding differential operator P . By the relation above and the commutativity of taking differentials, we may assume that P contains only differentials $\partial_{x_1}, \dots, \partial_{x_n}$.

Thus, by plugging $y_i = 0$, we get

$$b(s)\bar{f}^s = P(s, x_1, \dots, x_n, 0, \dots, 0, \partial_{x_1}, \dots, \partial_{x_n}) \bullet \bar{f}^{s+1}.$$

Let \bar{b} be the Bernstein-Sato polynomial of \bar{f} . By the above, we have that $\bar{b} \mid b$. Thus, by Theorem 2.21, we get $\text{LCT}(\bar{f}) \geq \text{LCT}(f)$.

The inequality in the other direction follows from Lemma 2.6. \square

2.5 The log canonical threshold via jet spaces

The following is based on [1]. Consider a smooth complex variety X .

Definition 2.23. We define the m -th space of jets of X to be

$$X_m := \text{Hom}(\text{Spec } \mathbb{C}[t]/(t^{m+1}), X).$$

We have the following theorem of Mustata, which relates jet spaces to the log canonical centre.

Theorem 2.24 ([1, Theorem 6.6]). *Let X be a smooth complex variety, and $D \subseteq X$ a closed subscheme. The following holds:*

$$\text{LCT}(X, Y) := \lim_{m \rightarrow \infty} \frac{\text{codim}(Y_m, X_m)}{m+1}.$$

Further, for m divisible enough the equality holds without taking a limit, that is

$$\text{LCT}(X, Y) := \frac{\text{codim}(Y_m, X_m)}{m+1}.$$

The proof is based on properties of closed cylinders and the usage of Change of Variable Theorem in motivic integration.

3 Singularities of theta divisors on abelian varieties

The aim of this section is to study singularities of theta divisors on principally polarized abelian varieties. All varieties in this section are defined over an algebraically closed field of characteristic 0.

3.1 Log canonicity of theta divisors

The following subsection is based on [9].

Definition 3.1. We say that (A, θ) is a principally polarized abelian variety, if A is an abelian variety, and $\theta \subseteq A$ is a symmetric ample divisor such that $H^0(A, \theta) = 1$.

The following theorem has been proven by Kollar.

Theorem 3.2 ([9, Theorem 10.1.6]). *A principally polarized abelian variety (A, θ) , treated as a log pair, has log canonical singularities.*

Proof. By definition of log canonicity, we need to show that $\mathcal{I}((1-\epsilon)\theta) = \mathcal{O}_A$ for all $0 < \epsilon \ll 1$. Assume by contradiction that $\mathcal{I}((1-\epsilon)\theta) \neq \mathcal{O}_A$.

Let $Z \subseteq A$ be the scheme defined by $\mathcal{I}((1-\epsilon)\theta)$. We have a natural exact sequence

$$0 \longrightarrow \mathcal{O}_A(\theta) \otimes \mathcal{I}((1-\epsilon)\theta) \longrightarrow \mathcal{O}_A(\theta) \longrightarrow \mathcal{O}_Z(\theta) \longrightarrow 0.$$

By Nadel vanishing theorem, we have that $H^1(\mathcal{O}_A(\theta) \otimes \mathcal{I}((1-\epsilon)\theta)) = 0$, and so

$$H^0(A, \mathcal{O}_A(\theta)) \longrightarrow H^0(Z, \mathcal{O}_Z(\theta))$$

is surjective.

Since $Z \subseteq \theta$ and $H^0(A, \mathcal{O}_A(\theta)) = 1$, we have $H^0(Z, \mathcal{O}_Z(\theta)) = 0$. This contradicts semicontinuity of cohomologies, because $H^0(Z, \mathcal{O}_Z(\theta_a)) \neq 0$ for $\theta_a = \theta + a$ and a general $a \in A$. \square

Further, Ein and Lazarsfeld proved the following:

Theorem 3.3 ([9, Theorem 10.1.8]). *Let (A, θ) be a principally polarized abelian variety with irreducible θ . Then θ is normal and has rational singularities.*

Before proceeding with the proof, we review the generic vanishing theorem and some properties of an adjoint ideal – the ideal that controls rationality of a singularity.

