Regularity and singularity in solutions of the three-dimensional Navier-Stokes equations

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Abstract

Higher moments of the vorticity field $\Omega_m(t)$ in the form of L^{2m}-norms $(1 \leq m < \infty)$ are used to explore the regularity problem for solutions of the three-dimensional incompressible Navier-Stokes equations on the domain $[0,\,L]^3_{per}$. It is found that the set of quantities

$$D_m(t) = \Omega_m^{\alpha_m}\,, \qquad \alpha_m = \frac{2m}{4m-3}\,,$$

provide a natural scaling in the problem resulting in a bounded set of time averages $\left<D_m\right>_T$ on a finite interval of time $[0,\,T]$. The behaviour of D_{m+1}/D_m is studied on what are called 'good' and 'bad' intervals of $[0,\,T]$ which are interspersed with junction points (neutral) τ_i. For large but finite values of m with large initial data $\big(\Omega_m(0) \leq \varpi_0 O(Gr^4)\big)$, it is found that there is an upper bound

$$\Omega_m \leq c_{av}^2 \varpi_0 Gr^4\,, \qquad \varpi_0 = \nu L^{-2}\,,$$

which is punctured by infinitesimal gaps or windows in the vertical walls between the good/bad intervals through which solutions may escape. While this result is consistent with that of Leray [1] and Scheffer [10], this estimate for Ω_m corresponds to a length scale well below the validity of the Navier-Stokes equations.

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1 Introduction

The challenge that analysts have faced in the last 75 years has been to prove the existence and uniqueness of the three-dimensional Navier-Stokes equations for arbitrarily long times [1, 2, 3, 4, 5, 6]. Its inclusion in the AMS Millenium Clay Prize list [7] has widely advertised the the nature of the problem but the elusiveness of a rigorous proof[1] and the severe resolution difficulties encountered in CFD, even at modest Reynolds numbers, are puzzles that have grown as the years progress.

Nevertheless, there is a long-standing belief in many scientific quarters, on the level of a folk-theorem, that the three-dimensional Navier-Stokes equations 'must' be regular. Mathematicians are more cautious and still take seriously the possibility that singularities may occur, at least in principle. Leray [1] and Scheffer [10] proved that the (potentially) singular set in time has zero half-dimensional Hausdorff measure [11]. The Leray-Scheffer result motivated Caffarelli, Kohn and Nirenberg [12] to introduce the idea of suitable weak solutions to study the singular set in space-time which they concluded has zero one-dimensional Hausdorff measure. Thus, if space-time singularities exist then they must be relatively rare events. These ideas have spawned a growing literature on the subject where more efficient routes to the construction of suitable weak solutions are in evidence [13, 14, 15, 16, 17, 18, 19, 20, 21].

It is worth remarking that the wider issue regarding the formation of singularities has been obscured by the very great difficulty that exists in distinguishing them from rough intermittent data. Intermittency is characterized by violent surges or bursts away from averages in the energy dissipation, resulting in the spiky data that is now recognized as a classic hallmark of turbulence [22, 23, 24, 25, 26]. At least three options are possible:

a) Solutions are always smooth with only mild excursions away from space and time averages;
b) Solutions are intermittent but, despite their apparent spikiness, remain smooth for arbitrarily long times when examined at very small scales;
c) Solutions are intermittent but spikes may be the manifestation of true singularities.

Options b) and c) are impossible to distinguish using known computational methods. The Leray-Scheffer result shows that potential singularities in time must be distributed as no more than points on the time axis, but it contains little other information. Both for analytical and computational reasons it would be desirable to understand the origin of these points and the structure of the solution near to them. The aim of this paper is to address this issue.

In the past generation physicists have used Kolmogorov's theory to examine intermittent events by studying anomalies in the scaling of velocity structure functions. This theory is based on a set of statistical axioms, not directly on the Navier-Stokes equations. Nevertheless, to make a comparison, the intermittent dynamics discussed above would lie deep in the dissipation range of the energy spectrum. Frisch's book [27] and the recent review by Boffetta, Mazzino and Vulpiani [28] contain readable accounts of these ideas.

[1] Cao and Titi [8] and Kobelkov [9] have recently proved the regularity of the primitive equations of the atmosphere and oceans, even though these have been considered by many to be a problem harder than the Navier-Stokes equations. The methods used unfortunately do not appear to successfully transfer to the Navier-Stokes equations.

1.1 General strategy

The main idea of this paper is to use higher moments of the vorticity field $\boldsymbol{\omega}$ instead of derivatives. Scaled by a system volume term L^{-3}, a set of moments with the dimension of a frequency are defined such that for $m \geq 1$

$$\Omega_m(t) = \left\{L^{-3}\int_{\mathcal{V}} |\boldsymbol{\omega}|^{2m}\,dV\right\}^{1/2m} + \varpi_0\,, \tag{1.1}$$

where $\varpi_0 = \nu L^{-2}$ is the basic frequency of the domain of side L. Ω_1 is synonymous with the H_1-norm and sits within the sequence of inequalities

$$\varpi_0 < \Omega_1(t) \leq \Omega_2(t) \leq \ldots \leq \Omega_m(t) \leq \Omega_{m+1}(t) \leq \ldots\,, \tag{1.2}$$

so control from above over Ω_m for any value of $m > 1$ implies control over the H_1-norm which, in turn, controls from above all derivatives of the velocity field [2, 3, 4, 5, 6].

A technical problem lies in how to differentiate the $\Omega_m(t)$ and manipulate them without the existence of strong solutions for arbitrarily large t. This difficulty can be circumvented my restricting estimates to a finite interval of time $[0, T]$ and then pursuing a contradiction proof in the following standard manner. Assume that there exists a *maximal* interval of time $[0, T_{max})$ on which solutions exist and are unique; that is, strong solutions are assumed to exist in this interval. If $[0, T_{max})$ is indeed maximal then $\Omega_1(T_{max}) = \infty$. The ultimate aim of such a calculation would then be to show that $\limsup_{T\to T_{max}}\Omega_m$ is finite for any $m \geq 1$; if this turned out to be the case it would lead to a contradiction because $[0, T_{max})$ would not be maximal. Thus T_{max} must either be zero or infinity: it cannot be zero because it is known that there exists a short interval $[0,\, t_0)$ on which strong solutions exist, so $T_{max} = \infty$.

