Heteroclinic bifurcations near Hopf-zero bifurcation in reversible vector fields in $\mathbb{R}^3$

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Abstract

We study the dynamics near a symmetric Hopf-zero bifurcation in a $\mathbb{Z}_2(R)$-reversible vector field in $\mathbb{R}^3$, with reversing symmetry $R$ satisfying $R^2 = I$ and $\dim \text{Fix}(R) = 1$. We focus on the case in which the normal form for this bifurcation displays a degenerate family of heteroclinics between two asymmetric saddle-foci. We study local perturbations of this degenerate family of heteroclinics within the class of reversible vector fields and establish the generic existence of hyperbolic basic sets (horseshoes), independent of the eigenvalues of the saddle-foci, as well as cascades of bifurcations of periodic, heteroclinic and homoclinic orbits.

Finally, we discuss the application of our results to the Michelson system, describing stationary states and travelling waves of the Kuramoto-Sivashinsky PDE.

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1 Introduction

In this paper we study the generic dynamics near a Hopf-zero bifurcation (steady-state/Hopf interaction) at a symmetric equilibrium solution of a $Z_2(R)$-reversible vector field in $\mathbb{R}^3$, where the reversing symmetry $R$ is such that $\dim \text{Fix}(R) = 1$.

Such a Hopf-zero bifurcation arises in the Michelson system [33]
\begin{align}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= c^2 - \frac{1}{2}x^2 - y.
\end{align}
(1.1)

which is reversible with respect to the time-reversal symmetry
\begin{equation}
R(x, y, z) = (-x, y, -z).
\end{equation}
(1.2)

Recall that reversibility means that $x(t)$ is a solution of (1.1) if and only if $Rx(-t)$ is a solution, or equivalently that the vector field anticommutes with $R$. The Hopf-zero bifurcation occurs precisely when $c = 0$. Namely, at $c = 0$, $(0, 0, 0)$ is an equilibrium at which the derivative has eigenvalues $\{0, \pm i\}$.

The Michelson system is obtained from a stationary-state and/or travelling wave reduction from the Kuramoto-Sivashinsky (KS) PDE *
\begin{equation}
\begin{align*}
\partial_t u + uu_x + u_{xx} + u_{xxxx} &= 0.
\end{align*}
\end{equation}
(1.3)

The reversibility of the Michelson system is a consequence of the $Z_2$-invariance of the KS system with respect to the transformation $(u, x, t) \rightarrow (-u, -x, t)$. The KS system has been intensively studied as a model PDE with complex behaviour. Equilibria and periodic solutions of the Michelson system correspond to spatially constant and spatially periodic stationary and travelling solutions of the KS equation. Likewise, homoclinic and heteroclinic solutions between equilibria of the Michelson system represent stationary and travelling solutions that converge to spatially constant solutions as $x \rightarrow \pm \infty$.

Our study of the reversible Hopf-zero bifurcation is partially motivated by the Michelson system, but also forms part of a program addressing the systematic study of the dynamics near local bifurcations in reversible (and reversible-equivariant) vector fields.

Many dynamical systems that arise in the context of applications possess robust structural properties, such as for instance symmetries or Hamiltonian structure. In order to understand the typical dynamics of such systems, their structure need to be taken into account, leading one to study phenomena that are generic among dynamical systems with the same structure.

Recently, there has been a surging interest in the study of systems with time-reversal symmetries, in particular since the group theoretical classification of linear reversible equivariant systems by Lamb & Roberts [29]. Since then, the linear normal form and unfolding theory of reversible equivariant linear systems has been developed by Hoveijn, Lamb & Roberts [22, 23]. Steady state bifurcation has been studied recently by Buono, Lamb & Roberts [6] and Hopf bifurcation by Buzzi & Lamb [7]. Such bifurcations are characterized by the appearance of a zero eigenvalue or a degenerate pair of purely imaginary eigenvalues of the linear part of the vector field at an equilibrium point. Whereas [6, 7] restrict to the description of elementary equilibria and periodic solutions, in this paper we describe more complicated locally recurrent dynamics.

We consider a one-parameter family of $Z_2(R)$-reversible dynamical systems
\begin{equation}
\dot{x} = F(x, \mu), \quad x = (x, y, z) \in \mathbb{R}^3,
\end{equation}
(1.4)

where $F \in \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$, $\mu \in \mathbb{R}$, with symmetric $(R$-invariant) equilibrium $0$ at $\mu = 0$, where $\dim \text{Fix}(R) = 1$. By Bochner’s Theorem [34], we may assume without loss of generality that $R$ acts (locally) linearly, eg as the twofold rotation around the $y$-axis in (1.2). $R$-Reversibility means that
\begin{equation}
F \circ R = -R \circ F.
\end{equation}
(1.5)

*It also arises in a 2-parameter model of a feedback system in [26].
A first important observation is that $R$-reversible vector fields of the type described above do not typically have symmetric equilibria. Namely, by virtue of the reversibility, $F$ maps Fix$(R)$ into Fix$(-R)$. Since, dim Fix$(R) = 1$ and dim Fix$(-R) = 2$ it follows that $F|_{\text{Fix}(R,0)}$ cannot hit 0 transversally. On the other hand, if we consider a one-parameter family of reversible vector fields, we find that if $F(0,0) = 0$, it generically hits 0 transversally so that the equilibrium point is typically persistent and isolated in Fix$(R) \times \mathbb{R}$ (where $\mathbb{R}$ denotes the parameter space).

We are therefore led to consider the one-parameter family $F$ with symmetric equilibrium $(0,0)$. The eigenvalues of $DxF(0,0)$ are either $\{0, \pm \alpha\}$ or $\{0, \pm \alpha i\}$, with $\alpha \in \mathbb{R}$. Varying the parameter $\mu$ reveals that the isolated symmetric equilibrium is typically (as long as $\alpha \neq 0$) a fold point, where two branches of asymmetric equilibria meet (see Section 2).

In case the eigenvalues of $DxF(0,0)$ are real, $(0, \pm \alpha)$ with $\alpha \in \mathbb{R}^+$, the recurrent dynamics is restricted to a one-dimensional centre manifold, on which the equations of motion are reversible and of the form

$$\dot{z} = \mu + az^2 + \cdots,$$

with $R(z) = -z$. the corresponding local dynamics is rather simple, featuring persistent heteroclinic connections between the two branches of asymmetric equilibria.

We are thus led to focus on the Hopf-zero case where the nonzero eigenvalues are purely imaginary $\pm \alpha i$. Without loss of generality, in this case we may take the linear part of the $R$-reversible vector field as

$$DxF(0,0) = \begin{pmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (1.6)

Associated with the pair of purely imaginary eigenvalues, varying $\mu$, one finds in addition to the branches of asymmetric equilibria, a one-parameter family of symmetric ($R$-invariant) periodic solutions with period near $2\pi/\alpha$ branching off the equilibrium point (see Section 2).

Since symmetric equilibria with the attached branches of equilibria and periodic solutions described above arise persistently in one-parameter families, it is natural to consider the Hopf-zero bifurcation as a codimension-one local bifurcation. However, as we shall see later, many aspects of the local dynamics depend sensitively on the nonlinear terms and are actually not finitely determined. In this situation, rather than aiming to describe the local dynamics in all its details, it is necessary to be less ambitious, and we correspondingly focus on certain generically robust aspects of the dynamics of one-parameter families of reversible vector fields in $\mathbb{R}^3$ passing at $\mu = 0$ through a Hopf-zero bifurcation point.

Our strategy is to study the local dynamics in first approximation by a normal form approach. It is a standard result of Birkhoff normal form theory [15] that by an $R$-equivariant coordinate transformation (preserving the $R$-reversibility), the vector field can be made to be $S^1$-equivariant up to arbitrary high order in its Taylor expansion, where $S^1 = \{\exp(Dx(0,0)s) | s \in \mathbb{R}\}$. Such $S^1$-equivariant vector fields can be studied from the symmetry reduced vector field on $\mathbb{R}^3/S^1 = \{\theta = 0\}$, taking $(r \cos \theta, r \sin \theta, z)$ as cylindrical coordinates for $\mathbb{R}^3$.

It turns out that the phase portraits of the reduced vector fields are finitely determined [36]. There are six different cases, most of them yielding relatively uncomplicated dynamics near the bifurcation point. However, in one of the cases (incidentally corresponding to the situation in the Michelson system) an elliptic point and heteroclinic cycle are simultaneously born and the situation is more complicated. The situation is illustrated in Figure 1, with the phase portraits of the reduced normal form vector field in $\mathbb{R}^3/S^1$ depicted when $\mu < 0$, $\mu = 0$ and $\mu > 0$. When $\mu > 0$ there is a heteroclinic cycle between the two asymmetric equilibria on the $z$-axis, and at the same time also a symmetric elliptic point on the $r$-axis. In $\mathbb{R}^3$ we find correspondingly a heteroclinic cycle between two asymmetric equilibria of saddle-focus type, whose one- and two-dimensional stable and unstable manifolds exactly coincide. The two one-dimensional manifolds coinciding in a straight line and the two-dimensional manifolds coinciding on a two-sphere. Also, in $\mathbb{R}^3$ we have an elliptic periodic solution, and the phase space between the heteroclinic cycle and the periodic solution is foliated by invariant two-tori.

It is essential to notice that the flow in the latter normal form approximation is very degenerate in the context of reversible flows in $\mathbb{R}^3$. The reason for the degeneracy is of course the $S^1$-equivariance it
Figure 1: The reduced normal form vector field on \( \mathbb{R}^3/S^1 = \{ \theta = 0 \} \) near the Hopf-zero bifurcation at \( \mu = 0 \), for the case \( a > 0, b = -1 \) referring to the coefficients in (3.3).

gained in the normal form approximation. The task is to understand the dynamics when the \( S^1 \) normal form symmetry is broken (by flat perturbations, ie perturbations that are small beyond any algebraic order).

Our first result concerns the generic occurrence of an infinite number of homoclinic and heteroclinic bifurcations in the neighbourhood of the reversible Hopf-zero bifurcation:

**Theorem 1.1** Denote by \( X^\mu_R \) the space of one parameter families of \( R \)-reversible vector fields (1.4) exhibiting the ‘Hopf-zero’ bifurcation as above at \( \mu = 0 \), endowed with the \( C^\infty \) topology. There exists an open subset \( U \in X^\mu_R \) containing the origin, which is determined by the 2-jet of the vector fields at \( (0,0) \in \mathbb{R}^3 \times \mathbb{R} \), such that the set of vector fields for which in a neighbourhood of the origin in \( \mathbb{R}^3 \times \mathbb{R} \) there exists a countable infinity of homoclinic orbits and heteroclinic cycles between the two saddle-focus fixed points, is residual in \( X^\mu_R \cap U \).

The proof of this result is obtained by an explicit perturbation argument and presented in Section 4. In fact, the homoclinic and heteroclinic orbits referred to in Theorem 1.1 are in fact homoclinic and heteroclinic orbits that lie close to the initial (degenerate) heteroclinic cycle, making no more than one revolution in its neighbourhood.

The (asymmetric) homoclinic cycles mentioned in Theorem 1.1 generically unfold following the classical treatment by Shilnikov, see \([21, 35]\). In particular, this means that one finds generically nontrivial hyperbolic basic sets (horseshoes) if the modulus of real part of the complex eigenvalues of the saddle-foci are smaller than the modulus of the real eigenvalues. This is determined by the 2-jet of the normal form: \( 0 < a < 2 \) in terms of coefficient of (3.3).

In this paper, however, we focus on the dynamics induced by the unfoldings of the heteroclinic cycles mentioned in Theorem 1.1. Their occurrence relies on the reversibility of the vector field. Apart from the fact that it supplies us with a lot of information on the local dynamics near the zero-Hopf bifurcation, we would like to emphasize that the study of the unfolding of such cycles is also of independent interest, for instance with relevance to the Michelson system, where at a special value of \( c \) an explicit expression for a 1D-heteroclinic solution has been obtained in \([24]\), see also Section 6.

The starting point of the analysis of the heteroclinic cycle bifurcations is sketched in Figure 2. We have two asymmetric saddle-foci \( p_0 \) and \( p_1 \) that are \( R \)-images of each other. The two-dimensional stable manifold of \( p_0 \) transversally intersects the two-dimensional unstable manifold of \( p_1 \) to yield a robust heteroclinic connection. At parameter value \( \lambda = 0 \), the one-dimensional unstable manifold of \( p_0 \) intersects \( \text{Fix}(R) \), giving rise to a coincidence with the one-dimensional stable manifold of \( p_1 \), resulting in a heteroclinic cycle. Clearly, the occurrence of such a heteroclinic cycle is generically persistent and isolated in one-parameter families of reversible vector fields of the type considered in this paper. We further assume that the one-dimensional heteroclinic connection unfolds generically along the \( \lambda \)-parameter family. A more technical description of the generic hypotheses is formulated in Section 5,
Figure 2: The heteroclinic cycle at $\lambda = 0$ with section planes and first hit maps.

where the dynamics near the cycle is studied using Poincaré maps $\phi, \psi_1, \psi_2, \phi', \psi_1', \psi_2'$ between the sections $\sigma_0, \sigma_1, \sigma_0', \sigma_1', \Sigma_0, \Sigma_1$ as indicated in Figure 2.

The analysis of the unfolding of the symmetric heteroclinic cycle yields many additional heteroclinic and homoclinic cycles, as well as periodic and aperiodic solutions.