3.2 The generic vanishing theorem

The following subsection is based on [7].

Theorem 3.4 (The generic vanishing theorem [8, Corollary 4.4.5]). *Let X be a smooth projective variety of dimension n and Albanese dimension $n - r$. Then*

$$H^i(X, K_X + L) = 0$$

for $i > r$ and a general $L \in \text{Pic}^0(X)$.

We give a brief sketch of an algebraic proof based on a detailed presentation from [7]. We assume that the reader is familiar with basic properties and notation from the theory of derived categories.

Proof. For a coherent sheaf \mathcal{F} on a variety A , define

$$S^i(A, \mathcal{F}) := \{L \in \text{Pic}^0(V) \mid H^i(A, \mathcal{F} \otimes L) \neq 0\}.$$

First, we make a general comment. Assume that A is an abelian variety and \mathcal{F} satisfies the following:

$$\mathbf{R}\phi_P(\mathcal{F}) \simeq \mathbf{R}\mathcal{H}\text{om}(\mathcal{G}, \mathcal{O}_{\hat{A}}),$$

where $\mathbf{R}\phi_P: D^b(A) \rightarrow D^b(\hat{A})$ is the Fourier-Mukai transform, \hat{A} is the dual abelian variety, and \mathcal{G} is some coherent sheaf on \hat{A} . Then properties of Ext imply that

$$\text{codim Supp}(R^i\phi_P(\mathcal{F})) \geq i$$

for all $i \geq 0$. It is not difficult to see that this gives

$$\text{codim } S^i(A, \mathcal{F}) \geq i.$$

Let $\text{alb}: X \rightarrow A := \text{Alb}(X)$ be the albanese map. Recall that $r = \dim(X) - \dim(\text{alb}(X))$. We apply the strategy above to $R^j\text{alb}_*(\omega_X)$ on A . Using Kodaira-type relative vanishing theorems, one can show that it satisfies the condition:

$$\mathbf{R}\phi_P(\text{alb}_*(R^j\omega_X)) \simeq \mathbf{R}\mathcal{H}\text{om}(\mathcal{G}, \mathcal{O}_{\hat{A}}),$$

for some coherent sheaf \mathcal{G} (which is different for different j), so we get that

$$\text{codim } S^i(A, R^j\text{alb}_*(\omega_X)) \geq i. \tag{1}$$

To obtain the generic vanishing theorem, we need to apply the theorem of Kollar which states that:

$$\mathbf{R}f_*\omega_Y = \bigoplus_{j=0}^r R^j f_*\omega_Y[-j],$$

where $f: Y \rightarrow X$ is a proper map between varieties, and $r = \dim(Y) - \dim(f(Y))$.

Now, take a general $L \in \text{Pic}^0(X)$. By properties of the Albanese map, we must have

$$L = \text{alb}^*(P_a)$$

for some $P_a \in \text{Pic}^0(A)$. Thus, by the theorem of Kollar

$$\begin{aligned} H^i(X, K_X + L) &= \mathcal{H}^i(A, \mathbf{R}\text{alb}_*\omega_X \otimes P_a) \\ &= \bigoplus_{j=0}^r H^{i-j}(A, R^j\text{alb}_*(\omega_X) \otimes P_a) \end{aligned}$$

Hence, in order to prove that $H^i(X, K_X + L) = 0$ for $i > r$, we need to show that

$$H^k(A, R^j\text{alb}_*(\omega_X) \otimes P_a) = 0$$

for a general $P_a \in \text{Pic}^0(A)$ and all $k > 0$. But this is an immediate consequence of (1). \square

In particular, the generic vanishing theorem implies that if X has maximal Albanese dimension, then

$$\chi(\mathcal{O}_X) = \chi(L) = H^0(X, L) \geq 0,$$

where $L \in \text{Pic}^0(X)$ used in the proof was taken as a general one.