The results in §2 have been estimated using this strategy. It turns out that there exists a natural scaling within the Navier-Stokes equations which makes the variable

$$D_m(t) = \left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m} \qquad \text{with} \qquad \alpha_m = \frac{2m}{4m-3}\,, \tag{1.3}$$

the most natural to choose. Then Theorem 1 shows that

$$\left<D_m\right>_T \leq c_{av}Gr^2 + O(T^{-1})\,, \tag{1.4}$$

with a uniform constant c_{av}. Two remarks are in order. Firstly it is not difficult to extract an estimate for a set of length scales from (1.4). Defining $\lambda_m^{-2\alpha_m} = \nu^{-\alpha_m}\left<\Omega_m^{\alpha_m}\right>_T$, this shows that

$$\left(L\lambda_m^{-1}\right)^{2\alpha_m} = \left<D_m\right>_T\,, \tag{1.5}$$

and therefore[2]

$$L\lambda_m^{-1} \leq (c_{av}Gr^2)^{1/2\alpha_m} + O(T^{-1})\,. \tag{1.6}$$

The exponent α_m within the definition of D_m appears to be a natural scaling consistent with that of the Sobolev inequalities. This paper suggests that the breaking of this scaling through stretching

[2] Doering and Foias [37] have shown that for Navier-Stokes solutions $Gr \leq c\,Re^2$ which would be valid if solutions were assumed to exist for large enough values of T. In this case the Gr^2-term on the right hand side of (1.4) would be replaced by Re^3 in which case the right hand side of (1.6) would be $Re^{3/2\alpha_m}$. Thus, $L\lambda_1^{-1} \leq c^{1/4}Re^{3/4}$ which is the Kolmogorov estimate. For large m, this becomes significantly larger running to $L\lambda_m^{-1} \leq c\,Re^3$.

between D_{m+1} and D_m may be required to make progress. This is gauged more specifically in Theorem 2 in §2 where it is shown that a finite interval $[0,\,T]$ of the time axis can be potentially broken down into three classes, denoted by *good* and *bad* intervals with set of junction points (or intervals) $\{\tau_i\}$ designated as *neutral*. In §3, it is found that the direction of the inequality is reversed on the good and bad intervals; that is

$$\frac{D_{m+1}}{D_m} \lesseqgtr c_{av} D_m^{-\mu_m} Gr^{p(T)} \qquad \begin{cases} < \text{(good)} \\ = \text{(neutral)} \\ > \text{(bad)} \end{cases} \tag{1.7}$$

In (1.7) $p(T)$ is a T-dependent exponent (> 2) of the Grashof number Gr and μ_m is a parameter in the range $0 < \mu_m < 1$. The universal inequality $\Omega_m \leq \Omega_{m+1}$ ultimately shows that on good and neutral intervals

$$D_m \leq \mathcal{G}_m^{\alpha_m}\,, \tag{1.8}$$

where $\mathcal{G}_m$ is a function of $p(T)$, Gr, α_m and μ_m. The main question lies in the nature of the transition from the good to the bad intervals through the neutral points τ_i. On bad intervals the application of the reverse inequality in (1.7) to the differential inequality for D_m in Proposition 1 results in regions smaller in amplitude than $\mathcal{G}_m$ in which solution trajectories remain bounded by

$$D_m \leq \mathcal{B}_m^{\alpha_m}\,. \tag{1.9}$$

The bad regions are not absorbing : solutions remain inside these regions if they enter inside, but they are not attracted into them if they lie outside. The key point is that for all *finite* values of $m \geq 1$, $\mathcal{B}_m < \mathcal{G}_m$, thereby leaving vertical gaps or windows through which trajectories can potentially escape to infinity – see Figures 1, 2 and 3. **However, while the gap between $\mathcal{G}_m$ and $\mathcal{B}_m$ closes for large m, the limit $m = \infty$ is forbidden and so these windows can only be reduced to infinitesimally small holes which puncture a general upper bound.** This result is consistent with that of Leray [1] and Scheffer [10]. In terms of Ω_m, this punctured bound turns out to be

$$\Omega_m \lesssim c_{av}^2 \varpi_0 Gr^4 \tag{1.10}$$

When converted into a length scale, this estimate shows that regular solutions may go as deep as near nuclear scales (10^{-2} angstroms) and therefore many orders of magnitude below the validity of the Navier-Stokes equations. The conclusion is that unless other unknown controlling mechanisms are shown to exist, the Navier-Stokes equations may formally possess solutions that either become singular or, if they continue to exist, may be unresolvable numerically.

1.2 Notation and functional setting

The setting is the incompressible ($\mathrm{div}\,\boldsymbol{u} = 0$), forced, three-dimensional Navier-Stokes equations for the velocity field $\boldsymbol{u}(\boldsymbol{x},\,t)$

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f}(\boldsymbol{x})\,, \tag{1.11}$$

with the equation for the vorticity expressed as

$$\boldsymbol{\omega}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = \nu\Delta\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} + \mathrm{curl}\boldsymbol{f}\,. \tag{1.12}$$

The properties of the forcing & other definitions are given in Table 1. The domain $\mathcal{V} = [0,\, L]^3$ is taken to be three dimensional and periodic. The forcing function $\boldsymbol{f}(\boldsymbol{x})$ is L^2-bounded and the Grashof number Gr is proportional to $\|\boldsymbol{f}\|_2$: see the paper by Doering and Foias [37] for a discussion of narrow-band forcing [37] : for simplicity the forcing is taken at a single length-scale $\ell = L/2\pi$.

Quantity	Definition	Remarks
Box length	L	
Forcing length scale	ℓ	$\ell = L/2\pi$
Average forcing	$f_{rms}^2 = L^{-3}\|\boldsymbol{f}\|_2^2$	
Narrow-band forcing	$\|\boldsymbol{f}\|_2^2 \approx \ell^{2n}\|\nabla^n\boldsymbol{f}\|_2^2$	$n \geq 1$
Grashof No	$Gr = \ell^3 f_{rms}\nu^{-2}$	
Box frequency	$\varpi_0 = \nu L^{-2}$	
Characteristic velocity	$u_0 = L\varpi_0$	
E-definition	$E(t) = \int_{\mathcal{V}} \|\boldsymbol{u}\|^2\, dV$	Energy
β_m-definition	$\beta_m = m(m+1)$	
α_m-definition	$\alpha_m = \frac{2m}{4m-3}$	
ρ_m-definition	$\rho_m = 2m(4m+1)/3$	

Table 1: *Definitions of the main parameters. The forcing is taken at a single length-scale* $\ell = L/2\pi$.