In order to facilitate the discussion, we introduce some terminology in order to distinguish different types of these orbits. A heteroclinic orbit connecting $p_0$ to $p_1$ will be called a 1D heteroclinic orbit, and is always symmetric ($R$-invariant). A 1D heteroclinic orbit that intersects the section $\Sigma_1$ (see Figure 2) $n$ times is called an $n$-1D heteroclinic cycle. A symmetric heteroclinic orbit intersects Fix($R$) exactly once. Note that if $n$ is odd it intersect Fix($R$) in $\Sigma_1$, and if $n$ is even in $\Sigma_0$. Correspondingly we refer to such symmetric heteroclinic orbits also as upper and lower 1D heteroclinic orbits. The 1D heteroclinic orbit we start from is a 1-1D heteroclinic cycle. Heteroclinic orbits connecting $p_1$ to $p_0$ are correspondingly called 2D heteroclinic orbits. We similarly call such connections which intersect $\Sigma_0$ $n$ times $n$-2D heteroclinics. Symmetric 2D heteroclinics are upper if $n$ is even and lower if is $n$ is odd. Note that 2D heteroclinic orbits need not always be symmetric. Likewise, one can characterize symmetric periodic orbits which always intersect Fix($R$) precisely twice by the number of times $n$ that $\Sigma_1$ (or $\Sigma_0$) is intersected. If $n$ is even then the intersections of periodic solutions with Fix($R$) are either upper or lower, whereas if $n$ is odd the intersections are always mixed (one upper and one lower).

There also may arise homoclinic solutions connecting $p_0$ to $p_0$ or $p_1$ to $p_1$. By symmetry, whenever there exists a homoclinic solution to $p_0$ then such a solution also arises to $p_1$. We call a homoclinic solution $n$-homoclinic if it intersects $\Sigma_0$ (and $\Sigma_1$) $n$ times. Analogously we call a heteroclinic cycle $n$-heteroclinic if it intersects $\Sigma_0$ (and $\Sigma_1$) $n$ times.

The main results of the heteroclinic cycle bifurcation analysis in this paper are summarised in the following theorem. Its proof is discussed in Section 5.

**Theorem 1.2** Consider a one-parameter family of $R$-reversible vector fields $F(x, \lambda)$ in $\mathbb{R}^3$, with at $\lambda = 0$ a symmetric heteroclinic cycle between two asymmetric saddle-foci. Then, generically, the following statements hold:

A. At $\lambda = 0$ for all $n > 1$ there is a countably infinite number of (upper and lower) symmetric, and asymmetric ($n > 2$), transverse $n$-2D heteroclinic orbits accumulating to the symmetric hetero-
Note that in combination with the 1-1D heteroclinic, each n-2D heteroclinic solution constitutes a heteroclinic cycle.

B. For each heteroclinic cycle there exists a countable infinity of periodic solutions converging to the heteroclinic cycle as their period goes to infinity. For small $\lambda$, these periodic solutions form a one parameter family in $\mathbb{R}^3 \times \mathbb{R}$, parameterised by $\lambda$ as an oscillating function of the period.†

The type of the periodic orbits is in correspondence to that of the heteroclinic cycle that is accumulated:

- If the heteroclinic cycle intersects $\Sigma_1$ $n$ times then so do the periodic solutions.
- If the heteroclinic cycle is symmetric, then so are the periodic solutions, and the periodic solutions intersect $\text{Fix}(R)$ is in the same section(s) $\Sigma_i$, as the heteroclinic cycle.

A similar set of periodic solutions accumulates each homoclinic solution, if the eigenvalues of the saddle-foci satisfy Shilnikov’s condition (that the modulus of the real eigenvalue is larger than the modulus of the real part of the complex eigenvalue) [19, 35].

C. For each $n > 1$ there exists a countably infinite set of parameters $\{\lambda_k^{(n)}\}$, converging exponentially to zero as $k \to \infty$ such that at $\lambda = \lambda_k^{(n)}$ there exists a (symmetric) $n$-1D heteroclinic orbit. These orbits converge to the initial 1-1D heteroclinic orbit as $k \to \infty$.

Consequently, the set of parameter values for which there exists a 1D-heteroclinic orbit forms an accumulation set.

D. For each $n > 1$ there exists a countably infinite set of parameters $\{\lambda_j^{(n)}\}$, converging exponentially to zero as $j \to \infty$ such that at $\lambda = \lambda_j^{(n)}$, $p_0$ and $p_1$ have (asymmetric) homoclinic solutions. These orbits converge to the initial heteroclinic cycle as $j \to \infty$.

E. At $\lambda = 0$ there exists an indecomposable $R$-invariant nonuniformly hyperbolic invariant set, containing a countable infinity of nontrivial hyperbolic basic sets (horseshoes), whose dynamics is topologically conjugate to a full shift on an infinite number of symbols. For small nonzero $|\lambda|$, an $R$-invariant uniformly hyperbolic basic set remains whose dynamics is topologically conjugate to a full shift on a finite number of symbols (tending to infinity as $|\lambda| \to 0$).

We would like to highlight the result under E in the above theorem, as it establishes the existence of nontrivial basic sets (horseshoes) in the unfolding of the symmetric heteroclinic cycle, independent of the eigenvalues at the saddle-foci. This is in sharp contrast to the Shilnikov condition for the existence of horseshoes in the unfolding of a homoclinic orbit to a saddle-focus (which is incidentally automatically satisfied in the case of volume preserving vector fields, cf [5]).

Many of the results in Theorem 1.2 on the dynamics near the heteroclinic cycle do not crucially rely on the reversibility of the system. In general dissipative (non-reversible) systems heteroclinic cycles between two saddle-foci of the type introduced above have codimension two. Apart from the fact that in such systems 1D heteroclinic connections can only be expected to be seen in two-parameter unfoldings, our results on the existence of horseshoes (independent of any Shilnikov condition) remain valid as their proof relies only on the geometry of the flow near the cycle. We note that Bykov [9, 10] also studied the unfolding of such cycles in two-parameter families of non-reversible vector fields. He obtained results on the existence of certain periodic, homoclinic and heteroclinic solutions but none on the existence of horseshoes.

The construction of the $R$-invariant non-uniformly hyperbolic invariant set at $\lambda = 0$ mentioned in Theorem 1.2.E, by the intersection of strips is sketched in Figure 3.

By combining Theorem 1.1 and Theorem 1.2 above, we obtain the following conclusion concerning the generic dynamics near the Hopf-zero bifurcation.

†See Figure 6 for a sketch.
Figure 3: Sketch of the construction of the invariant set conjugate to a full shift on an infinite number of symbols with horizontal strips $H_i$ and vertical strips $V_i$. The construction starts with considering intersecting strips $B_0$ and $B_1 = R_0(B_0)$ in $\Sigma_0$ adjacent to the traces of the two-dimensional stable and unstable manifolds of the saddle-foci. The images $S_0 := \psi_2 \circ \phi \circ \psi_1(B_0)$ and $S_1 := (\psi_1' \circ \phi' \circ \psi_2')^{-1}(B_1)$ in $\Sigma_1$ are logarithmic spiralling strips that are exactly each other’s $R_1$-image. Let \( \{M_i\} \) denote the countable set of intersections of $S_0$ and $S_1$ containing part of the Fix $R_1$-axis. The images $H_i := \psi_1' \circ \phi' \circ \psi_2'(M_i)$ and $V_i := (\psi_2 \circ \phi \circ \psi_1)^{-1}(M_i)$ form sets of horizontal and vertical strips.
Theorem 1.3 Denote by $\mathcal{X}_R^\mu$ the space of one parameter families of $R$-reversible vector fields (1.4) exhibiting the ‘Hopf-zero’ bifurcation as above at $\mu = 0$, endowed with the $C^\infty$ topology. There exists an open subset $U \subseteq \mathcal{X}_R^\mu$ containing the origin, which is determined by the 2-jet of the vector fields at $(0, 0) \in \mathbb{R}^3 \times \mathbb{R}$, such that the set of vector fields for which in a neighbourhood of the origin in $\mathbb{R}^3 \times \mathbb{R}$ there exists for each $n \in \mathbb{Z}^+$

- a countable infinity of $n$-homoclinic orbits,
- a countable infinity of symmetric $n$-heteroclinic cycles,
- a countable infinity of asymmetric $n$-heteroclinic cycles $n > 2$,
- a countable infinity of $n$-periodic orbits (accumulating to $n$-heteroclinic cycles),
- a countable infinity of hyperbolic basic sets (horseshoes),

is residual in $\mathcal{X}_R^\mu \cap U$.

The subset of $\mathcal{X}_R^\mu \cap U$ for which which in a neighbourhood of the origin in $\mathbb{R}^3 \times \mathbb{R} \cap U$ there exists a hyperbolic basic set (horshoe) is open and dense.

Our work has parallels with that of Broer & Vegter [5], Delshams & Martinez-Seara [11] and Gaspard [17], who studied the Hopf-zero bifurcation in the context of dissipative and volume preserving vector fields, displaying a similar family of degenerate heteroclinic cycles in normal form approximation. The main difference between the dissipative/volume preserving and reversible contexts is that in the latter context one finds codimension one heteroclinic cycles, whereas in the former context such heteroclinic cycles have codimension two. From our results we see that the heteroclinic cycles arising due to the reversibility have important implications for the local dynamics.

Our work also forms part of a systematic effort to study local bifurcations in reversible equivariant systems. The Hopf-zero bifurcation is one of the simplest examples of a codimension one local bifurcation in reversible vector fields. The behaviour of the equilibria in this example follows that predicted by the more general treatment of reversible equivariant steady-state bifurcations of Buono, Lamb and Roberts [6]. The branch of periodic solutions is reminiscent of a Liapunov Centre family of periodic solutions embedded in $\mathbb{R}^3 \times \mathbb{R}$.

This case study illustrates how some results that are well-known to hold in reversible systems in $\mathbb{R}^{2n}$ with $\dim \text{Fix}(R) = n$, do not hold without modification in odd dimensions. We noted above the Liapunov centre family of periodic solutions embedded in $\mathbb{R}^3 \times \mathbb{R}$. Another striking example encountered in this study is the absence of a one-parameter family of periodic solutions accumulating to symmetric heteroclinic cycles. In even dimensions, it is well known that such families exist in phase space, converging to a persistent heteroclinic [12]. Here, however, the heteroclinic cycle is only persistent in one-parameter families, and there exists a one-parameter family of periodic solutions converging to the heteroclinic in $\mathbb{R}^3 \times \mathbb{R}$. In fact, varying the parameter towards the heteroclinic cycle bifurcation point, there appears an increasing number of isolated periodic solutions approximating the heteroclinic cycle, with at the bifurcation point an infinite discrete set of isolated periodic solutions with growing period, approaching the heteroclinic cycle as the period goes to infinity (see Theorem 1.2 B).

As mentioned before, this paper is partially motivated by the Michelson system, which arises as a reduction to travelling wave and steady-state solutions in the Kuramoto-Sivashinsky partial differential equation. The model is an example of a reversible vector field in $\mathbb{R}^3$. However, at the same time the Michelson system has more structure: it is for instance, analytic (even quadratic), volume preserving and also it has the property that it can be written as a third order ODE in one variable. It is thus natural to ask what our analysis of the generic reversible Hopf-zero bifurcation can tell us about the Michelson system.

The Michelson system exhibits a Hopf-zero bifurcation of the type discussed in this paper. Importantly, Theorem 1.1 and Theorem 1.3 also hold if the vector field is not only reversible but also volume preserving. The following theorem illustrates that we can establish the validity of the most important conclusions of Theorem 1.1 and Theorem 1.3 also for the Michelson system.
Theorem 1.4 Consider the Michelson system (1.1) with parameter \( c \). Then, in every parameter interval \((0, \delta]\) with \( \delta > 0 \), there exists for each \( n \in \mathbb{Z}^+ \)

- a countable infinity of \( n \)-homoclinic orbits,
- a countable infinity of \( n \)-heteroclinic cycles \( n > 1 \),
- a countable infinity of \( n \)-periodic orbits (accumulating to \( n \)-heteroclinic and \( n \)-homoclinic cycles),
- a countable infinity of hyperbolic basic sets (horseshoes).

The proof of this result is discussed in Section 6. We note that Adams et al. [1] also proved the existence of heteroclinic cycles in this system arbitrarily close to the singularity in parameter space. We here show that this fact indeed coincides with the behaviour one would expect near reversible Hopf-zero bifurcation points.

2 Elementary stationary and periodic solutions

In this section we examine the simplest solutions emerging in the unfolding of the Hopf-zero equilibrium, i.e. those that relate to solutions of the linear approximation: stationary solutions and periodic solutions with period approximately equal to \( 2\pi/\alpha \). The existence of such solutions can be proven using Liapunov-Schmidt reduction.

We briefly set out the Liapunov-Schmidt reduction technique, along the lines of e.g. [20]. The first step is to introduce new functions \( u(t) = x(t(1+\tau)) \), so that (1.4) transforms into

\[
N(u, \mu, \tau) = (1 + \tau) \frac{du}{dt} - \frac{1}{\alpha} F(u, \mu) = 0. \tag{2.1}
\]

Restricting ourselves to \( 2\pi \)-periodic solutions, we can view \( N \) as a differential operator

\[
N : C^1_{2\pi} \times \mathbb{R} \times \mathbb{R} \to C_{2\pi},
\]

where \( C_{2\pi} \) and \( C^1_{2\pi} \) are Banach spaces of continuous, respectively continuously differentiable \( 2\pi \)-periodic functions into \( \mathbb{R}^3 \). By varying the newly introduced small variable \( \tau \), one keeps track not only of solutions of (1.4) with period \( 2\pi/\alpha \) but also of solutions with nearby period.