Further, we will need the following lemma.

Lemma 3.5 ([7, Theorem 27.2]). *Let X be a projective variety of maximal Albanese dimension. If $\chi(\mathcal{O}_X) = 0$, then the image of the Albanese mapping $\text{alb}(X)$ is fibered by tori.*

The idea of the proof is the following. The condition $\chi(\mathcal{O}_X) = 0$ implies that $S^0(X, \omega_X)$ is a proper subvariety of $\text{Pic}^0(X)$. A standard fact in the theory of generic vanishing implies that in this case some $S^i(X, \omega_X)$ has an irreducible component of dimension $\dim(X) - i$. This component is a translation of some abelian variety $B \subseteq \hat{A}$ by a point of finite order. By checking cohomological data, one can see that the map $\text{alb}(X) \rightarrow \hat{B}$ is exactly the fibration we were looking for.

3.3 Adjoint ideals

Let (X, D) be a log pair, where $D \subseteq X$ is a reduced integral divisor. Let $\mu: \bar{X} \rightarrow X$ be a resolution of singularities. Write $\mu^*D = \mu_*^{-1}D + F$, where $\mu_*^{-1}D$ is the strict transform of D , and F is an exceptional divisor.

Definition 3.6. With notation as above we define the adjoint ideal to be

$$\text{adj}(D) := \mu_*\mathcal{O}_{\bar{X}}(K_{\bar{X}/X} - F).$$

One can show that it is independent of the choice of a resolution.
The following proposition signifies the importance of adjoint ideals.

Proposition 3.7 ([9, Proposition 9.3.48]). *Let $\nu: \bar{D} \rightarrow D$ be a resolution of singularities. Then we have the following exact sequence*

$$0 \longrightarrow \mathcal{O}_X(K_X) \longrightarrow \mathcal{O}_X(K_X + D) \otimes \text{adj}(D) \longrightarrow \nu_* \mathcal{O}_{\bar{D}}(K_{\bar{D}}) \longrightarrow 0.$$

Further, $\text{adj}(D)$ is trivial if and only if D is normal and its singularities are rational.

3.4 The rationality of theta divisors

We follow the presentation in [7].

Proof of Theorem 3.3. Let $f: X \rightarrow \theta$ be a resolution of singularities. First, using Lemma 3.5, we would like to show that

$$\chi(X) \geq 1.$$

We need to prove that θ is not fibered by tori. Assume contrary. Then, by definition, there exists a torus $V \subseteq A$ such that $V + a$ for $a \in A$ is either contained in θ or disjoint from it. This is impossible, since θ is ample.

Recall that $K_A = \mathcal{O}_A$. We have the standard exact sequence of an adjoint ideal tensored by a general $L \in \text{Pic}^0(A)$.

$$0 \longrightarrow L \longrightarrow L \otimes \mathcal{O}_A(\theta) \otimes \text{adj}(\theta) \longrightarrow L \otimes f_* \mathcal{O}_X(K_X) \longrightarrow 0.$$

Since X has maximal Albanese dimension, the generic vanishing theorem implies that

$$\begin{aligned} H^0(A, f_* \mathcal{O}_X(K_X) \otimes L) &= H^0(X, \mathcal{O}_X(K_X) \otimes f^* L) \\ &= \chi(X, \mathcal{O}_X(K_X) \otimes f^* L) \\ &= \chi(X, \mathcal{O}_X(K_X)) \geq 1. \end{aligned}$$

Since θ is ample, $\text{Pic}^0(A)$ is generated by line bundles of the form $\mathcal{O}_A(\theta_a - \theta)$, where $\theta_a = \theta + a$ and $a \in A$. Therefore

$$H^0(A, \mathcal{O}_A(\theta_a) \otimes \text{adj}(\theta)) \neq 0$$

for a general $a \in A$.