Now define

$$J_m(t) = \int_{\mathcal{V}} |\boldsymbol{\omega}|^{2m} dV\,, \tag{1.13}$$

where the frequencies Ω_m are given by

$$\Omega_m(t) = (L^{-3}J_m)^{1/2m} + \varpi_0\,. \tag{1.14}$$

The term ϖ_0 in (1.13) provides a lower bound for Ω_m. Indeed it is easy to prove that

$$\varpi_0 \leq \Omega_1(t) \leq \Omega_2(t) \leq \ldots \leq \Omega_m(t) \leq \Omega_{m+1}(t) \leq \ldots \tag{1.15}$$

The symbol $\langle\,\cdot\,\rangle_T$ denotes the time average up to time T

$$\langle g(\cdot)\rangle_T = \limsup_{g(0)} \frac{1}{T}\int_0^T g(\tau)\, d\tau\,. \tag{1.16}$$

2 Some properties of the $\Omega_m(t)$

2.1 A differential inequality and a time average

This subsection firstly contains a result concerning the differential inequalities that govern the set of frequencies $\Omega_m(t)$. Secondly it contains a result that is an estimate for an upper bound on a set of time averages over the interval $[0,\, T]$. Finally it contains a result on the nature of exponential bounds on $[0,\, T]$. All of the proofs, which lie in Appendices A, B and C, are based on the contradiction strategy explained in §1.1. Firstly we define

$$D_m = (\varpi_0^{-1}\Omega_m)^{\alpha_m} \qquad\qquad \alpha_m = \frac{2m}{4m-3}\,. \tag{2.1}$$

Proposition 1 *On* $[O,T]$*, for* $1 \leq m < \infty$*,* $n = \frac{1}{2}(m+1)$ *and* $Gr \geq 1$*, the* D_m *satisfy*

$$(\varpi_0\alpha_m)^{-1}\dot{D}_m \leq D_m\left\{-\frac{1}{c_{1,m}}\left(\frac{D_{m+1}}{D_m}\right)^{\rho_m}D_m^2 + c_{2,m}D_n^2 + c_{3,m}Gr\right\}, \qquad (2.2)$$

where $\rho_m = 2m(4m+1)/3$*. For the unforced case the last term on the right hand side of (2.2) is proportional to* $c_{3,m}$*.*

Remark : Note the strict inequality $m < \infty$: the Riesz transform used in the proof in Appendix A requires the introduction of higher derivatives when $m = \infty$.

Theorem 1 : *For* $1 \leq m \leq \infty$ *and* $Gr \geq 1$

$$\left\langle D_m\right\rangle_T \leq c_{av}\left(Gr^2 + \frac{L^{-5}E_0}{\varpi_0^3 T}\right), \qquad (2.3)$$

where $E_0 = E(0)$ *is the initial value of the energy. For the unforced case, the estimate is*

$$\left\langle D_m\right\rangle_T \leq c\,\frac{L^{-5}E_0}{\varpi_0^3 T}\,. \qquad (2.4)$$

Remark : (2.3) can also be expressed as

$$\left\langle D_m\right\rangle_T \leq c_{av}Gr^p\,, \qquad (2.5)$$

where C is a uniform constant. The m-independent exponent $p(T,\,E_0, Gr)$ written as

$$p(T,\,E_0, Gr) = 2 + \ln\left\{1 + \frac{L^{-5}E_0}{\varpi_0^3 T}Gr^{-2}\right\}(\ln Gr)^{-1}\,. \qquad (2.6)$$

3 Trajectories on good, bad and neutral intervals

3.1 The ratio D_{m+1}/D_m

Given the result in Proposition 1, understanding the behaviour of the ratio D_m/D_{m+1} is an important step.

Theorem 2 *For the parameters* $\mu_m = \mu_m(T,\,p,\,Gr)$ *with values in the range* $0 < \mu_m < 1$*, the ratio* D_m/D_{m+1} *obeys the inequality*

$$\left\langle\left[\frac{D_m}{D_{m+1}}\right]^{(1-\mu_m)/\mu_m} - \left[c_{av}^{-1}Gr^{-p(T)}D_m^{\mu_m}\right]^{(1-\mu_m)/\mu_m}\right\rangle_T \geq 0\,. \qquad (3.1)$$

Remark 1 : The proof lies in Appendix C and is dependent on the result of Theorem 1.

Remark 2 : Theorem 2 implies that while there must be intervals where the integrand is positive, there could also be intervals where it is negative. While it tells us nothing about the interval size or distribution it is clear that these are T-dependent.

Formally the theorem leads to the conclusion that there exists at least one **good interval of time within** $[0,\, T]$ **on which:**

$$\frac{D_m}{D_{m+1}} > \left[c_{av}Gr^{p(T)}\right]^{-1} D_m^{\mu_m}\,, \tag{3.2}$$

while there potentially exist **bad intervals of time** on which

$$\frac{D_m}{D_{m+1}} < \left[c_{av}Gr^{p(T)}\right]^{-1} D_m^{\mu_m}\,. \tag{3.3}$$

Neutral points or intervals represented[3] by the zeros of the integrand in (3.1) lying at

$$\tau_i = \tau_i(\mu_m,\, p(T),\, Gr)\,. \tag{3.4}$$

In terms of Ω_{m+1} and Ω_m (3.2) and (3.3) become

$$\frac{\Omega_{m+1}}{\Omega_m} \quad \lesseqgtr \quad \left(\mathcal{G}_m\varpi_0\Omega_m^{-1}\right)^{\gamma_m} \qquad \begin{cases} \textit{good} \\ \textit{neutral} \\ \textit{bad} \end{cases} \tag{3.5}$$

where $\mathcal{G}_m$ and γ_m are defined by

$$\mathcal{G}_m = \left[c_{av}Gr^{p(T)}\right]^{1/\left(\alpha_m\tilde{\mu}_m\right)}\,, \tag{3.6}$$

$$\gamma_m\alpha_{m+1} = \alpha_m\tilde{\mu}_m\,, \tag{3.7}$$

$$\tilde{\mu}_m = \mu_m - \frac{3}{m(4m+1)}\,. \tag{3.8}$$

The positivity of γ_m requires that μ_m be bounded away from zero such that

$$\frac{3}{m(4m+1)} < \mu_m < 1\,. \tag{3.9}$$

Because $\Omega_{m+1} \geq \Omega_m$, (3.5) shows that on **good and neutral intervals**

$$D_{m,\,good} \leq \mathcal{G}_m^{\alpha_m} \qquad\qquad 1 \leq m \leq \infty\,. \tag{3.10}$$