\( N \) inherits the \( R \)-reversibility of \( F \), i.e. \( RN(u(t), \mu, \tau) = -N(Ru(-t), \mu, \tau) \). Moreover, due to the fact that \( F \) is autonomous, \( N \) is also \( S^1 \)-equivariant: \( \phi N(u, \mu, \tau) = N(\phi u, \mu, \tau) \), with \( \phi \in [0, 2\pi) \cong S^1 \) acting on \( C_{2\pi} \) and \( C^1_{2\pi} \) as \( \phi u(t) = u(t - \phi) \).

The next step is to consider the derivative \( D_u N(0, 0, 0) \) of \( N \):

\[
D_u N(0, 0, 0) = \frac{d}{dt} - \frac{1}{\alpha} D_u F(0, 0, 0).
\]

It follows that \( \dim \ker D_u N(0, 0, 0) = 3 \) since

\[
\ker D_u N(0, 0, 0) = \{ \exp(\frac{1}{\alpha} D_u F(0, 0, 0)t) u_0 \mid u_0 \in \mathbb{R}^3 \},
\]

\[
= \{ (\text{Im}(e^{-it}v), \text{Re}(e^{-it}v), z) \mid v \in \mathbb{C}, z \in \mathbb{R} \},
\]

with \( v = y + ix \in \mathbb{C} \) and \( z \in \mathbb{R} \).

Using the fact that \( D_u N(0, 0, 0) \) is a Fredholm operator of index zero [13, 20] we may write

\[
C_{2\pi} = \ker D_u N(0, 0, 0) \oplus \text{range } D_u N(0, 0, 0),
\]

\[
C^1_{2\pi} = \ker D_u N(0, 0, 0) \oplus M, \tag{2.2}
\]
Birkhoff normal form takes the form \( \tau \) and \( f \) are therefore equilibria, this equation must be invariant with respect to parameter translations of the normal form approximation.

In this section we derive a Birkhoff normal form for the reversible Hopf-zero bifurcation and its unfolding.

We consider a symmetric equilibrium point of an \( R \)-reversible vector field (1.4) in \( \mathbb{R}^3 \) with Hopf-zero linear part given by (1.6).

It is a standard result of Birkhoff normal form theory [15] that one can find coordinate transformations that render the vector field \( S^1 \)-equivariant up to arbitrarily high order, where \( S^1 = \{ \exp(D_\theta F(0,0)s) \mid s \in [0,2\pi/\alpha) \} \). Moreover, the corresponding coordinate transformations can be taken to be \( R \)-equivariant, preserving the \( R \)-reversibility of the vector field, see e.g. [25].

Employing cylindrical coordinates \( x = r \cos \theta, y = r \sin \theta \) the \( R \)-reversible (formal) \( S^1 \)-equivariant Birkhoff normal form takes the form

\[
\begin{align*}
\dot{\theta} &= f(v^2, z^2) \\
\dot{r} &= rzg(v^2, z^2) \\
\dot{z} &= h(v^2, z^2)
\end{align*}
\]
where \( h(0,0) = \frac{\partial h}{\partial z}(0,0) = 0 \) and \( f(0,0) = \alpha \). In fact, using a reparametrization of the vector field we may take without loss of generality \( f(r^2,z^2) = \alpha \) to be constant in a neighbourhood of \((0,0)\). We shall denote the \( S^1\)-equivariant normal form vector field by \( \tilde{X}^\mu \). Note that the original vector field \( F \) is not conjugate to \( \tilde{X}^\mu \), but conjugate to \( X^\mu = \tilde{X}^\mu + Y^\mu \), where \( Y^\mu(x,y,z) : \mathbb{R}^3 \to \mathbb{R}^3 \) is small beyond all algebraic orders (flat) in \((x,y,z)\).

Due to the \( S^1\)-equivariance the vector field is \( \theta \)-independent, and thus it is possible to consider the normal form vector field on the reduced phase space \( \mathbb{R}^3/S^1 = \{ \theta = 0 \} \), yielding up to third order:

\[
\begin{align*}
\dot{r} &= a_1 rz, \\
\dot{\theta} &= b_1 r^2 + b_2 z^2
\end{align*}
\]

where \( a_1, b_1, b_2 \) are constants. Takens [36] showed that the differential equation (3.2) is 2-determined up to \( C^0 \) orbital equivalence, under the generic conditions that \( a_1, b_1, b_2 \neq 0 \) and \( b_2 - a_1 \neq 0 \). Note that the \( S^1\)-equivariant 2-jets are the same in the generic (codimension 2) and the reversible (codimension 1) case. There are six topologically different phase portraits for the truncated vector fields and they can be found for instance in [21].

The according one-parameter reversible versal unfolding follows directly by restriction from the general two-parameter versal unfolding in [21, 36], cf [22]. After applying some additional rescalings, one obtains the normal form

\[
\begin{align*}
\dot{r} &= arz, \\
\dot{\theta} &= \mu + br^2 - z^2
\end{align*}
\]

where \( a \in \mathbb{R} \) and \( b \in \pm 1 \) are constants and \( \mu \) is the unfolding parameter. Phase portraits for this normal form can be found in [21, 36]. In Figure 1 the phase portraits before, at and after the Hopf-zero bifurcation are depicted in the case that \( a > 0, b = -1 \), which is the case arising in the Michelson system and the one we focus on in this paper.

We emphasize again that the normal form vector field, although a good approximation, is highly degenerate because of the \( S^1 \) normal form symmetry. Theorem 1.1 discusses some of the consequences of small \( S^1 \)-symmetry breaking perturbations. Its proof is discussed in the following section.

### 4 Proof of Theorem 1.1

We are interested in the reversible Hopf-zero bifurcation, when the 2-jet normal form coefficients in (3.3) satisfy \( a > 0, b = -1 \). These conditions form the open conditions mentioned in Theorem 1.1.

In this case, due to the \( S^1 \) normal form symmetry, the flow of the normal form when \( \mu > 0 \) gives rise to a highly degenerate approximation in which the stable and unstable manifolds of the saddle-foci coincide in a line and a 2-sphere. A generic \( R \)-reversible but \( S^1 \)-symmetry breaking perturbation would remove such degeneracies, and the question is what remains. Our first observation is that due to the fact that \( \text{Fix} \ R \) intersects the heteroclinic 2-sphere of the normal form flow transversely in two points, in the break up of such a 2-sphere by some small perturbation at least two symmetric (ie setwise \( R \)-invariant) 2D heteroclinic orbits will remain to exist. The situation is analogous to the illustration of the ‘perturbed globe’ in [5].

Our proof of Theorem 1.1 is constructive and relies on the following lemma.

**Lemma 4.1** Let \( \tilde{X}^\mu \) be an \( S^1 \)-symmetric vector field in \( \mathbb{R}^3 \) with a degenerate heteroclinic cycle as arising in the normal form (3.3) with \( a > 0, b = -1 \) and \( \mu > 0 \), cf Figure 1. There exists a flat perturbation \( Y^\mu \) such that the perturbed vector field \( \tilde{X}^\mu + Y^\mu \) has a sequence of Shilnikov homoclinic bifurcations at a discrete set of parameter values \( \mu_i \), which accumulate at \( \mu = 0 \). There also is a sequence of parameter values \( \mu_j \) accumulating at \( \mu = 0 \) for which the two saddle-foci are connected by a heteroclinic cycle.
Recall that the Shilnikov homoclinic bifurcation is a generic codimension one homoclinic bifurcation of a homoclinic orbit to a saddle-focus (in $\mathbb{R}^3$). The proof closely follows the constructions by Broer & Vegter [5] of an infinite sequence of Shilnikov homoclinic bifurcations near Hopf-zero bifurcation in volume-preserving vector fields, involving an explicit description of flat perturbations $Y^\mu$.

We consider flat perturbations $Y^\mu$ that are the compositions of two flat perturbations $Y^\mu = Y_1^\mu + Y_2^\mu$, such that the perturbation $Y_1^\mu$ controls the position of the one-dimensional stable and unstable manifolds and the perturbation $Y_2^\mu$ yields a transversal intersection between the two-dimensional stable and unstable manifolds.

We consider first the perturbation $Y_2^\mu$. We set $Y_2^\mu = \delta(\mu)P_2^\mu$, where $\delta(\mu)$ is a flat function in $\mu$ at $\mu = 0$. This perturbation is designed to cause a transversal intersection between the two dimensional manifolds $W^s(p_0(\mu))$ and $W^u(p_1(\mu))$. First note that the invariant 2-sphere for the normal form $\tilde{X}_1^\mu$, formed by $W^s(p_0(\mu))$ and $W^u(p_1(\mu))$, is given by the equation [21]

$$\frac{r}{1 + a} + z^2 = \mu.$$ 

We choose the support of $P_2^\mu$ to be a torus centred on the circle $r^2 = (1 + a)\mu$, in $\mathbb{R}^3$. The following lemma implies that we can construct a perturbation $Y_2^\mu = \delta_2(\mu)P_2^\mu$ such that $\tilde{X}_1^\mu + Y_2^\mu$ has transverse heteroclinic connections lying in $W^s(p_0(\mu)) \cap W^u(p_1(\mu))$. We refer to [38] for the proof.

**Lemma 4.2 ([38])** Let $X_R$ be the space of $R$-reversible vector fields in $\mathbb{R}^3$ endowed with the $C^\infty$ topology, where $R(x, y, z) = (-x, y, -z)$, and let $\mathcal{S}_R \subset X_R$ be the subset for which all fixed points have transversally intersecting invariant manifolds. Then $\mathcal{S}_R$ is residual in $X_R$.

The flat perturbation $Y_1^\mu$ is constructed in order to manipulate the one-dimensional stable and unstable manifolds. We write

$$Y_1^\mu(x, y, z) = \delta_1(\mu)P_1^\mu(x, y, z),$$

where $\delta_1(\mu)$ is some appropriate flat function in $\mu$ at $\mu = 0$ (appropriate in a sense to be specified later) and

$$P_1^\mu(x, y, z) = \left(\frac{\partial}{\partial y}(y\beta_\mu(\xi)), -\frac{\partial}{\partial x}(y\beta_\mu(\xi)), 0\right)$$

(4.1) where $\beta^\mu : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$\beta^\mu(x, y, z) = \gamma\left(\frac{1}{\mu}r\right), \gamma\left(\frac{1}{\mu^2}z\right), \quad \mu > 0 \text{ small.}$$

(4.2) with $x = r \cos \theta$, $y = r \sin \theta$, and $\gamma : \mathbb{R} \to \mathbb{R}$ is an even bump function with support $\text{supp}(\gamma) = [-2, 2]$ and $\gamma(s) \equiv 1$ for $s \in [-1, 1]$, cf [5]. Observe that $\beta^\mu$ is thus constructed to be $R$-invariant, where $R(x, y, z) = (-x, y, -z)$. The support of $P_1^\mu$ in $\mathbb{R}^3$ is a cylinder $\eta^\mu$, given by

$$\eta^\mu = \{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} = r \leq 2\mu, \quad |z| \leq 2\mu^2\}.$$ 

We define two subsets $\nu_1^\mu, \nu_2^\mu \subset \eta^\mu$ by

$$\nu_1^\mu = \eta^\mu \cap \{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} = r \leq \mu\}, \quad \text{and} \quad \nu_2^\mu = \nu_1^\mu \cap \{(x, y, z) \in \mathbb{R}^3 : |z| \leq \mu^2\}.$$ 

See Figure 4 for a sketch.

It follows from (4.1) and (4.2) that for $(x, y, z) \in \nu_1^\mu$, $Y_1^\mu(x, y, z) = \delta_1(\mu)\gamma\left(\frac{1}{\mu^2}z\right)\frac{\partial}{\partial y}$ and for $(x, y, z) \in \nu_2^\mu$, $Y_1^\mu(x, y, z) = \delta_1(\mu)\frac{\partial}{\partial x}$. Note that $Y_1^\mu$ is a $C^\infty$ reversible divergence free flat perturbation.

The flat perturbation $Y_1^\mu$ described above has the following consequences for the positions of the one-dimensional stable and unstable manifolds of the saddle-foci $p_0(\mu)$ and $p_1(\mu)$ [4, 5]:

**Lemma 4.3 ([4])** Consider vector field $\tilde{X}^\mu + Y_1^\mu$ as considered above where $\tilde{X}^\mu$ is the $S^1$-equivariant normal form vector field with for $\mu > 0$ a heteroclinic cycle between saddle-foci $p_0(\mu)$ and $p_1(\mu)$, and
\(Y^{\mu}_1\) is the flat perturbation introduced above. Let \(r^*_\mu\) be the \(r\)-coordinate of the 1-dimensional unstable manifold \(W^u(p_0)\) when it exits \(\eta^\mu\), with \(\mu > 0\) sufficiently small. Then\(^1\)

\[ r^*_\mu \sim \mu \delta_1(\mu) \quad (4.3) \]

For the proof of this lemma, we refer to [4].

Lemma 4.3 demonstrates that we can control the order of magnitude of the \(r\)-coordinate of the one-dimensional unstable manifold of \(p_0(\mu)\) by using a flat perturbation. Note that we also have \(\theta^*_\mu = o(1)\) as \(\mu \to 0^+\), where \(\theta^*_\mu\) is the \(\theta\)-coordinate of the 1-dimensional unstable manifold \(W^u(p_0)\) when it exits \(\eta^\mu\).