Let Z be the subscheme defined by $\text{adj}(\theta)$. Given that $h^0(A, \mathcal{O}_A(\theta_a)) = 1$, we get $Z \subseteq \theta_a$ for a general $a \in A$. This is a contradiction. \square

4 Moduli spaces of abelian varieties and singularities of the theta-null divisor

4.1 Moduli spaces of principally polarized abelian varieties

We recall the basic facts about moduli spaces of abelian varieties.

Definition 4.1. We denote the *moduli space of complex principally polarized abelian varieties* (ppav in short) of dimension g as

$$\mathcal{A}_g = \mathbb{H}_g / \mathrm{Sp}(2g, \mathbb{Z}),$$

where

- \mathbb{H}_g is the *Siegel upper half-space* – the set of symmetric complex $g \times g$ matrices, whose imaginary part is positive definite,
- an element of the symplectic matrices group

$$\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$$

acts on $\tau \in \mathbb{H}_g$ by

$$\sigma \cdot \tau := (a\tau + b)(c\tau + d)^{-1}.$$

Further, we define the *universal family*

$$\mathcal{X}_g := \mathbb{H}_g \times \mathbb{C}^g / (\mathrm{Sp}(2g, \mathbb{Z}) \times \mathbb{Z}^{2g}),$$

where the group acts in an obvious way.

This moduli space parametrizes, up to isomorphism, principally polarized abelian varieties (A_τ, θ_τ) , where $A_\tau := \mathbb{C}^g / (\mathbb{Z}^g \tau + \mathbb{Z}^g)$, $\tau \in \mathbb{H}_g$, and θ_τ is defined via the Riemann theta function

$$\theta(\tau, z) := \sum_{m \in \mathbb{Z}^g} \exp \pi i (m^t \tau m + 2m^t z).$$

One can calculate, that $\theta(\tau, z)$ defined in such a way is not invariant under the action of $\mathrm{Sp}(2g, \mathbb{Z})$. One of the problems is that this group acts on the corresponding line bundle by translations by 2-torsion points.

Let us give a brief overview how one can circumvent this problem. First, we define a *full level l subgroup* of $\mathrm{Sp}(2g, \mathbb{Z})$ to be

$$\Gamma_g(l) := \{\gamma \in \mathrm{Sp}(2g, \mathbb{Z}) \mid \gamma \equiv \mathrm{id}_{2g} \pmod{l}\},$$

that is, the kernel of the projection $\mathrm{Sp}(2g, \mathbb{Z}) \rightarrow \mathrm{Sp}(2g, \mathbb{Z}/l\mathbb{Z})$.

We represent \mathcal{A}_g as a quotient stack via a Galois map of varieties $\mathcal{A}_g(l) \rightarrow \mathcal{A}_g$, with Galois group $\mathbb{P}\mathrm{Sp}(2g, \mathbb{Z}/l\mathbb{Z})$, where

$$\mathcal{A}_g(l) := \mathbb{H}_g / \Gamma_g(l).$$

Further, we define the universal family

$$\mathcal{X}_g(l) := \mathbb{H}_g \times \mathbb{C}^g / (\Gamma_g(l) \times (l\mathbb{Z})^{2g}).$$

One can check that the Riemann theta function $\theta(\tau, z)$ defines a divisor on $\mathcal{X}_g(l)$, which descends to a divisor Θ_g on the universal stack family \mathcal{X}_g over the stack \mathcal{A}_g . We call it the *universal theta divisor*.

As mentioned before, Θ_g is not defined on \mathcal{X}_g in the category of analytic varieties.

4.2 Theta characteristics and the theta-null divisor

In what follows we denote the l -torsion points of an abelian variety A , by $A[l]$.

Consider a ppav (A_τ, θ_τ) and take $\epsilon, \sigma \in A_\tau[2] \simeq (\mathbb{Z}/2\mathbb{Z})^g$. Then, one can check that

$$x = \tau\epsilon/2 + \sigma/2$$

lies in $A_\tau[2]$, and $t_x^*\theta$ also gives a symmetric polarization. This divisor is the zero of the theta function with characteristic $[\epsilon, \sigma]$.

