Jordan’s Lemma

Jordan’s Lemma deals with the problem of how a contour integral behaves on the semi-circular arc $H_R^+$ of a closed contour $C$.

**Lemma 1 (Jordan)** If the only singularities of $F(z)$ are poles, then

$$\lim_{R \to \infty} \int_{H_R} e^{imz} F(z) \, dz = 0$$

provided that $m > 0$ and $|F(z)| \to 0$ as $R \to \infty$.

**Proof:** Since $H_R$ is the semi-circle $z = Re^{i\theta} = R(\cos \theta + i \sin \theta)$ and $dz = iRe^{i\theta} d\theta$

$$\lim_{R \to \infty} \left| \int_{H_R} e^{imz} F(z) \, dz \right| = \lim_{R \to \infty} \left| \int_{H_R} e^{imR \cos \theta - mR \sin \theta} F(z)R e^{i\theta} d\theta \right|$$

$$\leq \lim_{R \to \infty} \int_{H_R} e^{-mR \sin \theta} |F(z)|R d\theta$$

having recalled that $|e^{i\alpha}| = 1$ for any real $\alpha$ and $|\int f(z) \, dz| \leq \int |f(z)| \, dz$. Note that in the exponential term on the RHS of (2), $\sin \theta > 0$ in the upper half plane. Hence, provided $m > 0$, the exponential ensures that the RHS is zero in the limit $R \to \infty$ (see remarks below). $\square$

**Remarks:**

a) When $m > 0$ forms of $F(z)$ such as $F(z) = \frac{1}{z}$, $F(z) = \frac{1}{z+a}$ or rational functions of $z$ such as $F(z) = \frac{z^p}{z^q + a}$ (for $0 \leq p < q$ and $p$ and $q$ integers) will all work as these all have simple poles and $|F(z)| \to 0$ as $R \to \infty$.

b) If, however, $m = 0$ then a modification is needed: e.g. if $F(z) = \frac{1}{z}$ then $|F(z)| \to 0$ but the $R|F(z)| = 1$. We need to alter the restriction on the integers $p$ and $q$ to $0 \leq p < q - 1$ which excludes cases like $F(z) = \frac{1}{z}$, $F(z) = \frac{1}{z+a}$.

c) What about $m < 0$? To ensure that the exponential is decreasing for $R \to \infty$ we need $\sin \theta < 0$. This is true in the lower half plane. Hence in this case we take our contour in the lower half plane (call this $H_R^-$ as opposed to $H_R^+$ in the upper) but still in an anti-clockwise direction.

A contour in the lower $\frac{1}{2}$-plane with semi-circle $H_R^-$. 

 [$H_R^+$ diagram]

[$H_R^-$ diagram]