**Proof of Lemma 4.1** The perturbed vector field \(\tilde{X}^\mu + Y^\mu_1 + Y^\mu_2\) has transverse heteroclinic connections for all \(\mu\). We follow the behaviour of the two-dimensional manifold \(W^s(p_0(\mu))\) in negative time, as it gets close to the one-dimensional manifold \(W^s(p_1(\mu))\). By the \(\lambda\)-lemma, \(W^s(p_0(\mu))\) wraps itself tightly around \(W^s(p_1(\mu))\) in a logarithmic spiral. Consider the top of the support box \(\eta^\mu\), that is \(\eta^\mu \cap \{z = 2\mu^2\}\). We assume here that the angle \(\theta\) is lifted to \(\mathbb{R}\). Then \(W^s(p_1)\) intersects \(\eta^\mu \cap \{z = 2\mu^2\}\) along the \(z\)-axis (the perturbations \(Y^\mu_1\) and \(Y^\mu_2\) do not affect the relevant part of \(W^s(p_1)\)), and \(W^s(p_0)\) will trace out a 1-dimensional curve in this section, with equation \(r(\theta, \mu) \sim \delta_3(\mu)e^{a\theta\sqrt{\pi}}\), where \(\delta_3(\mu)\) is a flat function depending on \(\mu\). Note that the perturbation \(Y^\mu_1\) does not affect this logarithmic spiral. Since \(\theta^*_\mu = o(1)\) and from (4.3), if we set, for example \(\mu \delta_1(\mu) = \delta_3(\mu)e^{-1/\mu}\) we obtain a sequence of Shilnikov homoclinic bifurcations. Finally, choosing \(Y^\mu_1 = \sin\left(\frac{1}{\mu}\right)\delta_1(\mu)P^\mu_1\) will create a sequence of 1-D heteroclinic connections (identical to the connection in the \(S^1\)-equivariant normal form) whenever \(\sin\left(\frac{1}{\mu}\right) = 0\). When \(\sin\left(\frac{1}{\mu}\right) = 1\) we have \(r^*_\mu \sim \mu \delta_1(\mu)\) as before. This completes the proof of Lemma 4.1.

**Proof of Theorem 1.1** It remains to verify the generic occurrence of sequences of global bifurcations as described in Lemma 4.1. From Section 3 we know that \(\{\tilde{X}^\mu\}\)—the set of \(S^1\)-symmetric vector fields—is dense in \(X_R\) in the \(C^\infty\) topology. Since the perturbations we have used to prove existence are flat, this proves that our bifurcation sequences are dense in \(X_R\). Let \(B^k_{\text{hom/het}}\) be the set of vector fields with \(k\) homoclinic (resp. heteroclinic) bifurcations close to the Hopf-zero bifurcation point. Given any element \(\tilde{X}^\mu + Y^\mu\) in the dense set which has an infinite sequence of homoclinic and heteroclinic bifurcations accumulating at the bifurcation point, this family will have \(k\) such bifurcations for all \(\mu > 0\) sufficiently small. Each such bifurcation in persistent in the \(C^1\) topology, thus \(B^k_{\text{hom/het}}\) is open in the

\(^1\)More precisely, the relation (4.3) means there exist constants \(C_1, C_2\) such that \(r^*_\mu \leq C_1 \mu \delta(\mu)\) and \(\mu \delta(\mu) \leq C_2 r^*_\mu\).
$C^1$ topology. Note that this set says nothing about what happens in a neighbourhood of the bifurcation point itself as any finite number of bifurcations is bounded away from zero. Hence, for each vector field in the set $\bigcap_k B^k_{\text{hom/het}}$, which is residual in the $C^1$-topology, we have a countable infinity of homoclinic and heteroclinic bifurcations. This completes the proof of Theorem 1.1.

**Remark 4.4** Theorem 1.1 also holds in the case of volume-preserving vector fields [5], and in the case of reversible volume-preserving vector fields, since the flat perturbation used in the argument may be chosen to be reversible and volume preserving at the same time.

In the next section, we show that typically, in the neighborhood of the heteroclinic bifurcations described above, there exist countably many other homoclinic and heteroclinic bifurcations, all accumulating on the heteroclinic bifurcation.

## 5 Heteroclinic cycle bifurcation

In this section we study the dynamics near a heteroclinic cycle bifurcation. We consider a one-parameter family of $R$-reversible vector fields $F : \mathbb{R}^3 \times R \to \mathbb{R}^3$, with $R$ as before, satisfying the following hypotheses:

[H1] $F$ has two fixed points $p_0$ and $p_1$, such that $R(p_0) = p_1$.

[H2] $Df(p_0)$ has one real eigenvalue $\mu > 0$ and a complex pair of eigenvalues $-\rho \pm i\omega$ with $\rho, \omega > 0$.

[H3] There exists an isolated symmetric heteroclinic solution $h(t)$ contained in the (transversal) intersection of the two-dimensional stable manifold of $p_0$ and the two-dimensional unstable manifold of $p_1$, ie $h(t) \in W^s(p_0) \cap W^u(p_1)$ for all $\lambda \in [-\pi, \pi]$ for $\pi$ sufficiently small.

[H4] at $\lambda = 0$ the unstable manifold of $p_0$ coincides with the stable manifold of $p_1$, ie $W^u(p_0) = W^s(p_1)$, so that $F(\cdot, 0)$ has a symmetric heteroclinic loop. Additionally $W^u(p_0)$ passes (with positive speed) through $\text{Fix}(R)$ at $\lambda = 0$.

Hypothesis [H1-H4] are robust, ie satisfied in an open subset of one-parameter families of smooth $R$-reversible vector fields in $\mathbb{R}^3$. See Figure 2 for a sketch of the situation.

In this section we detail some important aspects of the dynamics near such a heteroclinic cycle and its unfolding, as formulated in Theorem 1.2.

Our results include the existence of certain symmetric heteroclinic and periodic solutions close to the original heteroclinic cycle.

This section is organized as follows. In Subsection (a) we introduce first hit maps between surfaces of sections and return maps. In Subsection (b) we discuss the some of the tranformation properties of these return maps, which are used in Subsection (c), (d), (e), (f) to discuss the existence of respectively 2D-heteroclinic orbits, symmetric periodic solutions, 1D-heteroclinic and homoclinic solutions and horseshoes, at and near the bifurcation point $\lambda = 0$.

(a) **Sections and return maps**

In this section we define the surfaces of sections that we employ to define return maps to study the dynamics near the heteroclinic cycle.

We define two main local sections, $\Sigma_0$ and $\Sigma_1$, satisfying:

- The sections are setwise invariant under $R$: $R(\Sigma_0) = \Sigma_0$ and $R(\Sigma_1) = \Sigma_1$. Consequently, $\text{Fix}(R)$ bisects $\Sigma_0$ and $\Sigma_1$. We distinguish between the local actions of the time-reversal symmetry $R$: $R_0 = R|_{\Sigma_0}$ and $R_1 = R|_{\Sigma_1}$.

- The sections are locally transverse to the the flow of $F$ at $\lambda = 0$ (and hence also at sufficiently small values of $\lambda$).
The surfaces of section intersect the heteroclinic cycle transversally. We thus define the Poincaré return map \( \psi \). We define the local section \( \Sigma \) so that the local stable and unstable manifolds are straightened:

\[
\psi = \psi_1 \circ \phi \circ \psi_2 \circ \phi \circ \psi_1,
\]

where the maps \( \psi_1 : \Sigma \to \sigma_0, \phi : \sigma_0 \to \sigma_1, \psi_2 : \sigma_1 \to \Sigma_1, \psi_2' : \Sigma_1 \to \sigma_1', \phi' : \sigma_1' \to \sigma_0', \) and \( \psi_1' : \sigma_0' \to \Sigma_0 \) are first hit maps (see the illustration in Figure 2).

We define some more sections close to the saddle-foci \( p_0 \) and \( p_1 \). We subsequently choose local coordinates around \( p_0 \) and \( p_1 \) such that the saddle fixed point \( p_0 \) is at the origin. Recall the eigenvalues of the total configuration and the position of the surfaces of section is given in Figure 2.

The next step is to construct a return map that is built from the composition of first hit maps between the surfaces of section. Such first hit maps are locally well defined (for small \( \lambda \)) since at \( \lambda = 0 \) the surfaces of section intersect the heteroclinic cycle transversally. We thus define the Poincaré return map \( F_0 : \Sigma_0 \to \Sigma_0 \):

\[
F_0 = \psi_1' \circ \phi' \circ \psi_2' \circ \psi_2 \circ \phi \circ \psi_1,
\]

where the maps \( \psi_1 : \Sigma_0 \to \sigma_0, \phi : \sigma_0 \to \sigma_1, \psi_2 : \sigma_1 \to \Sigma_1, \psi_2' : \Sigma_1 \to \sigma_1', \phi' : \sigma_1' \to \sigma_0', \) and \( \psi_1' : \sigma_0' \to \Sigma_0 \) are first hit maps (see the illustration in Figure 2).

We may use the reversibility of the vector field to express the maps \( \psi_1', \psi_2', \phi' \) in terms of \( \psi_1, \psi_2, \phi, R, R_0, R_1 \). Namely:

\[
\psi_1' = R_0 \circ \psi_1^{-1} \circ R, \quad \psi_2' = R \circ \psi_2^{-1} \circ R_1, \quad \phi' = R \phi^{-1} R.
\]

Consequently, we have:

\[
F_0 = R_0 \circ \psi_1^{-1} \circ \phi^{-1} \circ \psi_2^{-1} \circ R_1 \circ \psi_2 \circ \phi \circ \psi_1.
\]

and it is readily verified that \( F_0 \) is a \( R_0 \)-reversible map, ie

\[
F_0^{-1} = R_0 \circ F_0 \circ R_0^{-1}.
\]

Similarly it is easy to show that the Poincaré return map \( F_1 : \Sigma_1 \to \Sigma_1 \) satisfies

\[
F_1 = \psi_2 \circ \phi \circ \psi_1 \circ R_0 \circ \psi_1^{-1} \circ \phi^{-1} \circ \psi_2^{-1} \circ R_1,
\]

and that hence \( F_1 \) is a \( R_1 \)-reversible map, ie

\[
F_1^{-1} = R_1 \circ F_1 \circ R_1^{-1}.
\]

We now focus on the local return maps. We consider the local map \( \phi \) about the saddle point \( p_0 \), the corresponding properties for the map \( \phi' \) can be deduced from the form of \( \phi \) and the fact that \( \phi' = R \phi^{-1} R \). This local map will provide the key to understanding the features of the dynamics we are interested in.

Since the saddle fixed point \( p_0 \) is hyperbolic, there is a unique hyperbolic saddle point \( p_0^0 \) for each \( |\lambda| \) sufficiently small, and without loss of generality we may assume that the equilibrium \( p_0 \) is at \( (-p, 0, 0) \) for all \( \lambda \) small. Similarly we may choose \( R \) to be spanned by \( (0, 1, 0) \).

We now choose local coordinates around \( p_0 \) such that \( p_0 \) is at the origin. Recall the eigenvalues of the fixed point \( p_0 \) are \( -\rho(\lambda) \pm i \omega(\lambda), \mu(\lambda) \), with \( \rho(\lambda), \mu(\lambda) > 0 \). From now on we suppress the argument \( \lambda \). It can be shown [3] that there exists a local \( C^1 \) change of coordinates (and a reparametrization of time), such that in these coordinates the flow in an \( \epsilon \)-neighbourhood near the saddle point \( p_0 \) is linear, so that the local stable and unstable manifolds are straightened:

\[
\begin{align*}
\dot{x}_L &= -(\rho/\mu)x_L + (\omega/\mu)y_L \\
\dot{y}_L &= -(\omega/\mu)x_L - (\rho/\mu)y_L \\
\dot{z}_L &= z_L
\end{align*}
\]

We define the local section \( \sigma_0 \) as follows:

\[
\sigma_0 = \{(x_L, y_L, z_L) \in \mathbb{R}^3 \mid x_L = 0, y_L = y^* + \delta \},
\]

where the point of first intersection of \( h(t) \) and \( \sigma_0 \) is \( (0, y^*, 0) \), and \( \delta \) is sufficiently small so that \( (0, y^*, 0) \) is the only intersection of \( h(t) \) with \( \sigma_0 \). Note that \( \{(0, y, 0) \} \subset \sigma_0 \) is the trace of the two dimensional stable manifold \( W^s(p_0) \) in \( \sigma_0 \). We subsequently choose

\[
\sigma_1 = \{(x, y, z) \in \mathbb{R}^3 \mid z = d \}.\]
Note that \( \{(0,0,d)\} \in \sigma_1 \) is the trace of the one dimensional unstable manifold \( W^u(p_0) \).