Definition 4.2. For $\epsilon, \sigma \in (\mathbb{Z}/2\mathbb{Z})^g$ we define the *theta function with characteristics* $[\epsilon, \sigma]$ to be

$$\theta \begin{bmatrix} \epsilon \\ \sigma \end{bmatrix} (\tau, z) := \sum_{m \in \mathbb{Z}^g} \exp \pi i \left[\left(m + \frac{\epsilon}{2}\right)^t \tau \left(m + \frac{\epsilon}{2}\right) + 2 \left(m + \frac{\epsilon}{2}\right)^t \left(z + \frac{\sigma}{2}\right) \right].$$

For characteristics equal to 0, this is just the standard Riemann theta function $\theta(\tau, z)$. One can see theta functions with characteristics as certain modular forms.

We say that $[\epsilon, \sigma]$ has an *even* (respectively an *odd*) characteristic if $\epsilon^t \sigma \in \mathbb{Z}/2\mathbb{Z}$ is zero (respectively one).

One can show that the action of $\mathrm{Sp}(2g, \mathbb{Z})$ on \mathbb{H}_g , transforms $\theta \begin{bmatrix} \epsilon \\ \sigma \end{bmatrix} (\tau, z)$ into another theta function, but (possibly) with a different characteristic. It motivates the following definition.

Definition 4.3. We define the *theta-null divisor* $\theta_{\mathrm{null}} \subseteq \mathcal{A}_g$ to be the zero locus of

$$F_g(\tau) := \prod_{\mathrm{even}} \theta \begin{bmatrix} \epsilon \\ \sigma \end{bmatrix} (\tau, 0),$$

where the product is taken over all characteristics which are even.

One can show that the theta-null divisor is well defined on the stack \mathcal{X}_g and is the locus of principally polarized abelian varieties whose one of the even two-torsion points lies on the theta divisor.

The following proposition signifies θ_{null} . Let $N_0 \subseteq \mathcal{A}_g$ be the locus of ppav with a singular theta divisor. It is called the *Andreotti-Mayer divisor*. Let N'_0 be the closure of the locus of ppavs, whose theta divisor does not have a singularity at a 2-torsion point.

Proposition 4.4 ([10]). *The locus N_0 is a divisor, and it has two irreducible components θ_{null} and N'_0 . More precisely*

$$N_0 = \theta_{\text{null}} + 2N'_0.$$

We are interested in understanding what are the singularities of θ_{null} . Since the question is local, we may start by considering only one branch of θ_{null} , that is a local branch defined by one function $\theta \begin{bmatrix} \epsilon \\ \sigma \end{bmatrix} (\tau, 0)$.

The crucial property of a theta function is that it satisfies the following *heat equation*

$$\frac{\partial^2 \theta \begin{bmatrix} \epsilon \\ \sigma \end{bmatrix} (\tau, z)}{\partial z_j \partial z_k} = 2\pi i (1 + \sigma_{j,k}) \frac{\partial \theta \begin{bmatrix} \epsilon \\ \sigma \end{bmatrix} (\tau, z)}{\partial \tau_{jk}}.$$

It suggests that the singularities of a branch defined by $\theta \begin{bmatrix} \epsilon \\ \sigma \end{bmatrix} (\tau, 0)$ should be related to singularities of $\theta \begin{bmatrix} \epsilon \\ \sigma \end{bmatrix} (\tau_0, z)$ for fixed τ_0 , that is, translates of theta divisors on abelian varieties. This suggests the following problem, raised by Prof. Shepherd-Barron.