Now we turn to the **bad intervals**: consider (3.3) in (2.2), in which case ($\Omega_n \leq \Omega_m$)

$$\left(\varpi_0\alpha_m\right)^{-1}\dot{D}_m \leq D_m\left\{-\frac{1}{c_{1,m}}\left(c_{av}Gr^{p(T)}D_m^{-\mu_m}\right)^{\rho_m}D_m^2 + c_{2,m}D_m^{2\alpha_n/\alpha_m} + c_{3,m}Gr\right\}\,, \tag{3.11}$$

where $\rho_m = 2m(4m+1)/3$ but $m = \infty$ is forbidden. The range of validity of μ_m in (3.9) can be re-written as $\rho_m > \mu_m\rho_m > 2$. Thus $\dot{D}_m \leq 0$ if, at the time of entry τ_i into a bad interval

$$\left(c_{av}Gr^{p(T)}D_{m,bad}(\tau_i)^{-\mu_m}\right)^{\rho_m}D_m^2 \geq c_{1,m}c_{2,m}D_{m,bad}(\tau_i)^{2\alpha_n/\alpha_m} + c_{3,m}Gr\,. \tag{3.12}$$

Given that $\rho_m\mu_m > 2$ and $\alpha_n \geq \alpha_m$, the first term on the right hand side of (3.12) is dominant. Using the lower bound $D_m \geq 1$ it is found that

$$D_{m,\,bad}(\tau_i) \leq \mathcal{B}_m^{\alpha_m}\,, \tag{3.13}$$

where

$$\mathcal{B}_m = \left\{\frac{1}{c_{1,m}c_{2,m}}\left[c_{av}Gr^{p(T)}\right]^{\rho_m} - \frac{c_{3,m}}{c_{1,m}c_{2,m}}Gr\right\}^{1/a_m}\,, \tag{3.14}$$

$$a_m = 2(\alpha_n - \alpha_m) + \alpha_m\rho_m\mu_m \qquad\qquad b_m = \alpha_m\rho_m\mu_m\,. \tag{3.15}$$

[3] There is no information on how the τ_i are distributed.

3.2 How large are $\mathcal{G}_1^2$ and $\mathcal{B}_1^2$?

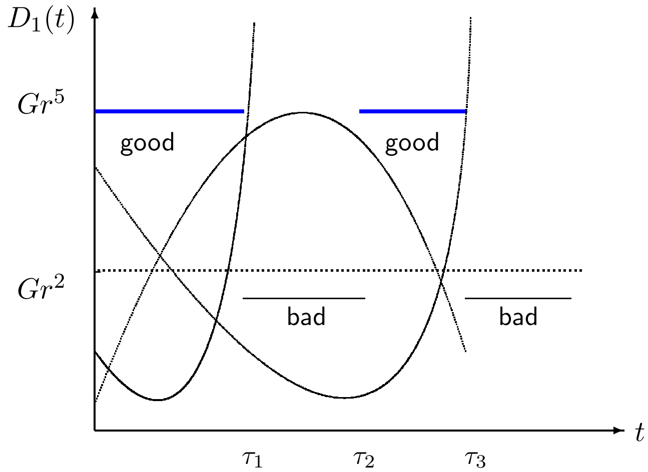

Figure 1 : *From a variety of initial conditions for $m = 1$ the cartoon above shows how solutions may potentially escape at or near neutral points $t = \tau_1$ or a later value $t = \tau_3$, or even return at $t = \tau_2$. However, all must satisfy the bound on the time-average.*

For $m = 1$ we have $b_1/a_1 = 1$ and $\rho_1 = 10/3$; the difference in the sizes of $\mathcal{G}_1$ and $\mathcal{B}_1$ lies in the upper bounds on μ_1 and on $\tilde{\mu}_1$. The latter has been defined in (3.8)

$$\mu_1 < 1\,, \qquad\qquad \tilde{\mu}_1 < 1 - 3/5 = 2/5\,. \tag{3.16}$$

From (3.6) and (3.10) we have

$$D_{1,\,good} \leq \left(c_{av}Gr^p\right)^{1/\tilde{\mu}_1} \tag{3.17}$$

which, on minimization of the right hand side, gives

$$D_{1,\,good} \leq \left(c_{av}Gr^2\right)^{5/2} = c_{av}^{5/2}Gr^5\,. \tag{3.18}$$

The equivalent estimate for $D_{1,\,bad}$ is

$$D_{1,\,bad} \leq \frac{c_{av}}{(c_{1,1}c_{2,1})^{3/10}}Gr^2 - O(Gr^{3/10})\,. \tag{3.19}$$

It is useful to re-work these estimates in terms of a point-wise inverse[4] length-scale $\eta_1^{-4} = \nu^{-3}\epsilon$ with a point-wise energy dissipation rate $\epsilon = \nu\Omega_1^2 = \nu\varpi_0^2 D_1$. The result,

$$L\eta_1^{-1} \leq c_{av}^{1/4}Gr^{1/2} \tag{3.20}$$

is shown in Figure 3 where the constant on the bad estimate is slightly smaller.

[4] The context of this is the estimate for the inverse length $L\lambda_m^{-1} \leq c_{av}^{1/4}Gr^{1/2}$ of §1.

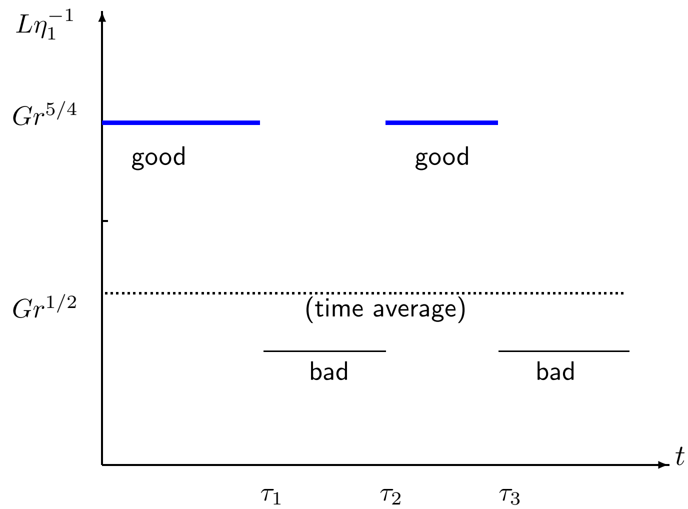

Figure 2: *Bounds on $L\eta_1^{-1}$: notice the large size of the gaps between the good and bad intervals. Based on the constants, the upper bound on the time average is larger than that on the bad intervals.*