The flow of (5.6) can be integrated explicitly to give:

\[
\begin{align*}
\frac{dx_L}{dt} &= x_L(0) \exp(-\frac{dL}{\mu}) \cos(\frac{\omega t}{\mu}) + y_L(0) \exp(-\frac{dL}{\mu}) \sin(\frac{\omega t}{\mu}), \\
\frac{dy_L}{dt} &= y_L(0) \exp(-\frac{dL}{\mu}) \cos(\frac{\omega t}{\mu}) - x_L(0) \exp(-\frac{dL}{\mu}) \sin(\frac{\omega t}{\mu}), \\
\frac{dz_L}{dt} &= z_L(0) \exp(t),
\end{align*}
\]

from which the time of flight from \( \sigma_0 \) to \( \sigma_1 \) can be calculated to be \( t^* = -\ln(z_L(0)/d) \). We thus find the following expression for the first hit map \( \phi : \sigma_0 \to \sigma_1 \):

\[
\phi(y_L, z_L) = \left(y_L \left(\frac{z_L}{d}\right)^{\rho/\mu} \sin \left(\frac{\omega}{\mu} \ln \left(\frac{z_L}{d}\right)\right), y_L \left(\frac{z_L}{d}\right)^{\rho/\mu} \cos \left(\frac{\omega}{\mu} \ln \left(\frac{z_L}{d}\right)\right)\right).
\]

The analysis near the equilibrium point \( p_1 \) is analogous, yielding the \( \phi = \phi'^{-1} \) in terms of local coordinates \( (x'_L, y'_L, z'_L) \) near \( p_1 \). Throughout the remainder we use \( (x_L, y_L, z_L) \) to denote local coordinates near \( p_0 \) and \( (x'_L, y'_L, z'_L) \) to denote local coordinates near \( p_1 \).

To complete the construction, we choose the remaining surfaces of section in terms of local coordinates

\[
\Sigma_{0,1} = \{(x_i, y_i, z_i) \in \mathbb{R}^3 \mid x = 0\},
\]

where \( (y_i, z_i) \) are chosen such that \((0,0)\) is a point of intersection of the heteroclinic cycle and the sections \( \Sigma_i \) at \( \lambda = 0 \).

For the construction of the global maps, we may write, for example \( \psi_1 : \Sigma_0 \to \sigma_0 \) as

\[
\psi_1 \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} y_L \\ z_L \end{pmatrix} = \begin{pmatrix} y^* \\ 0 \end{pmatrix} + A \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} + \cdots
\]

where the dots denote terms of higher order. Since the map \( \psi_1 \) is a diffeomorphism, \( A \) is a nonsingular matrix. We can similarly write

\[
\psi_2 \begin{pmatrix} x_L \\ y_L \end{pmatrix} = \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = B \begin{pmatrix} x_L \\ y_L \end{pmatrix} + \cdots
\]

\[
\psi'_2 \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x'_L \\ y'_L \end{pmatrix} = RB^{-1}R_1 \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} + \cdots
\]

\[
\psi'_1 \begin{pmatrix} y'_L \\ z'_L \end{pmatrix} = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} = R_0A^{-1}R \begin{pmatrix} y'_L \\ z'_L \end{pmatrix} + \cdots
\]

It should be noted that with the above choices the compositions \( \psi'_2 \circ \psi_2 \) and \( \psi'_2 \circ \psi_2 \) appear to be orientation reversing if we identify the local coordinates \( (x_L, y_L, z_L) \) near \( p_0 \) with the local coordinates \( (x'_L, y'_L, z'_L) \) near \( p_1 \), but this is just due to the choices of local coordinates.

Finally, the local reversing symmetries \( R_{0,1} \) act on the sections \( \Sigma_{0,1} \) as:

\[
R_{0,1} : \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \to \begin{pmatrix} 0 \\ y \\ -z \end{pmatrix},
\]

(b) Dynamics of the first hit maps

In this section we establish three lemmas that are central to the proof of Theorem 1.2. Throughout this section we assume that the hypotheses [H1]-[H4] are satisfied.

The first result concerns the fact that line segments transversal to the local stable manifold in \( \sigma_0 \) get mapped by \( \phi \) to a logarithmic spiral in \( \sigma_1 \). Its proof is immediate from (5.8).
Lemma 5.1 Let \((y_L(s), z_L(s))\) be a line segment in \(\sigma_0\), parameterised by \(s\), such that \(y_L(0)\) is close to \(y^*\), and \((y_L(s), z_L(s))\) transversely intersects the local stable manifold \(W^s(p_0)\), i.e. \(z_L(0) = 0\) and \(\frac{\partial z_L(s)}{\partial s}_{s=0} \neq 0\).

Then the image of \((y_L(s), z_L(s))\) under the local map \(\phi\) is a logarithmic spiral in \(\sigma_1\). That is, in polar coordinates \(x_L = r \sin \theta, y_L = r \cos \theta\), the image of \((y_L(s), z_L(s))\) takes the form

\[
(r, \theta) = \left( y_L(s) \left( \frac{z_L(s)}{d} \right)^{\rho/\mu}, \frac{\omega}{\mu} \ln \left( \frac{z_L(s)}{d} \right) \right). \tag{5.15}
\]

The second result discusses how the image of the above logarithmic spiral under \(\psi'_2 \circ \psi_2\) (in which is still a logarithmic spiral) is subsequently mapped to \(\sigma'_0\) by the first hit map \(\phi'\).

Lemma 5.2 Consider a logarithmic spiral \(\Gamma = (x'_L(s), y'_L(s))\) in \(\sigma'_1\) with

\[
x'_L(s) = ay_L(s) \left( \frac{z_L(s)}{d} \right)^{\rho/\mu} \sin \left( \frac{\omega}{\mu} \ln \left( \frac{z_L(s)}{d} \right) \right) + by_L(s) \left( \frac{z_L(s)}{d} \right)^{\rho/\mu} \cos \left( \frac{\omega}{\mu} \ln \left( \frac{z_L(s)}{d} \right) \right) + \ldots
\]

\[
y'_L(s) = cy_L(s) \left( \frac{z_L(s)}{d} \right)^{\rho/\mu} \sin \left( \frac{\omega}{\mu} \ln \left( \frac{z_L(s)}{d} \right) \right) + dy_L(s) \left( \frac{z_L(s)}{d} \right)^{\rho/\mu} \cos \left( \frac{\omega}{\mu} \ln \left( \frac{z_L(s)}{d} \right) \right) + \ldots
\]

where \(a, b, c, d\) are constants, \(ad - bc = -1\), and the remainders \(\ldots\) denote terms of higher order in \(z_L(s)\). The coordinates \((y_L(s), z_L(s))\) refer to the coordinates of the logarithmic spiral in Lemma 5.1.

Then the image of \(\Gamma\) under \(\phi'\) in \(\sigma'_0\) consists of a countably infinite set of lines, accumulating exponentially fast (in the \(C^1\)-topology) to the trace of the unstable manifold \(W^u(p_1) \cap \sigma'_0 = \{z'_L = 0\}\), with exponent \(-\frac{\omega}{\mu} \pi\).

Proof Recall that \(\phi'^{-1} = R \phi R\), so the pre-image of a point \((y'_L, z'_L)\) in \(\sigma'_0\) is, in polar coordinates \((x'_L, y'_L) = (r' \sin \theta', r' \cos \theta')\):

\[
(r', \theta') = \left( y'_L \left( \frac{z'_L}{d} \right)^{\rho/\mu}, \frac{\omega}{\mu} \ln \left( \frac{z'_L}{d} \right) \right). \tag{5.17}
\]

We first consider the case where the linear part \(C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) of \(\psi'_2 \circ \psi_2\) equals \(C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\) and neglect the higher order terms. We note here that due to the reversibility (and our choice of coordinates which accounts for the minus sign) we always have \(\det(C) = -1\).

Then, in \((r', \theta')\) coordinates \(\Gamma\) has the form

\[
(r', \theta') = \left( y_L(s) \left( \frac{z_L(s)}{d} \right)^{\rho/\mu}, -\frac{\omega}{\mu} \ln \left( \frac{z_L(s)}{d} \right) \right). \tag{5.18}
\]

We note that the equations for \(\theta'\) are modulo \(2\pi\), and that the \(\theta'\) equation in (5.17) is valid for \(-\theta'\) sufficiently large, and the equation for \(\theta'\) in (5.18) is valid for \(\theta'\) sufficiently large. We first consider \(y'_L \in \sigma'_0\) fixed, and search for values of \(z'_L \in \sigma'_0\) that are in the image of \(\Gamma\) under \(\phi'\).

By equating the radius coordinates of (5.17), (5.18), we obtain

\[
z'_L = \left( \frac{y_L(s)}{y'_L} \right)^{\mu/\rho} z_L(s). \tag{5.19}
\]

Note that (5.19) gives \(z'_L\) as a function of \(s\). \(y_L(s)\) is \(O(1)\) in \(s\) as \(s \to 0\), so \(z'_L(s)\) and \(z_L(s)\) are of the same order as \(s \to 0\).

Now recall that the equations for the arguments in (5.17), (5.18) are modulo \(2\pi\). Hence, with \(z'_L(s)\) and \(z_L(s)\) sufficiently small, equating the angle equations in (5.17) and (5.18) yields

\[
z'_L(s) = d \left( \frac{y_L(s)}{y'_L} \right)^{\mu/2\rho} \exp \left( -\frac{\mu}{\omega} n \pi \right). \tag{5.20}
\]
Figure 5: The spirals $\Gamma$ and $D(\Gamma)$ in the section $\sigma'_1$. The line $L$ shown is spanned by the expanding eigenvector of $D$. The dashed spiral is the preimage of $(y'_L, z'_L) \in \sigma'_2$ under $\phi'$, for fixed $y'_L$.

for $n \in \mathbb{N}$. As $s \to 0$, $y_L(s)$ tends to a constant and $z'_L(s) \to 0$. Then for a fixed large $n \in \mathbb{N}$, (5.20) has a solution for $s$ close to zero. This is a point at which the curves $\Gamma$ and the preimage of $(y'_L, z'_L)$ (for fixed $y'_L$) in $\sigma'_1$ intersect. Moreover since $\frac{dz_L(s)}{ds}$ (and hence $\frac{dz'_L(s)}{ds}$) is bounded away from zero for $s$ sufficiently close to zero, and $\frac{dy_L(s)}{ds}$ is approximately constant for $s$ sufficiently close to zero, for $n$ large enough this intersection is transverse. Substituting (5.20) into (5.17), we see that these intersections (for each $n \in \mathbb{N}$) occur every $\pi$ in the angle argument, asymptotically as $s \to 0$. We may use the Implicit Function Theorem to show that as $y_L$ is varied, we can still find a unique value for $s$ such that the two curves intersect transversally. Then the image of $\Gamma$ under the map $\phi'$ is a countable set of lines that exponentially accumulate to $z'_L = 0$ (the trace of the unstable manifold of $p_1$). They accumulate with the order of $\exp \left( -\frac{n}{2} \pi \right)$ for $n \in \mathbb{N}$.

We now consider the general case that $C = D \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, where $\det(D) = 1$ since $\det(C) = -1$. $D$ has the effect of a linear transformation of the curve $\Gamma$ that we previously obtained. If $D$ is elliptic, the situation is analogous to $D = I$, so we focus on the case that $D$ is hyperbolic. As before, we first fix $y'_L$ and then consider the intersections of $\Gamma$ with the preimage of $(y'_L, z'_L)$. These intersections are approximately $\pi$ apart in the angle $\theta'$. Now consider the line $L$ in $\sigma'_1$ that is spanned by the expanding eigenvector of $D$, such that this line bisects these intersections, see Figure 5. We use $L$ to divide $\Gamma$ into disjoint, countably many arcs, each of length $2\pi$ in the angle variable, such that each endpoint of each arc is in $L$. The effect of $D$ on each of these arcs is similar, so we just consider one of these arcs. In fact, by the $\mathbb{Z}_2$-rotational symmetry of $\Gamma$ and $D$ (both commute with $-I$), the effect of the transformation is similar on each half of these arcs, each of which parameterized by a parameter interval of length $\pi$.

The transformed curve $D(\Gamma)$ cannot have fewer intersections with the preimage of $(y'_L, z'_L)$ than $\Gamma$. Namely, each intersection can be followed while we linearly continuously deform the curve $\Gamma$ to $B_1(\Gamma)$. Moreover, for the reasons of symmetry just mentioned, the intersections will remain to be $\pi$ apart in the angle variable.

By using similar arguments as before, the image of $\Gamma$ in $\sigma'_0$ is a set of lines exponentially accumulating to $z'_L = 0$ (the trace of the unstable manifold of $p_1$), with the exponential rate $\sim \exp \left( -\frac{n}{2} \pi \right)$, $n \to \infty$ with $n \in \mathbb{N}$.

The final lemma of this section describes the image of a line segment in $\sigma'_1$ under $\phi'$, and is reminiscent of Lemma 5.2:
Lemma 5.3 Consider a smooth \((C^1)\) line segment \(\gamma = \{(x^0_L(s), y^0_L(s)) \mid s \in [0, \ell]\}\) (some \(\ell > 0\)) in \(\sigma^{1}_b\) with \((x^0_L(s), y^0_L(s)) = (0, 0)\). The image of \(\gamma\) under \(\phi'\) in \(\sigma^{0}_b\) consists of a countably infinite set of lines which accumulate exponentially fast (in the \(C^1\)-topology) to the trace of \(W^u(p_1)\), with exponent \(-\frac{\pi}{2}\).

Proof The proof is very similar to that of Lemma 5.2. The line \(\gamma\) and the preimage of \((y^0_L, z^0_L)\) in \(\sigma^{1}_b\) for constant \(y^0_L\), intersect transversally in countably many points. Moreover, as we vary \(y^0_L\), the image of \(\gamma\) in \(\sigma^{0}_b\) is a countable set of lines accumulating exponentially to \(z^1_L = 0\) at the rate \(z^1_L \sim \exp(-\frac{\pi}{2}n\pi)\) with \(n \to \infty\) and \(n \in \mathbb{N}\).