Question 4.5. *Let H be a polynomial in variables z_1, \dots, z_n and τ_{jk} for $1 \leq j \leq k \leq n$. We write $H \in \mathbb{C}[\tau_{jk}, z_i]$. Assume that H satisfies the heat equation*

$$\frac{\partial^2 H(\tau, z)}{\partial z_j \partial z_k} = \frac{\partial H(\tau, z)}{\partial \tau_{jk}}.$$

Set $f(\tau) := H(\tau, 0)$ and $g(z) := H(0, z)$, that is we plug $z_i = 0$ and $\tau_{jk} = 0$ respectively into H . Assume that $\{g(z) = 0\}$ has normal, log canonical, rational singularities, and is, further, symmetric. What can we say about singularities of $\{f(\tau) = 0\}$?

Via a change of coordinates, this relates to the problem of understanding singularities of θ_{null} .

4.3 Singularities of the theta-null divisor

In this section we discuss Question 4.5.

Proposition 4.6. *Let*

$$g(z) = \sum_{\alpha_1, \dots, \alpha_n} c_{\alpha_1 \dots \alpha_n} \frac{z_1^{\alpha_1}}{\alpha_1!} \cdots \frac{z_n^{\alpha_n}}{\alpha_n!}.$$

Then

$$f(\tau) = \sum_{\alpha_1, \dots, \alpha_n} c_{\alpha_1 \dots \alpha_n} \sum_{\beta_{jk}} \prod_{j \leq k} \frac{\tau_{jk}^{\beta_{jk}}}{\beta_{jk}!},$$

where the second sum is taken over β_{jk} such that $\sum_{1 \leq k \leq n} \beta_{jk} = \alpha_j$, in which we take $\beta_{jk} = \beta_{kj}$.

Proof. It follows from the heat equation, since

$$\frac{\partial^{\sum \alpha_i} H(\tau, z)}{\prod \partial^{\alpha_i} z_i} = \frac{\partial^{\sum \alpha_i} H(\tau, z)}{\prod_{j \leq k} \partial^{\beta_{jk}} \tau_{jk}}$$

□

Corollary 4.7. *It holds that*

$$\text{mult}_0(g) = 2 \text{mult}_0(f)$$

Intuitively, the more variables and the lower multiplicity, the weaker a singularity is. This motivates a hypothesis, that if $g(z)$ has log canonical singularities, then $f(\tau)$ should have such, too.

Further, we can present f and g in a more admissible way. First, we consider the following lemma.

Lemma 4.8. *Every polynomial $h \in \mathbb{C}[x_1, \dots, x_n]$ can be presented as a sum of powers of linear functions, that is, in the form*

$$h(x_1, \dots, x_n) = \sum_i \left(\sum_j a_{ij} x_j \right)^{n_i},$$

for $a_{ij}, n_i \in \mathbb{Z}_{\geq 0}$.

Proof. It follows from the following well known identity

$$n! x_1 \dots x_n = \sum_{1 \leq i_1 < \dots < i_j \leq n} (-1)^{n-j} (x_{i_1} + \dots + x_{i_j})^n.$$

□

Hence, it is enough to understand how powers of linear series are transformed the heat equation. Note, that $g(z)$ is symmetric, so it is enough to consider only even-degree powers.

Proposition 4.9. *Let*

$$g(z_1, \dots, z_n) = \sum_i \left(\sum_j c_{ij} z_j \right)^{2n_i}.$$

Then

$$f(\tau_{jk}) = \sum_i \frac{(2n_i)!}{n_i!} \left(\sum_{j \leq k} c_{ij} c_{ik} \tau_{jk} \right)^{n_i}$$

Proof. It is easy to see, that

$$h_1 := \binom{2n_i}{\alpha_1, \dots, \alpha_n} \prod_j (c_{ij} z_j)^{\alpha_j} = (2n_i)! \prod_j \frac{(c_{ij} z_j)^{\alpha_j}}{\alpha_j!}$$

for $\sum_j \alpha_j = 2n_i$, and

$$h_2 := \frac{(2n_i)!}{n_i!} \binom{n_i}{\dots, \beta_{jk}, \dots} \prod_{j \leq k} (c_{ij} c_{ik} \tau_{jk})^{\beta_{jk}} = (2n_i)! \prod_{j \leq k} \frac{(c_{ij} c_{ik} \tau_{jk})^{\beta_{jk}}}{\beta_{jk}!}$$

for $\sum_k \beta_{jk} = \alpha_j$, satisfy

$$\frac{\partial^{\sum \alpha_i} h_1(z)}{\prod \partial^{\alpha_i} z_i} = \frac{\partial^{\sum \alpha_i} h_2(\tau)}{\prod_{j \leq k} \partial^{\beta_{jk}} \tau_{jk}}.$$