3.3 How large are $\mathcal{G}_m^{\alpha_m}$ and $\mathcal{B}_m^{\alpha_m}$ for large m?

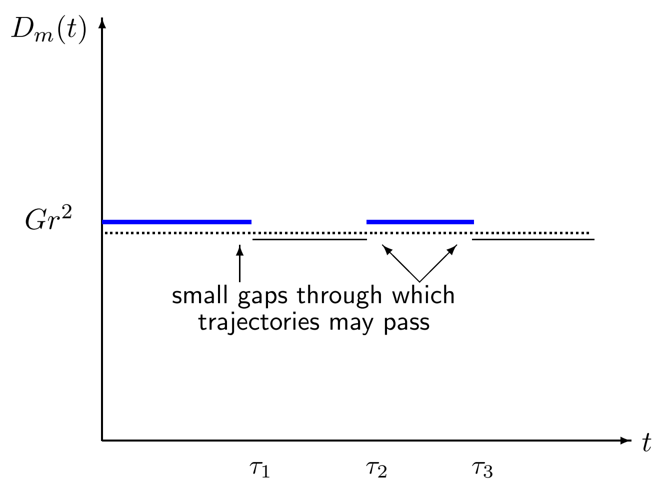

Figure 3: *For large m, the gap between $\mathcal{G}_m^{\alpha_m}$ and $\mathcal{B}_m^{\alpha_m}$ is infinitesimally small but the limit $m = \infty$ is forbidden. The upper bound on the time-average is the horizontal line of dots. At τ_1 and τ_3 a solution must enter the corresponding bad interval within the upper bound to remain inside.*

From the definitions of (3.6) and (3.14) and the fact that $\tilde{\mu}_m < \mu_m$, it is clear that $\mathcal{G}_m^{\alpha_m} - \mathcal{B}_m^{\alpha_m} > 0$, keeping in mind that the limit $m = \infty$ is forbidden. Firstly the $c_{i,m}$ are polynomial in m and $\rho_m \sim O(m^2)$ for large m. Therefore

$$\Big(c_{1,m}c_{2,m}\Big)^{-1/\rho_m} \nearrow 1\,, \qquad b_m/a_m \nearrow 1\,, \qquad \text{and} \qquad \mu_m^{-1} \nearrow \tilde{\mu}_m^{-1}\,. \tag{3.21}$$

Hence, for large m

$$\mathcal{G}_m^{\alpha_m} - \mathcal{B}_m^{\alpha_m} \searrow 0\,. \tag{3.22}$$

Specifically for D_m, for very large m, the upper bounds on μ_m and $\tilde{\mu}_m$ can now be taken arbitrarily close to unity provided that $\mu_m < 1$ and $\tilde{\mu}_m < 1$. From (3.6), minimization of the right hand side gives

$$D_{m,\,good} \leq c_{av} Gr^2 \quad (\searrow)\,. \tag{3.23}$$

The equivalent estimate for $D_{1,bad}$ is

$$D_{m,\,bad} \leq \frac{c_{av}}{\left(c_{m,1}c_{m,2}\right)^{1/\rho_m}} Gr^2 \nearrow c_{av} Gr^2\,. \tag{3.24}$$

4 Conclusion: what are the length scales corresponding to the upper bounds?

The key feature of this paper is the closure of the gaps between the good/bad intervals as $m\to\infty$ but with the actual limit $m=\infty$ forbidden. The origin of this lies in Proposition 1 in the use of the inequality ($p=\frac{1}{2}(m+1)$)

$$\|\nabla\boldsymbol{u}\|_p \quad\leq\quad c_p\|\boldsymbol{\omega}\|_p \qquad\qquad p\in(1,\,\infty)\,, \tag{4.1}$$

whereas, when $m=\infty$

$$\|\nabla\boldsymbol{u}\|_\infty \quad\leq\quad c\,\|\boldsymbol{\omega}\|_\infty\left(1+\ln H_3\right)\,. \tag{4.2}$$

(4.1) has its origin in a double Riesz transform while (4.2) arises from the work of Beale, Kato and Majda [38] on the three-dimensional Euler equations – see also Kato and Ponce [39]. The $\ln H_3$ term in (4.2) prevents the closure of the set of inequalities for D_m. While the $m=\infty$ limit is valid for good intervals, it is not valid for the bad because of the necessary use of Proposition 1. Thus it is not possible to completely close the gaps between the two sets of intervals, although they can become arbitrarily small. This allows for the possibility of the escape of trajectories. The m-dependence of the τ_i means that the junction points can, in principle, lie at different places on the time-axis as m varies. If the gaps fall randomly with respect to m then a trajectory would have to thread its way through these to escape to infinity. However, an unknown but subtle alignment of the gaps cannot entirely be ruled out.

The closeness of the upper-bounds on both the time average and on point-wise values of D_m ($m\gg 1$) away from the gaps, poses the question whether there exists dynamics that naturally lie either close to these bounds or even fulfill them. The point-wise energy dissipation rate per unit volume is

$$\varepsilon = \nu\Omega_1^2 \leq \nu^3 L^{-4} D_m^{2/\alpha_m} \to \nu^3 L^{-4} c_{av}^4 Gr^8\,. \tag{4.3}$$

Defining a *local* Kolmogorov length as $\lambda_{k,loc} = \left(\varepsilon/\nu^3\right)^{1/4}$ we obtain

$$L\lambda_{k,loc}^{-1} \leq c_{av} Gr^2\,, \tag{4.4}$$

which is consistent with the estimate in (1.5) for large m. If the solution survives for large enough T to make sense of a Reynolds number based on $U_0^2 = L^{-3}\left\langle\|\boldsymbol{u}\|_2^2\right\rangle_T$, then the Doering-Foias result for

Navier-Stokes solutions [37], $Gr \leq c\,Re^2$, can be invoked to give an estimate for a *local* Kolmorgorov scale[5]

$$L\lambda_{k,loc}^{-1} \leq c\,Re^3\,. \tag{4.5}$$

In the atmosphere, for instance, this length-scale would be of $O(10^{-12})$ metres – about 10^{-2} angtroms – which is about the scale of the nucleus (!) and is thus outside the validity of the Navier-Stokes equations. *Because the bounds on the good and bad intervals are very close to the time average then solutions could, in principle spend long periods of time close to this bound and remain regular, yet such a scale is not only unreachable computationally but is outside the validity of the NS equations. Thus, a singularity is not necessary to produce unresolvable solutions.*

Acknowledgements: I would like to express very warm thanks to Claude Bardos, Matania Benartzi, Toti Daskalopoulos, Darryl Holm, Roger Lewandowski, Gustavo Ponce, James Robinson and Edriss Titi for discussions on this topic.