The above results on properties of the first hit maps form the basis of the proofs of the statements in Theorem 1.2, which we continue to discuss below.

(c) Symmetric and asymmetric 2D heteroclinic orbits

In this subsection we focus on the heteroclinic connections from \(p_1\) to \(p_0\), consisting of the intersections of the twodimensional stable and unstable manifolds of \(p_0\) and \(p_1\), respectively. The symmetric heteroclinics admit a simple characterization, the proof of which is folklore, see e.g [28].

Proposition 5.4 A heteroclinic solution connection \(p_0\) to \(p_1\), or vice-versa, is symmetric \((R\text{-invariant})\) if and only if it intersects \(\text{Fix}(R)\) (precisely once).

We are now ready to prove the first part of Theorem 1.2.

Proof of Theorem 1.2 A We start from an isolated symmetric 2D heteroclinic connection formed by the transversal intersection of \(W^s(p_0)\) and \(W^u(p_1)\), which does not pass through \(\Sigma_1\).

We consider the unstable manifold of \(p_1\) to track down 2D heteroclinics at \(\lambda = 0\). The trace of the local unstable manifold of \(p_1\) in \(\sigma^{0}_b\) is given by the set \(z^1_L = 0\).

By (5.13) and the symmetry (5.14), the \(\psi'\)-image of \(W^u(p_1) \cap \sigma^0\) is a \(C^1\) (in general curved) line segment in \(\Sigma_0\), transverse to \(\text{Fix} R\) and hence also transverse to the \(\psi^{-1}\)-image of the trace of the stable manifold of \(p_0\), \(W^s(p_0) \cap \sigma^0\). By (5.10), the image of the line segment \(\psi_1'((W^u(p_1) \cap \sigma^0) \in \Sigma_0\) under \(\psi_1\) is a line segment in \(\sigma^0\), which satisfies the hypotheses of Lemma 5.1, by which in turn \(\phi\) maps this line segment to a logarithmic spiral in \(\sigma^{1}\). \(\psi_2\) maps this spiral subsequently diffeomorphically to \(\Sigma_1\). Denote this spiral in \(\Sigma_1\) as \(\Sigma\).

The spiral \(\Sigma\) intersects \(\text{Fix} R_1 \subset \Sigma_1\) in countably infinitely many points, that exponentially accumulate to the centre of the spiral on \(\text{Fix} R_1\). By Lemma 5.4, each of these points represents a symmetric 2-2D heteroclinic orbit. By reversibility, the image of \(W^s(p_0)\) under \((\psi_2^{-1} \circ (\phi')^{-1} \circ (\psi_1')^{-1} \circ \psi^{-1}\) in \(\Sigma_1\) is the \(R_1\)-image of \(\Sigma\), intersecting \(\Sigma\) along \(\text{Fix} R\). These intersections are generically transverse. It should be noted that depending on the spirals, there may be additional intersections of these two spirals, giving rise to additional asymmetric 2-2D heteroclinic orbits.

In a similar fashion we can find 3-2D heteroclinics. Consider \(\psi_2^{-1}(\Sigma) \subset \sigma^{1}_b\) and its image in \(\sigma^{1}_b\) under \(\psi'\). By (5.11), (5.12), the linear part of this map has determinant \(-1\) so that the spiral satisfies the hypotheses of Lemma 5.2. By application of Lemma 5.2 the image of the unstable manifold of \(p_1\) accumulates on itself exponentially as a set of lines in \(\sigma^0\). By (5.13), any one of these lines (sufficiently close to \(z^1_L = 0\) in \(\sigma^0\)) maps to \(\Sigma_0\) by \(\psi'_1\) as a line segment which is transversal to \(\text{Fix} R_0\) and \(W^s(p_0)\). Where it intersects \(\text{Fix} R_1\) we have a symmetric 3-2D heteroclinic orbit, and where it intersects \(W^s(p_0)\) we have a 2-2D heteroclinic orbit. By the reversing symmetry, we may apply the same procedure to the stable manifold of \(p_0\) to produce an exponentially accumulating set of lines in \(\Sigma_0\) which are the \(R\)-images of those for the unstable manifold of \(p_1\). By choosing two lines (one in the image of \(W^u(p_1)\) and one in the pre-image of \(W^s(p_0)\)) that are not symmetric images of each other, for \((y, z)\) sufficiently small, these lines will have an intersection that produces an asymmetric 3-2D heteroclinic orbit.

By induction, these constructions can be carried out \(ad\ infinitum\) to reveal the existence of \(n\)-2D heteroclinic orbits, for any \(n\). Similar arguments to those above show that for \(n \geq 3\) we find symmetric and asymmetric \(n\)-2D heteroclinics, and there are countably infinitely many of each.
(d) Periodic solutions

We use the return maps $F_0$ and $F_1$ to study the occurrence of $R$-symmetric periodic solutions. To that effect, we recall some simple characterization of symmetric periodic orbits for reversible maps. It is readily verified that symmetric periodic solutions of the reversible vector field near the heteroclinic cycle correspond to symmetric (setwise $R$-invariant) periodic orbits of the return maps $F_0$ and $F_1$. The following observation is folklore, dating back at least to Birkhoff and Poincaré, cf [28]:

**Proposition 5.5** Let $F$ be an $R$-reversible map, then an orbit of $F$ is $k$-periodic and $R$-symmetric if and only if it intersects $\text{Fix}(R) \cup \text{Fix}(R^{-1} \circ F^k)$ precisely twice.

It is important to note that in the present situation, $F_0 = R_0 \circ P_1$ where $P_1 : \Sigma_0 \rightarrow \Sigma_0$ is an involution, i.e. $P_1^2 = \text{Id}$. In fact, $\text{Fix}(P_1)$ is precisely the pull-back by the flow of $\text{Fix}(R_1)$ inside $\Sigma_1$ to $\Sigma_0$, so that $\dim \text{Fix}(P_1) = \dim \text{Fix}(R_1) = 1$. Similarly, we may define the involution $P_0 : \Sigma_1 \rightarrow \Sigma_1$ so that $F_1 = P_0 \circ R_1$. Note that with our one-parameter family of vector fields $F(\cdot, \lambda)$ it is natural to think of $P_0$ and $P_1$ being nonlinear involutions depending on a parameter $\lambda$.

With the above interpretation of $P_0$ and $P_1$, we may reformulate the result on periodic solutions as follows:

**Proposition 5.6** A periodic solution of $F(\cdot, \lambda)$ near the heteroclinic cycle is $R$-symmetric if and only if it intersects $\text{Fix}(R_0) \cup \text{Fix}(R_1)$ precisely twice.

Of course, this property coincides with the observation that for an $R$-reversible vector field a solution is periodic and $R$-symmetric if and only if it intersects $\text{Fix}(R)$ precisely twice, cf [28].

**Proof of Theorem 1.2 B** We first consider symmetric periodic solutions accumulating to symmetric heteroclinic cycles.

We first consider symmetric periodic solutions that are close to the 1-heteroclinic cycle consisting of the isolated transversal 1-2D heteroclinic connection and the 1D heteroclinic connection that exists at $\lambda = 0$. To find symmetric periodic solutions then, we consider $\psi_2 \circ \phi \circ \psi_1(\text{Fix}(R_0))$. By Lemma 5.1 we find

$$
\psi_2 \circ \phi \circ \psi_1 \left[ \begin{array}{c} y \\ 0 \end{array} \right] = B \left[ \begin{array}{c} \hat{y}_L \left( \frac{a_1}{d} \right)^{\rho/\mu} \sin \left( \frac{\omega}{\mu} \ln \left( \frac{a_1}{d} \right) \right) \\ \hat{y}_L \left( \frac{a_2}{d} \right)^{\rho/\mu} \cos \left( \frac{\omega}{\mu} \ln \left( \frac{a_2}{d} \right) \right) \end{array} \right] + O(z^{2\rho/\mu}),
$$

where $\hat{y}_L = y^* + a_1 y + O(y^2)$, $\hat{z}_L = a_3 y + O(y^2)$, which may be rewritten as

$$
\psi_2 \circ \phi \circ \psi_1 \left[ \begin{array}{c} y \\ 0 \end{array} \right] = \left[ \begin{array}{c} y^* \left( \frac{a_1}{d} \right)^{\rho/\mu} \left( b_1^2 + b_2^2 \right)^{1/2} \sin \left( \frac{\omega}{\mu} \ln \left( \frac{a_1}{d} \right) + \Phi_1 \right) \\ y^* \left( \frac{a_2}{d} \right)^{\rho/\mu} \left( b_3^2 + b_4^2 \right)^{1/2} \sin \left( \frac{\omega}{\mu} \ln \left( \frac{a_2}{d} \right) + \Phi_2 \right) \end{array} \right] + O(y^{\min\{1+(\rho/\mu),2\rho/\mu\}}),
$$

(5.21)

where $\Phi_1 = \tan^{-1} \left( \frac{b_1}{b_2} \right)$ and $\Phi_2 = \tan^{-1} \left( \frac{b_3}{b_4} \right)$. Hence symmetric periodic solutions correspond to solutions of the equation

$$
y^{\rho/\mu} \sin \left( \frac{\omega}{\mu} \ln \left( \frac{\hat{z}_L}{d} \right) + \Phi_2 \right) + O(y^{\min\{1+(\rho/\mu),2\rho/\mu\}}) = 0.
$$

(5.22)

Note that the fact that $B = \left( \begin{array}{cc} b_1 & b_2 \\ b_3 & b_4 \end{array} \right)$ is nonsingular implies that $b_3, b_4$ are not both zero, and $\Phi_1 \neq \Phi_2$. From (5.22) we find that at $\lambda = 0$ there are countably many intersections of $\text{Fix}(R_1)$ and $\psi_2 \circ \phi \circ \psi_1(\text{Fix}(R_0))$, and so countably many symmetric periodic orbits. Moreover, these periodic orbits are asymptotically $\pi/\omega$ apart in the time of passage from $\Sigma_0$ to $\Sigma_1$. As the intersections are also transversal, we may continue them as we vary the parameter $\lambda$. By application of the Implicit Function Theorem and a rescaling of the parameter we thus find that for $|\lambda|$ sufficiently small, symmetric periodic solutions are in one-to-one correspondence to solutions of the equation

$$
\lambda + y^{\rho/\mu} \sin \left( \frac{\omega}{\mu} \ln \left( \frac{\hat{z}_L}{d} \right) + \Phi_2 \right) = 0.
$$

20
Consequently, the set of symmetric periodic solutions forms a one parameter family, parametrised by period. This family is parametrised along the spiral that is the image of $\text{Fix } R_0$ in $\Sigma_1$ under $\psi_2 \circ \phi \circ \psi_1$. The effect of perturbing the parameter $\lambda$ is effectively to move this spiral transversally to $\text{Fix } R_1$, and so by oscillating the parameter $\lambda$ about zero we can follow the spiral into the centre, where the period tends to infinity, see Figure 6. Clearly as the period tends to infinity, the periodic orbit converges to the 1-heteroclinic cycle.

The same analysis can be applied for any symmetric heteroclinic cycle, yielding symmetric periodic solutions intersecting $\text{Fix}(R)$ in the same section $\Sigma_i$ as the heteroclinic cycle.

The above discussion concerned symmetric periodic solutions accumulating to symmetric heteroclinic cycles. In the case of asymmetric heteroclinic cycles the analysis is standard and very similar (we leave the details to the reader), leading to the analogous conclusion: a one-parameter family of periodic orbits oscillating towards the heteroclinic cycle. In the case of homoclinic cycles we also obtain the existence of an oscillating family of periodic solutions if Shilnikov’s condition is met [19, 35] (in which case we also obtain associated horseshoes).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{Sketch of a one-parameter family of symmetric periodic solutions accumulating to a heteroclinic cycle (at $\lambda = 0$) in a parameter ($\lambda$) versus period ($T$) plot. Note the countable infinity of periodic orbits accumulation to the heteroclinic cycle at the heteroclinic bifurcation point $\lambda = 0$. As $T \to \infty$ the graph intersects $\lambda = 0$ asymptotically $\pi/\omega$-periodically. Also depicted is a sketch of the image of $\text{Fix } R_0$ in $\Sigma_1$.}
\end{figure}

\section*{(e) 1D heteroclinic orbits and homoclinic orbits}

In this subsection we prove Theorem 1.2 C and D, that deals with the occurrence of 1D heteroclinic and homoclinic orbits in the unfolding of the symmetric heteroclinic cycle.

\textbf{Proof of Theorem 1.2 C and D} For $\lambda \neq 0$ the local and global maps will change slightly, but not significantly. What is most important is that by hypothesis [H4] for $\lambda \neq 0$ (small enough), $W^s(p_1)$ and $W^u(p_0)$ do not coincide.

In order to study the 1D heteroclinic orbits, for each $\lambda \neq 0$, we consider the trace $\zeta$ of the first intersection of $W^u(p_0)$ with $\Sigma_1$, as a function of a parameter $\lambda$ restricted to a small interval around 0 such that $\zeta(\lambda)$ is the trace of $W^u(p_0)$ in $\Sigma_1$ at parameter value $\lambda$. By [H4] and smoothness of our system in the parameter, $\zeta$ is a smooth curve intersecting $\text{Fix } R_1$ at $\lambda = 0$ transversally.
By studying $\zeta$ under the return maps, we are thus following the trace of the unstable manifold of $p_0$ in $\Sigma_1$ for a set of parameter values close to zero. To be precise, we need to follow the point $\zeta(\lambda)$ under the return maps at parameter value $\lambda$, for a small interval of $\lambda$ values including 0. We can here use the fact that the return maps change only little for the parameter values in such a small interval (the main issue being that only at $\lambda = 0$ we have a 1-1D heteroclinic connection). In fact, we can treat the maps as being almost constant. By studying the intersections of the image of $\zeta$ under the return maps with $\text{Fix}(R_0) \cup \text{Fix}(R_1)$ we obtain 1D heteroclinic orbits. Its intersections with $W^s(p_0)$ yield homoclinic orbits.

