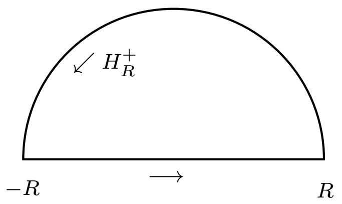
Jordan's Lemma



Jordan's Lemma deals with the problem of how a contour integral behaves on the semi-circular arc H_R^+ of a closed contour C.

Lemma 1 *(Jordan) If the only singularities of $F(z)$ are poles, then*

$$\lim_{R\to\infty} \int_{H_R} e^{imz} F(z)\, dz = 0 \tag{1}$$

provided that $m > 0$ and $|F(z)| \to 0$ as $R \to \infty$.

Proof: Since H_R is the semi-circle $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$ and $dz = iRe^{i\theta}d\theta$

$$\begin{aligned} \lim_{R\to\infty} \left| \int_{H_R} e^{imz} F(z)\, dz \right| &= \lim_{R\to\infty} \left| \int_{H_R} e^{imR\cos\theta - mR\sin\theta} F(z) R\, e^{i\theta} d\theta \right| \\ &\leq \lim_{R\to\infty} \int_{H_R} e^{-mR\sin\theta} |F(z)| R\, d\theta \end{aligned} \tag{2}$$

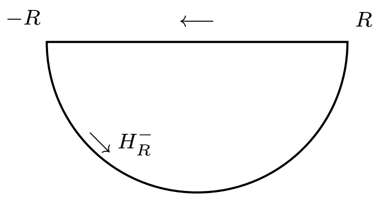
having recalled that $|e^{i\alpha}| = 1$ for any real α and $\left|\int f(z)\, dz\right| \leq \int |f(z)|\, dz$. Note that in the exponential term on the RHS of (2), $\sin\theta > 0$ in the upper half plane. Hence, provided $m > 0$, the exponential ensures that the RHS is zero in the limit $R \to \infty$ (see remarks below). □

Remarks:

a) When $m > 0$ forms of $F(z)$ such as $F(z) = \frac{1}{z}$, $F(z) = \frac{1}{z+a}$ or rational functions of z such as $F(z) = \frac{z^p \ldots}{z^q + \ldots}$ (for $0 \leq p < q$ and p and q integers) will all work as these all have simple poles and $|F(z)| \to 0$ as $R \to \infty$.

b) If, however, $m = 0$ then a modification is needed: e.g. if $F(z) = \frac{1}{z}$ then $|F(z)| \to 0$ but the $R|F(z)| = 1$. We need to alter the restriction on the integers p and q to $0 \leq p < q - 1$ which excludes cases like $F(z) = \frac{1}{z}$, $F(z) = \frac{1}{z+a}$.

c) What about $m < 0$? To ensure that the exponential is decreasing for $R \to \infty$ we need $\sin\theta < 0$. This is true in the *lower* half plane. Hence in this case we take our contour in the *lower* half plane (call this H_R^- as opposed to H_R^+ in the upper) but still in an anti-clockwise direction.



A contour in the lower $\frac{1}{2}$-plane with semi-circle H_R^-.