A Proof of Proposition 1

Consider the time derivative of J_m defined in (1.13)

$$\frac{1}{2m}\dot{J}_m = \int_{\mathcal{V}} |\boldsymbol{\omega}|^{2(m-1)}\boldsymbol{\omega}\cdot\{\nu\Delta\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} + \mathrm{curl}\boldsymbol{f}\}\,dV\,. \tag{A.1}$$

Bounds on each of the three constituent parts of (A.1) are dealt with in turn, culminating in a differential inequality for J_m. In what follows, c_m is a generic m-dependent constant.

1) The Laplacian term: Let $\phi = \omega^2 = \boldsymbol{\omega}\cdot\boldsymbol{\omega}$. Then

$$\begin{aligned}\int_{\mathcal{V}} |\boldsymbol{\omega}|^{2(m-1)}\boldsymbol{\omega}\cdot\Delta\boldsymbol{\omega}\,dV &= \int_{\mathcal{V}} \phi^{m-1}\boldsymbol{\omega}\cdot\Delta\boldsymbol{\omega}\,dV\\ &= \int_{\mathcal{V}} \phi^{m-1}\{\Delta(\tfrac{1}{2}\phi) - |\nabla\boldsymbol{\omega}|^2\}\,dV\\ &\leq \int_{\mathcal{V}} \phi^{m-1}\Delta(\tfrac{1}{2}\phi)\,dV\,.\end{aligned} \tag{A.2}$$

Using the fact that $\Delta(\phi^m) = m\{(m-1)\phi^{m-2}|\nabla\phi|^2 + \phi^{m-1}\Delta\phi\}$ we obtain

$$\begin{aligned}\int_{\mathcal{V}} |\boldsymbol{\omega}|^{2(m-1)}\boldsymbol{\omega}\cdot\Delta\boldsymbol{\omega}\,dV &\leq -\tfrac{1}{2}(m-1)\int_{\mathcal{V}}\phi^{m-2}|\nabla\phi|^2\,dV + \frac{1}{2m}\int_{\mathcal{V}}\Delta(\phi^m)\,dV\\ &= -\frac{2(m-1)}{m^2}\int_{\mathcal{V}}|\nabla\phi^{\frac{1}{2}m}|^2\,dV\\ &= -\frac{2(m-1)}{m^2}\int_{\mathcal{V}}|\nabla(\omega^m)|^2\,dV\,.\end{aligned} \tag{A.3}$$

Thus we have

$$\int_{\mathcal{V}} |\boldsymbol{\omega}|^{2(m-1)}\boldsymbol{\omega}\cdot\Delta\boldsymbol{\omega}\,dV \leq \begin{cases} -\int_{\mathcal{V}}|\nabla\boldsymbol{\omega}|^2]\,dV & m=1\,,\\ -\frac{2}{\tilde{c}_{1,m}}\int_{\mathcal{V}}|\nabla A_m|^2\,dV & m\geq 2\,.\end{cases} \tag{A.4}$$

[5] The correspondence is that Gr^2 is replaced by Re^3.

where

$$A_m = \omega^m \qquad\qquad \tilde{c}_{1,m} = \frac{m^2}{m-1}\,, \tag{A.5}$$

where there is equality for $m=1$. The negativity of the right hand side of (A.4) is important. Both $\|\nabla A_m\|_2$ and $\|A_m\|_2$ will appear later in the proof.

2) The nonlinear term in (A.1): The second term in (A.1) is

$$\begin{aligned}\left|\int_{\mathcal{V}}|\boldsymbol{\omega}|^{2(m-1)}\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}\,dV\right| &\leq c_m\left(\int_{\mathcal{V}}|\boldsymbol{\omega}|^{2(m+1)}\,dV\right)^{\frac{m}{m+1}}\left(\int_{\mathcal{V}}|\nabla\boldsymbol{u}|^{m+1}\,dV\right)^{\frac{1}{m+1}}\\ &\leq c_m\left(\int_{\mathcal{V}}|\boldsymbol{\omega}|^{2(m+1)}\,dV\right)^{\frac{m}{m+1}}\left(\int_{\mathcal{V}}|\boldsymbol{\omega}|^{m+1}\,dV\right)^{\frac{1}{m+1}}\end{aligned} \tag{A.6}$$

where the inequality $\|\nabla\boldsymbol{u}\|_p \leq c_p\|\boldsymbol{\omega}\|_p$ for $p\in(1,\,\infty)$ has been used[6]: this can be proved in the following way: write $\boldsymbol{u} = \text{curl}(-\Delta)^{-1}\boldsymbol{\omega}$. Therefore $u_{i,j} = R_jR_i\,\omega_i$ where R_i is a Riesz transform.

Together with (A.2) this makes (A.1) into

$$\frac{1}{2m}\dot{J}_m \leq -\frac{\nu}{\tilde{c}_{1,m}}\int_{\mathcal{V}}|\nabla(\omega^m)|^2\,dV + c_m J_{m+1}^{\frac{m}{m+1}}J_{\frac{1}{2}(m+1)}^{\frac{1}{m+1}} + \int_{\mathcal{V}}|\boldsymbol{\omega}|^{2(m-1)}\boldsymbol{\omega}\cdot\text{curl}\boldsymbol{f}\,dV\,. \tag{A.7}$$

3) The forcing term in (A.1): Now we use the narrow-band property of the forcing (see the Table in §1.2) to estimate the last term in (A.7)

$$\begin{aligned}\int_{\mathcal{V}}|\boldsymbol{\omega}|^{2(m-1)}\boldsymbol{\omega}\cdot\text{curl}\boldsymbol{f}\,dV &= \int_{\mathcal{V}}|\boldsymbol{\omega}|^{2(m-1)}\boldsymbol{\omega}\cdot\text{curl}\,\boldsymbol{f}\,dV\\ &\leq \left(\int_{\mathcal{V}}|\omega|^{2m}dV\right)^{(2m-1)/2m}\left(\int_{\mathcal{V}}|\nabla\boldsymbol{f}|^{2m}\,dV\right)^{1/2m}.\end{aligned} \tag{A.8}$$