First we map $\zeta$ under $\psi'_2$. Then $\psi'_2(\zeta)$ is a line segment in $\sigma'_1$ that satisfies the hypotheses of Lemma 5.3. Namely, let $\psi'_2(\lambda, \cdot)$ denote the first hit map at parameter value $\lambda$, then $\psi'_2$ is a diffeomorphism and $\psi'_2(0, \zeta(0)) = (0, 0)$. Applying Lemma 5.3, $\phi' \circ \psi'_2(\zeta)$ is a countable set of exponentially accumulating lines to $z'_L = 0$ in $\sigma'_0$. Any such line sufficiently close to $z'_L = 0$ maps by $\psi'_1$ to $\Sigma_0$ so that it intersects both $\text{Fix}(R_0)$ and $W^s(p_0)$ transversally. Since there are a countably infinite number of lines with $z'_L$ sufficiently small, there is a countable infinity of 2-1D heteroclinic orbits and 1-homoclinic orbits, exponentially accumulating to $\lambda = 0$ in parameter space from both sides.

Now consider one of the lines in $\sigma'_0$ sufficiently close to $z'_L = 0$, and its image under $\psi_1 \circ \psi'_2$. This appears in $\sigma_0$ as a line segment satisfying the hypotheses of Lemma 5.1. By application of this lemma $\phi$ maps this line segment to $\sigma_1$ as a logarithmic spiral, which in turn is diffeomorphically mapped by $\psi_2$ to $\Sigma_1$. We now recall that we should follow each point $\zeta(\lambda)$ under the return maps at parameter value $\lambda$. The return map at $\lambda$ produces a logarithmic spiral that is centred on $\text{Fix}(R_1)$ in $\Sigma_1$ if and only if $\lambda = 0$. Considering the image of the points in $\zeta$ under the return maps in a small $\lambda$ subinterval, we thus find as the image of each of the lines in $\sigma'_0$ a logarithmic spiral centered outside $\text{Fix}(R_1)$, but tending to $\text{Fix}(R_1)$ as we chose the lines tending to $z'_L = 0$. Each of these spirals has a large finite number of intersections with $\text{Fix}(R_1)$, tending to infinity as the distance to $z'_L = 0$ goes to zero. As each intersection with $\text{Fix}(R_1)$ yields a 3-1D heteroclinic we thus obtain a countable infinity of 3-1D heteroclinic orbits accumulating to $\lambda = 0$. That this accumulation is exponential follows from the logarithmic nature of the spirals.

Then subsequently mapping one of the above mentioned spirals in $\Sigma_1$ to $\sigma'_1$ by $\psi'_2$, we obtain a logarithmic spiral that does not quite satisfy the conditions of Lemma 5.2, but which may be chosen arbitrarily close to it, by choosing a corresponding $\lambda$-interval sufficiently close to zero). By transversality, any finite number of the intersections of this spiral with the preimage of $(y'_L, z'_L)$ for fixed $y'_L$ persists, as in the proof of the Lemma. We thus conclude that for $\lambda$ sufficiently close to zero, the spiral maps into $\sigma'_0$ as a finite set of lines which get as close to $z'_L = 0$ as we desire. The $\psi'_1$-images of these lines in $\Sigma_0$ yield transverse intersections with both $\text{Fix}(R_0)$ and $W^s(p_0)$ and thus 1D-heteroclinic and homoclinic orbits.

This procedure can be repeated indeﬁnitely to yield a countably inﬁnite number of $n$-1D heteroclinic orbits for $n \geq 2$ and $n$-homoclinic orbits for $n \geq 1$, occuring for unique parameter points, all accumulating exponentially to $\lambda = 0$ in parameter space from both sides of $\lambda = 0$.

(f) **Horseshoes**

It is well known from the work of Shilnikov that under certain eigenvalue conditions, a homoclinic orbit to a saddle-focus such as described in Theorem 1.2 D may give rise to chaotic dynamics. In the context of the homoclinic bifurcations identified in the previous section, this condition is $\rho < \mu$. Here, however, it turns out that (generically) horseshoes arise due to the heteroclinic cycle, independent of any kind of Shilnikov condition on eigenvalues at the saddle-foci. In this subsection we prove this result, which is stated in Theorem 1.2 E.

**Proof of Theorem 1.2 E** For reference, please consider the sketches in Figure 3.

We consider an open set $B_0$ in $\Sigma_0$ such that one side of $B_0$ coincides with $W^s(p_0)$, and such that if we map this strip to $\sigma_0$ by $\psi_1$, it appears as the set

$$\psi_1(S) = \{ (y_L, z_L) : 0 < z_L < \varepsilon_1, y^* - \varepsilon_2 < y_L < y^* + \varepsilon_2 \},$$  \hspace{1cm} (5.23)

for some small $\varepsilon_2 \gg \varepsilon_1 > 0$. $B_0$ appears in $\Sigma_0$ as a thin strip along $W^s(p_0)$. By Lemma 5.1 $\psi_1(B_0)$ is
mapped to a thickened logarithmic spiral in $\sigma_1$, which in turn is mapped diffeomorphically by $\psi_2$ to a logarithmic spiral $S_0$ in $\Sigma_1$.

Now consider $B_1 := R_0 \circ B_0$, this strip lies along $W^u(p_1)$ in $\Sigma_0$. Note that the leaves of $B_0$, defined as the preimage of the lines $\{(y_L, z_L) : y_L = \text{constant}\}$ in $\Sigma_0$, generically intersect the leaves of $B$ (defined similarly, or simply by letting $R_0$ act on the leaves on $B_0$) transversally everywhere.

By the reversibility, the map $(\psi'_2)^{-1} \circ (\phi')^{-1} \circ (\psi'_1)^{-1}$ acts on $B$ to produce a thickened spiral $S_1$ in $\Sigma_1$, which is the $R_1$ image of $S_0$.

Define the leaves of $S_0$ to be the leaves of $B_0$ under the map $\psi_2 \circ \phi \circ \psi_1$. Similarly for the leaves of $S_1$. Now arguments similar to those used in the proof of Lemma 5.2 can be used to show that for $\varepsilon_1, \varepsilon_2$ sufficiently small, any two leaves of $S_0$ and $S_1$ intersect each other in countably infinitely many points, and that generically each of these intersections is transverse. We now define a countable set $M_i$ ($i \in \mathbb{N}$) of consecutive disjoint areas where the two thickened spirals intersect in $\Sigma_1$, such that $M_i$ approaches the centre of the spirals as $i \to \infty$, and $M_i$, $M_{i+1}$ are approximately $\pi$ apart from each other in the angle coordinate.

We subsequently consider the following images of $M_i$, which provide sets of horizontal and vertical strips in $\Sigma_0$:

$$H_i := \psi_1^{-1} \circ \phi^{-1} \circ \psi_2^{-1}(M_i),$$

$$V_i := \psi'_1 \circ \phi' \circ \psi'_2(M_i).$$  \hspace{1cm} (5.24)  

Consider a finite number of the $M_i$, $H_i$ and $V_i$ for $i$ sufficiently large. By the symmetry, we have $F_0(H_i) = V_i$. Also, if we consider $B := B_0 \cap B_1$ as a topological square $\overline{B} = \{(\overline{y}, \overline{z}) \in \mathbb{R}^2 \mid 0 < \overline{y} < 1, 0 < \overline{z} < 1\}$ then $H_i$ and $V_i$ can be considered ‘horizontal’ and ‘vertical’ strips in $\overline{B}$ respectively, in correspondence with the definitions in [21, section 5.2]. We thus obtain by virtue of the existence of the horizontal and vertical strips a topological horseshoe. It remains to establish their hyperbolicity.

In pursuit of hyperbolic horseshoes, we follow the line of argument of [21]. Importantly, in order to obtain symmetric hyperbolic horseshoes, in addition to hypotheses [H1-H4] introduced before, we need to insist on avoiding tangencies between the spirals formed by the traces of the two-dimensional stable and unstable manifolds of $p_0$ and $p_1$ in $\text{Fix}R_1 \subset \Sigma_1$. Indeed, they may become tangent, but due to the symmetry such tangencies cannot be quadratic or of any other even order. As these tangencies are easily perturbed away, they are generically avoided ($C^4$ open and dense). However, in unfoldings new intersections may arise. This is illustrated in Figure 7.

We thus add the hypothesis:

[H5] The first intersection of $W^u(p_1)$ does not have tangencies with the first intersection of $W^s(p_0)$ on $\text{Fix}R_1 \subset \Sigma_1$.  

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Proof We first focus on \( \psi_2 \circ \phi \circ \psi_1(H_i) \). We can write the map as

\[
\psi_2 \circ \phi \circ \psi_1 \left( \begin{array}{c} y \\ z \end{array} \right) = B \left( \begin{array}{c} y_L \left( \frac{\rho}{\mu} \right)^{\rho/\mu} \sin \left( \frac{\pi}{\mu} \ln \left( \frac{z}{\lambda} \right) \right) \\ y_L \left( \frac{\rho}{\mu} \right)^{\rho/\mu} \cos \left( \frac{\pi}{\mu} \ln \left( \frac{z}{\lambda} \right) \right) \end{array} \right) + \ldots ,
\]

(5.26)

where \( B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \). We may write

\[
D\psi_2 \circ \phi \left( \frac{y_L}{z_L} \right) = \left( \begin{array}{c} y_L \left( \frac{\rho}{\mu} \right)^{\rho/\mu} \left( b_1 \sin \left( \frac{\pi}{\mu} \ln \left( \frac{z}{\lambda} \right) \right) + b_2 \cos \left( \frac{\pi}{\mu} \ln \left( \frac{z}{\lambda} \right) \right) \right) \\ y_L \left( \frac{\rho}{\mu} \right)^{\rho/\mu} \left( b_3 \sin \left( \frac{\pi}{\mu} \ln \left( \frac{z}{\lambda} \right) \right) + b_4 \cos \left( \frac{\pi}{\mu} \ln \left( \frac{z}{\lambda} \right) \right) \right) \end{array} \right),
\]

where \( \Phi_1 = \tan^{-1} \left( \frac{b_2}{b_1} \right) \), \( \Phi_2 = \tan^{-1} \left( \frac{b_4}{b_3} \right) \). It may be verified that the angle that \( D\psi_2 \circ \phi \left( \frac{y_L}{z_L} \right) \) (where \( y_L \) is a constant close to \( y^* \)) intersects the line \( z = 0 \) is equal to

\[
\Theta_0 := \tan^{-1} \left( \frac{(b_3 + b_4)^{1/2} \sin(\pi - \Phi_2 + \Phi_3)}{(b_1^2 + b_2^2)^{1/2} \sin(\pi - \Phi_2 + \Phi_3)} \right)
\]

(5.27)

where \( z_L = d \exp \left( \frac{\mu}{\omega}(n\pi - \Phi_2) \right) \), \( n \in \mathbb{N} \), \( \Phi_3 = \tan^{-1} \left( \frac{b_1 \rho - b_3 \omega}{b_1 \omega + b_2 \rho} \right) \) and \( \Phi_4 = \tan^{-1} \left( \frac{b_2 \rho - b_4 \omega}{b_2 \omega + b_3 \rho} \right) \). It may also be shown that the map \( D\psi_2 \circ \phi \) maps lines \( z_L = \) constant to radial lines in the \( (y, z) \) plane.

It follows that \( \Theta_0 \neq 0 \), and we use hypothesis [H5] to assure that \( \Theta_0 \neq \frac{\pi}{2} \) (yielding the transversal intersection of the spiralling traces of the two-dimensional stable and unstable manifolds in \( \text{Fix}(R_1) \subset \Sigma_1 \)). This condition holds generically (open and dense condition on the first derivatives of the return maps), and since we have used \( C^1 \) linearisation, the condition is corresponds to a \( C^k \) open and dense condition on the underlying \( (C^k) \) smooth vector field.