Thus the proposition follows from Proposition 4.6 and the binomial identity. \square

Consider a decomposition of g into homogenous pieces

$$g(z) = \sum_i g_i(z),$$

where $g_i \in \mathbb{C}[z_1, \dots, z_n]$ is a homogenous polynomial of degree $2i \in \mathbb{N}$. We have the following interesting identity, which says that some subvariety of $\{f(\tau) = 0\}$ has close properties to $\{g(z) = 0\}$.

Proposition 4.10. *Let $\bar{f} \in \mathbb{C}[z_1, \dots, z_n]$ be such that*

$$\bar{f}(z_1, \dots, z_n) = f(\bar{\tau}_{jk}),$$

where $\bar{\tau}_{jk} = (2 - \sigma_{jk}) z_j z_k$. Then

$$\bar{f}(z_1, \dots, z_n) = \sum_i \frac{(2i)!}{i!} g_i(z_1, \dots, z_n).$$

In particular, it gives the following corollary.

Corollary 4.11. *If monomials of the polynomial $g(z)$ have general coefficients, then $\{f(\tau) = 0\}$ is log canonical. Further, if $g(z)$ is a sum of two homogenous polynomials, then the log canonicity of $\{f(\tau) = 0\}$ also holds.*

Proof. The first statement follows from the proposition above together with Lemma 2.6 and Proposition 2.7.

For the second statement, notice, that we can modify \bar{f} from the proposition above, by making a change of coordinates, so that

$$\bar{f} = \sum_i g_i.$$

Hence, the corollary follows from Lemma 2.6. □

Berstein-Sato polynomials and other techniques

The fact, that the relation between f and g is defined via a differential equation, suggests using Bernstein-Sato polynomials to understand relations between their singularities.

Unfortunately, we cannot use the same technique as in Proposition 2.22. The reason is that the heat equation relates differentials of degree one with differentials of degree two.

Moreover, the following example shows, that if a relation between Bernstein-Sato polynomials of f and g exists, then it must be nontrivial.

Example 4.12. Calculations using Macaulay2 shows that

- For $g = z_1^2 z_2^4$ we have

$$\begin{aligned} b_g &= s^4 + 4s^5 + \frac{103}{16}s^4 + \frac{85}{16}s^3 + \frac{151}{64}s^2 + \frac{17}{32}s + \frac{3}{64} = \\ &= \frac{1}{64}(s+1)^2(2s+1)^2(4s+1)(4s+3). \end{aligned}$$

- For $f = 12(r_{11}r_{22}^2 + r_{12}^2r_{22})$ we have

$$\begin{aligned} b_f &= s^4 + 4s^3 + \frac{95}{16}s^2 + \frac{31}{8}s + \frac{15}{16} = \\ &= \frac{1}{16}(s+1)^2(4s+3)(4s+5). \end{aligned}$$

It is easy to see that there exists $H \in \mathbb{C}[z_i, \tau_{jk}]$ for which $f(\tau)$ and $g(z)$ are of this form. Unfortunately, the complexity of the algorithm seems to be too high to calculate the Bernstein-Sato polynomial of such H .

Similar problem occurs when one wants to use the arithmetic definition of multiplier ideal sheaves. Theoretically, we could consider a reduction modulo big enough prime number and then apply Fedder's criterion 2.19. In order to control the log canonical threshold of a hypersurface defined by a polynomial f , one needs to control powers f^{p-1} . In our case there is a nice description of a relation between f and g , but it seems much more difficult to obtain a reasonable comparison between f^{p-1} and g^{p-1} .

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