However, by going up to at least 3-derivatives in a Sobolev inequality it can easily be shown that $\|\nabla\boldsymbol{f}\|_{2m} \leq c\,\|\boldsymbol{f}\|_2 L^{\frac{3-5m}{2m}}$, because of the narrow-band property. (A.8) becomes

$$\begin{aligned}\left|\int_{\mathcal{V}}|\boldsymbol{\omega}|^{2(m-1)}\boldsymbol{\omega}\cdot\text{curl}\boldsymbol{f}\,dV\right| &\leq c\,\left(L^3\Omega_m^{2m}\right)^{\frac{2m-1}{2m}}\|\boldsymbol{f}\|_2 L^{\frac{3-5m}{2m}}\\ &\leq c\,\Omega_m^{2m-1}f_{rms}L^2\\ &\leq c\,\Omega_m^{2m-1}L^3\varpi_0^2 Gr\end{aligned} \tag{A.9}$$

4) A differential inequality for J_m: Recalling that $A_m = \omega^m$

$$J_{m+1} = \int_{\mathcal{V}}|\boldsymbol{\omega}|^{2(m+1)}\,dV = \int_{\mathcal{V}}|A_m|^{2(m+1)/m}\,dV = \|A_m\|_{2(m+1)/m}^{2(m+1)/m}\,. \tag{A.10}$$

A Gagliardo-Nirenberg inequality yields

$$\|A_m\|_{\frac{2(m+1)}{m}} \leq c_m\,\|\nabla A_m\|_2^{3/2(m+1)}\|A_m\|_2^{(2m-1)/2(m+1)} + L^{-\frac{3}{2(m+1)}}\|A_m\|_2\,, \tag{A.11}$$

[6] I am grateful to G. Ponce for pointing this result out to me. Note that the $m=\infty$ case is forbidden because an extra $\log H_3$-term is needed [38, 39]. It is this forbidden limit that ultimately prevents the closure of the gaps in the figures in §3, which allows trajectories to escape.

which means that

$$J_{m+1} \le c_m \left\{ \left(\int_{\mathcal{V}} |\nabla(\omega^m)|^2\, dV \right)^{3/2m} J_m^{(2m-1)/2m} + L^{-3/m} J_m^{\frac{m+1}{m}} \right\}. \tag{A.12}$$

With $\beta_m = m(m+1)$, (A.12) can be used to form Ω_{m+1}

$$\begin{aligned} \Omega_{m+1} = \left(L^{-3}J_{m+1}\right)^{1/2(m+1)} + \varpi_0 \quad &\le \quad c_m \left(L^{-1}\int_{\mathcal{V}} |\nabla(\omega^m)|^2\, dV + L^{-3}J_m + \varpi_0^{2m} \right)^{3/4\beta_m} \\ &\times \quad \left[\left(L^{-3}J_m\right)^{1/2m} + \varpi_0 \right]^{(2m-1)/2(m+1)} \end{aligned} \tag{A.13}$$

which converts to

$$c_m \left(L^{-1}\int_{\mathcal{V}} |\nabla(\omega^m)|^2\, dV + L^{-3}J_m + \varpi_0^{2m} \right) \ge \left(\frac{\Omega_{m+1}}{\Omega_m} \right)^{4\beta_m/3} \Omega_m^{2m}. \tag{A.14}$$

This motivates us to re-write (A.7) as

$$\begin{aligned} \frac{1}{2m}(L^{-3}\dot{J}_m) \quad &\le \quad -\frac{\varpi_0}{\tilde{c}_{1,m}} \left(L^{-1}\int_{\mathcal{V}} |\nabla(\omega^m)|^2\, dV + L^{-3}J_m + \varpi_0^{2m} \right) \\ &+ \quad c_{2,m}(L^{-3}J_{m+1})^{\frac{m}{m+1}} \left(L^{-3}J_{\frac{1}{2}(m+1)} \right)^{\frac{1}{m+1}} \\ &+ \quad c_{3,m}\varpi_0 L^{-3}J_m + c_{4,3}\varpi_0^{2m+1} + c_{5,m}\varpi_0^2\Omega_m^{2m-1}Gr\,. \end{aligned} \tag{A.15}$$

Converting the J_m into Ω_m and using $Gr \ge 1$

$$\dot{\Omega}_m \quad \le \quad \Omega_m \left\{ -\frac{\varpi_0}{c_{4,m}} \left(\frac{\Omega_{m+1}}{\Omega_m} \right)^{4m(m+1)/3} + c_{5,m} \left(\frac{\Omega_{m+1}}{\Omega_m} \right)^{2m} \Omega_{\frac{1}{2}(m+1)} + c_{6,m}\varpi_0 Gr \right\} \tag{A.16}$$

Using a Hölder inequality on the central term on the right hand side (A.16) finally becomes

$$\dot{\Omega}_m \le \Omega_m \left\{ -\frac{\varpi_0}{c_{1,m}} \left(\frac{\Omega_{m+1}}{\Omega_m} \right)^{\frac{4\beta_m}{3}} + c_{2,m}\varpi_0^{-\frac{3}{2m-1}} \Omega_{\frac{1}{2}(m+1)}^{\frac{2(m+1)}{2m-1}} + c_3\varpi_0 Gr \right\}. \tag{A.17}$$

With no forcing the final term in (A.17) is proportional to ϖ_0^2. Converting to the dimensionless quantity $D_m = \left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m}$ already defined in (2.1) with $\alpha_m = 2m/(4m-3)$, finally gives

$$(\varpi_0\alpha_m)^{-1}\dot{D}_m \le D_m \left\{ -\frac{1}{c_{1,m}} \left(\frac{D_{m+1}}{D_m} \right)^{2m(4m+1)/3} D_m^2 + c_{2,m}D_n^2 + c_{3,m}Gr \right\} \tag{A.18}$$

with $n = \frac{1}{2}(m+1)$. ■

B Proof of Theorem 1

There exists a result of Foias, Guillopé and Temam [32], which uses higher derivatives. Define H_n for $n \ge 1$

$$H_n = \int_{\mathcal{V}} |\nabla^n \boldsymbol{u}|^2\, dV\,, \tag{B.1}$$

together with an integration of Leray's energy inequality

$$\varpi_0^{-2}L^{-3}\left<H_1\right>_T = \left<D_1\right>_T \leq Gr^2 + \frac{L\nu^{-3}E_0}{T}\,. \tag{B.2}$$

Then the result of Foias, Guillopé and Temam [32] for $n \geq 3$ is

$$\left<H_n^{\frac{1}{2n-1}}\right>_T \leq c_n \nu^{\frac{2}{2n-1}} L^{-1}\left[Gr^2 + \frac{L\nu^{-3}}{T}E_0\right]\,, \tag{B.3}$$

where $E_0 = E(0) = H_0(0)$ is the initial energy. In the unforced case

$$\left<H_n^{\frac{1}{2n-1}}\right>_T \leq c_n \nu^{-\frac{6n-5}{2n-1}}\frac{E_0}{T}\,. \tag{B.4}$$