The derivative of the map \( \phi : \sigma_0 \rightarrow \sigma_1 \) is given by

\[
D\phi(y_L, z_L) = y_L \left( \frac{z_L}{\lambda} \right)^{\rho/\mu} \left[ \begin{array}{c} \frac{1}{y_L} \sin \left( \frac{\pi}{\mu} \ln \left( \frac{z}{\lambda} \right) \right) \\ \frac{1}{y_L} \cos \left( \frac{\pi}{\mu} \ln \left( \frac{z}{\lambda} \right) \right) \end{array} \right] \left( \begin{array}{c} \frac{\omega \cos \left( \frac{\pi}{\mu} \ln \left( \frac{z}{\lambda} \right) \right)}{\mu z_L} \\ \frac{\omega \cos \left( \frac{\pi}{\mu} \ln \left( \frac{z}{\lambda} \right) \right)}{\mu z_L} - \omega \sin \left( \frac{\pi}{\mu} \ln \left( \frac{z}{\lambda} \right) \right) \end{array} \right) + \phi_0.
\]

(5.28)
Then the derivative map $D(\psi_2 \circ \phi \circ \psi_1)$ is given by

$$D(\psi_2 \circ \phi \circ \psi_1)(y, z) = \tilde{y}_L \left( \frac{\tilde{z}_L}{d} \right)^{\rho/\mu} B \begin{bmatrix} \frac{1}{y_L} \sin \left( \frac{\omega}{\mu} \ln \left( \frac{\tilde{z}_L}{d} \right) \right) & \frac{\omega \cos \left( \frac{\omega}{\mu} \ln \left( \frac{\tilde{z}_L}{d} \right) \right) + \rho \sin \left( \frac{\omega}{\mu} \ln \left( \frac{\tilde{z}_L}{d} \right) \right)}{\mu \tilde{z}_L} \\ \frac{1}{y_L} \cos \left( \frac{\omega}{\mu} \ln \left( \frac{\tilde{z}_L}{d} \right) \right) & \frac{\rho \cos \left( \frac{\omega}{\mu} \ln \left( \frac{\tilde{z}_L}{d} \right) \right) - \omega \sin \left( \frac{\omega}{\mu} \ln \left( \frac{\tilde{z}_L}{d} \right) \right)}{\mu \tilde{z}_L} \end{bmatrix} A,$$

(5.29)

where $(\tilde{y}_L, \tilde{z}_L) = \psi_1(y, z)$, $B = D \psi_2(\phi \circ \psi_1(y, z))$ and $A = D \psi_1(y, z)$. Equation (5.29) may be rewritten in the form

$$\tilde{z}_L^{(-1+\rho/\mu)} C \begin{bmatrix} \tilde{z}_L \\ 0 \end{bmatrix} \tilde{y}_L = A,$$

(5.30)

where

$$C = d^{-\rho/\mu} B \begin{bmatrix} \sin \left( \frac{\omega}{\mu} \ln \left( \frac{\tilde{z}_L}{d} \right) \right) - \cos \left( \frac{\omega}{\mu} \ln \left( \frac{\tilde{z}_L}{d} \right) \right) \\ \cos \left( \frac{\omega}{\mu} \ln \left( \frac{\tilde{z}_L}{d} \right) \right) + \omega \sin \left( \frac{\omega}{\mu} \ln \left( \frac{\tilde{z}_L}{d} \right) \right) \end{bmatrix} \begin{bmatrix} 1 & \frac{\rho}{\mu} \\ 0 & -\frac{\omega}{\mu} \end{bmatrix}$$

(5.31)

Note that the regions $M_i$, $M_{i+1}$ have the property that their preimages (under $\psi_2 \circ \phi$) have $\tilde{z}_L$ values in $\sigma_0$ (respectively $\tilde{z}_L^1$, $\tilde{z}_L^{i+1}$) that satisfy $(\omega/\mu) \ln \tilde{z}_L^j = j \pi$, where $j \in \mathbb{N}$. Consider the strips $Z_i$ in $\Sigma_0$ which are formed by the preimages of the $M_i$. We shall also denote by $Z_i$ the preimages of $M_i$ in $\sigma_0$ where the meaning is clear.) In these strips the value of the matrix $C$ varies approximately by multiplication by

$$-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$  

We shall denote

$$C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}.$$  

From our previous calculation, we observe that the image of the line $y = -\frac{\omega}{\mu} z$ under $D(\psi_2 \circ \phi \circ \psi_1)$ intersects the line $z = 0$ at an angle $\Theta_0 \neq \frac{\pi}{2}$, in the regions $M_i$, asymptotically as $i \to \infty$. These conditions control the image under $C$ of the least contracting eigenvector of

$$\tilde{z}_L^{(-1+\rho/\mu)} C \begin{bmatrix} \tilde{z}_L \\ 0 \end{bmatrix} \tilde{y}_L = A.$$  

They ensure that this eigenvector is not mapped in the direction of Fix $R$ or Fix $(-R)$. These conditions ensure that $c_2, c_4 \neq 0$. It is also important to note that by the transversality hypothesis [H3], both $a_4, a_3 \neq 0$.

It can be verified that the map $DF_0 = R \circ D(\psi_2 \circ \phi \circ \psi_1)^{-1} \circ R \circ D(\psi_2 \circ \phi \circ \psi_1)$ is given by

$$DF_0 = \frac{1}{\det A \det C \tilde{y}_L \tilde{z}_L} \begin{bmatrix} 2a_3a_4c_2c_4\tilde{y}_L^2 + O(\tilde{z}_L) & 2a_2^2c_2c_4\tilde{y}_L + O(\tilde{z}_L) \\ 2a_2^2c_2c_4\tilde{y}_L + O(\tilde{z}_L) & 2a_3a_4c_2c_4\tilde{y}_L^2 + O(\tilde{z}_L) \end{bmatrix},$$

and that when $a_3, a_4, c_2, c_4 \neq 0$, the eigenvalues are $\lambda_1'(\tilde{z}_L), \lambda_2'(\tilde{z}_L)$, with corresponding eigenvectors $(-a_4/a_3 + O(\tilde{z}_L), 1)$ and $(a_4/a_3 + O(\tilde{z}_L), 1)$. Thus the map $DF_0$ is hyperbolic for $i$ sufficiently large, with eigenvalues tending to zero and infinity respectively as $i \to \infty$. Hence we may construct sector bundles $S^h$ and $S^c$ that satisfy the properties laid out in Proposition 5.7.

By application of Proposition 5.7, we have thus established the existence of countable many horseshoes at the critical parameter value $\lambda = 0$. This implies the existence of a hyperbolic invariant set which is topologically conjugate to a full shift on a countable infinity ($\mathbb{N}$) of symbols (represented by $i$). It is important to note that the closure of all these sets is not uniformly hyperbolic. However, any subset containing a finite number of these horseshoes is uniformly hyperbolic. This completes the proof of Theorem 1.2 E.
We note that the hyperbolic invariant set obtained above is not necessarily minimal: the regions \( S_0 \) and \( S_1 \) may have more intersections in \( \Sigma_1 \) than illustrated in figure 3, cf for instance the situation sketched in the rightmost diagram of Figure 7 where there are additional intersections away from \( \text{Fix} R_1 \).

Finally, we would like point out that our analysis can be carried out also for heteroclinic cycles that are not symmetric, with the 2D and/or 1D heteroclinic connections constituting the cycle not being \( R \)-invariant, yielding analogous conclusions. Notably, symmetry is not the deciding factor in the creation of the horseshoes.

When \( \lambda \) is varied from 0, many of the heteroclinic orbits will be removed in saddle-node-type bifurcations: as long as the unfolding by \( \lambda \) breaks the initial heteroclinic cycle, there remain only finitely many \( n \)-2D heteroclinic orbits for each \( n \).

6 Consequences for the Michelson system

We finally discuss the application of our results on the Hopf-zero bifurcation to the Michelson system (1.1). We recall that the Michelson system has a reversible Hopf-zero bifurcation at \( c = 0 \). We note that the Michelson system is in fact also volume preserving, but - as already mentioned in various places before - our results hold as well for reversible volume preserving vector fields in \( \mathbb{R}^3 \).

We concluded in Theorem 1.2 that generic unfoldings (in the \( C^\infty \) topology) of the Hopf-zero bifurcation in reversible (volume-preserving) vector fields exhibit many heteroclinic cycle bifurcations accumulating to the singularity. Hence we are led to ask whether heteroclinic cycle bifurcations occur in the Michelson system for small \( c \).

The normal form for the Michelson system satisfies the open conditions guaranteeing the Hopf-zero bifurcation where for small \( c \) the system has in normal form an invariant 2-sphere consisting of the coinciding two-dimensional (un)stable manifolds of the two newborn saddle-foci. As this sphere is transverse to \( \text{Fix} R \), at least two symmetric 2D heteroclinic orbits persist under any small perturbation and they thus really exist in the Michelson system for small \( c \). In fact, [1, 40] show that all 1-2D heteroclinic orbits must be symmetric, implying that for sufficiently small \( c \) there are at most two 1-2D symmetric heteroclinic orbits. We thus establish that there are precisely two 1-2D heteroclinic orbits in the Michelson system for small \( c \).

In turn, the above result implies that the invariant sphere arising in the normal form indeed breaks up when no truncation is made. This is also consistent with [1], where it is proved that for most parameter values, the one-dimensional invariant manifolds of the saddle-foci escape to infinity. Moreover, as the Michelson system is analytic, it follows that the 1-2D heteroclinic orbits are locally isolated in phase space. These intersections of two-dimensional stable and unstable manifolds are either transversal (generic) or arise at some tangency of the invariant manifolds that is of some (finite!) odd degree.

If there is a tangency, we cannot carry over all our conclusions and in particular the hyperbolicity condition on the symmetric topological horseshoes fails to hold. However, importantly, the occurrence of an odd symmetric tangency of the traces of the two-dimensional stable and unstable manifolds in \( \Sigma_0 \) implies the nearby existence of transversal (asymmetric) intersections of these manifolds, see Figure 8. Thus, even in the case of an odd order symmetric tangency of the 1-2D heteroclinic orbits, we find asymmetric transversal 2D heteroclinic orbits that we may use as the starting point of our analysis (taking into account our remark at the end of the previous section regarding the fact that asymmetric horseshoes are also found near asymmetric heteroclinic cycles). Analogously, if hypothesis [H5] fails to hold in \( \Sigma_1 \), we still find asymmetric transversal intersections of two-dimensional stable and unstable manifolds in \( \Sigma_1 \).

The subsequent question is whether the Michelson system admits a 1-1D heteroclinic connection for small \( c \). We expect the generic occurrence (in the \( C^\infty \)-topology) of an infinity of small values of \( c \) where such 1-1D heteroclinic orbit exists. However, as we employ a fast oscillating non-analytic flat perturbation to prove this result, it is not so surprising that in fact can be proven that the (analytic) Michelson system does not admit 1-1D heteroclinic orbits [1, 27, 40] (referred to as monotonic 1D heteroclinic orbits in these references). In fact, [27] prove that this result holds for all \( c > 0 \).

In [1] the existence of many 2-1D heteroclinic connections for small \( c \) is proved, and our results
Figure 8: Sketch of an odd order tangency between the traces of the two-dimensional stable and unstable manifolds of $p_0$ and $p_1$ (solid curves) in $\text{Fix}R_0 \subset \Sigma_0$ and the subsequent transversal intersections of other parts of these manifolds which accumulate onto them after having gone around the heteroclinic cycle (dashed curves.)

apply to the heteroclinic cycles formed involving these. In fact, we note from our analysis also that the occurrence of such $n$-1D heteroclinic connections ($N \geq 2$) is unavoidable in any small perturbation of the normal form, whereas in order to obtain many 1-1D heteroclinic connections we had to choose our flat perturbation carefully.

Another one of our hypotheses that remains unverifiable is that the 1D heteroclinic connections pass through $\text{Fix}(R)$ with positive speed. However, the majority of our results are in fact insensitive to such details. It is important that because of the analyticity of the vector field (also in the parameter) the 1D heteroclinics arise at locally isolated values of $\lambda$. The passing and positive speed assumptions are not so important. In fact, the latter assumption is merely necessary to obtain exponential accumulation of bifurcation points in parameter space. Such accumulations are exponential as long as the approach of the 1D manifold to $\text{Fix}(R)$ is not flat in the parameter.

In [1], it is shown that in the Michelson system the 1D manifolds approach each other as $c \to 0$ in a flat manner, without ever coinciding. Thus we may think of the Hopf-zero singularity as the bifurcation point for a 1-1D heteroclinic cycle, which is approached as a flat function of $c$. In fact, [1] prove that the accumulation of 1-homoclinic orbits to the singularity of the Michelson system is polynomial: in their notation $\epsilon \sim (2m)^{-\frac{1}{2}}, \ m \in \mathbb{N}$.

We thus establish the result formulated in Theorem 1.4.

We finalize our discussion of the Michelson system with a discussion of a heteroclinic cycle far away from $c = 0$. Kuramoto & Tsuzuki found that the Michelson systems has a 1D heteroclinic connection when $c = c_{KT} = \alpha \sqrt{2} \approx 0.84952$, with the following explicit expression:

$$x_{KT}(t) = \alpha(-9 \tanh \beta t + 11 \tanh^3 \beta t), \quad (6.1)$$

where $\alpha = 15 \sqrt{\frac{11}{19}}, \beta = \frac{1}{2} \sqrt{\frac{11}{19}}$.

If there is a 2D heteroclinic orbit at this parameter value, we would thus have a heteroclinic cycle. Unfortunately there do not exist any analytical proofs of existence of 2D-heteroclinic orbits at $c = c_{KT}$. However, there is strong numerical evidence that they exist and that they are transverse [30]. Also, McCord [32] has shown that for sufficiently large $c$ there exists a unique 2D heteroclinic connection. The existence of a 2D heteroclinic connection for all $c > 0$ remains an open problem. It seems likely that a computational proof of existence of such a 2D heteroclinic orbit would be tractable, and in this case the results of our analysis would again apply to this parameter range.

By analyticity of the Michelson system, the Kuramoto-Tsuzuki exact solution is locally isolated in
the parameter space. It is not so important to verify that the unfolding is generic (as this only yields the exponential properties of the accumulations in our results). As before, with only the hypothesis on local isolatedness of the heteroclinic cycle, most of the conclusions of Theorem 1.2 hold: many heteroclinic, homoclinic and periodic solutions and many horseshoes near $c = c_{KT}$.

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