A Sobolev inequality gives

$$\|\boldsymbol{\omega}\|_{2m} \leq c\,\|\nabla^2\boldsymbol{\omega}\|_2^a\|\boldsymbol{\omega}\|_2^{1-a} \tag{B.5}$$

where $a = 3(m-1)/4m$ for $m \geq 1$. Moreover, the constant c can be taken as finite for each finite m because the $m = \infty$ case it is a bounded. Thus, taking $n = 3$ in (B.3), which fixes the constant c_n, we have

$$\begin{aligned}\left<\|\boldsymbol{\omega}\|_{2m}^{\frac{2m}{4m-3}}\right>_T &\leq c\left<\left(H_3^{1/5}\right)^{\frac{15(m-1)}{4(4m-3)}} H_1^{\frac{m+3}{4(4m-3)}}\right>_T \\ &\leq c\left<\left(H_3^{1/5}\right)\right>_T^{\frac{15(m-1)}{4(4m-3)}}\left<H_1\right>_T^{\frac{m+3}{4(4m-3)}}\,. \end{aligned}\tag{B.6}$$

Using (B.2) and (B.4) this gives

$$\left<\|\boldsymbol{\omega}\|_{2m}^{\frac{2m}{4m-3}}\right>_T \leq c_{av}\nu^{\frac{2m}{4m-3}}\left(L^{-1}Gr^2 + \frac{\nu^{-3}}{T}E_0\right)\,, \tag{B.7}$$

and thus the final result with an m-independent constant. In the unforced case

$$\left<\|\boldsymbol{\omega}\|_{2m}^{\frac{2m}{4m-3}}\right>_T \leq c\,\varpi_0^{\alpha_m}\left(\frac{L^{-5}E_0}{T\varpi_0^3}\right)L^{3\alpha_m/2m}\,. \tag{B.8}$$

There is also a way of reproducing the Gr^2-estimate from Proposition 1 but with worse constants. Based on $\Omega_n^{m+1} \leq \Omega_m^m\,\Omega_1$ for $n = \frac{1}{2}(m+1)$, the relation in terms of the D_n and D_m is

$$D_n^2 \leq D_m^{\frac{4m-3}{2m-1}} D_1^{\frac{1}{2m-1}}\,. \tag{B.9}$$

Inequality (A.18) is now divided by $D_m^{2-\delta}$ where $\delta \geq \frac{1}{(2m-1)}$. Noting that $D_m \geq 1$ the D_n^2-term is handled as follows

$$\begin{aligned}\left<D_n^2 D_m^{\delta-2}\right>_T &\leq \left<\left(D_m^\delta\right)^{\frac{(2m-1)\delta-1}{(2m-1)\delta}}\left(D_1^\delta\right)^{\frac{1}{(2m-1)\delta}}\right>_T \\ &\leq \left(\frac{(2m-1)\delta-1}{(2m-1)\delta}\right)\left<D_m^\delta\right>_T + \frac{1}{(2m-1)\delta}\left<D_1^\delta\right>_T\,. \end{aligned}\tag{B.10}$$

It follows that

$$\left\langle\left(\frac{D_{m+1}}{D_m}\right)^{2m(4m+1)/3} D_m^\delta\right\rangle \;\leq\; c_{4,m}\left\langle D_m^\delta\right\rangle + c_{5,m}\left\langle D_1^\delta\right\rangle + c_{6,m}Gr + O(T^{-1}) \qquad \text{(B.11)}$$

where the coefficients from the Hölder inequality have been absorbed into the constants. Define $\Delta_m = 2m(4m+1)/3$, and consider

$$\begin{aligned}\left\langle D_{m+1}^\delta\right\rangle &= \left\langle\left[\left(\frac{D_{m+1}}{D_m}\right)^{\Delta_m} D_m^\delta\right]^{\delta/\Delta_m} (D_m^\delta)^{\frac{\Delta_m-\delta}{\Delta_m}}\right\rangle\\ &\leq \frac{\delta}{\Delta_m}\left\langle\left(\frac{D_{m+1}}{D_m}\right)^{\Delta_m} D_m^\delta\right\rangle + \left(\frac{\Delta_m-\delta}{\Delta_m}\right)\left\langle D_m^\delta\right\rangle \qquad \text{(B.12)}\end{aligned}$$

where a Hölder inequality has been used at the last step. The end result is

$$\left\langle D_{m+1}^\delta\right\rangle \leq c_{7,m}\left\langle D_m^\delta\right\rangle + c_{8,m}\left\langle D_1^\delta\right\rangle + c_{9,m}Gr + O(T^{-1})\,. \qquad \text{(B.13)}$$

Because $n = \frac{1}{2}(m+1)$, when $m = 1$ then $n = 1$. Moreover, only when $\delta = 1$ does an estimate exist for $\langle D_1\rangle$ through (B.2), then (B.13) is a generating inequality gives the Gr^2-estimate but with worse constants. ■

C Proof of Theorem 2

With $0 < \mu_m < 1$ we write

$$\begin{aligned}\langle D_m^{1-\mu_m}\rangle_T &= \left\langle\left(\frac{D_m}{D_{m+1}}\right)^{1-\mu_m} D_{m+1}^{1-\mu_m}\right\rangle\\ &\leq \left\langle\left(\frac{D_m}{D_{m+1}}\right)^{\frac{1-\mu_m}{\mu_m}}\right\rangle_T^{\mu_m} \langle D_{m+1}\rangle_T^{1-\mu_m}\,, \qquad \text{(C.1)}\end{aligned}$$

which becomes

$$\left\langle\left(\frac{D_m}{D_{m+1}}\right)^{\frac{1-\mu_m}{\mu_m}}\right\rangle_T \geq \left(\frac{\left\langle D_m^{1-\mu_m}\right\rangle_T}{\langle D_{m+1}\rangle_T}\right)^{\frac{1-\mu_m}{\mu_m}} \langle D_m^{1-\mu_m}\rangle_T\,. \qquad \text{(C.2)}$$

The estimate for the time average of $\langle D_{m+1}\rangle_T$ from (2.3) and the lower bound $D_m \geq 1$ are now used to give

$$\left\langle\left[\frac{D_m}{D_{m+1}}\right]^{\frac{1-\mu_m}{\mu_m}} - \left[c_{av}^{-1}Gr^{-p}D_m^{\mu_m}\right]^{\frac{1-\mu_m}{\mu_m}}\right\rangle_T \geq 0\,. \qquad \text{(C.3)}$$

This ends the proof of Theorem 2. ■

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