EE2 Mathematics
Solutions to Sheet 2
LINE INTEGRALS & INDEPENDENCE OF PATH

1) The equation of the helix is \( \mathbf{r} = \hat{i}\cos t + \hat{j}\sin t + \hat{k}t \) which means that in terms of the parameter \( t \) we have \( x = \cos t \); \( y = \sin t \) and \( z = t \). Hence \( dx = -\sin t\, dt \); \( dy = \cos t\, dt \) and \( dz = dt \). Thus

\[
(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (\sin^2 t + \cos^2 t + 1)\, dt^2 = 2\, dt^2
\]

and

\[
\int_C (x^2 + y^2 + z^2)\, ds = \int_0^{2\pi} (t^2 + 1) \sqrt{2}\, dt = \sqrt{2}(2\pi + 8\pi^3/3)
\]

2) Find \( \oint_C xy\, ds \) where \( C \) is the closed path of straight lines from \((0,0)\) to \((1,0)\) to \((0,1)\) and then back to \((0,0)\).

- \( C_1: y = 0 \) so \( ds = dx \) and \( \int_{C_1} xy\, ds = 0 \).
- \( C_3: x = 0 \) so \( ds = dy \) and \( \int_{C_3} xy\, ds = 0 \).
- \( C_2: y = 1 - x \) so \( ds = \sqrt{2}\, dx \).

Therefore

\[
\int_{C_2} xy\, ds = \sqrt{2} \int_0^1 (1 - x)\, dx = -\sqrt{2}/6
\]

Hence \( \oint xy\, ds = -\sqrt{2}/6 \).

3a) To evaluate \( \int_C [(x^2 + y^2)\, dx - 2xy\, dy] \) where \( C \) is the straight line:

On \( C \), which is the line \( y = x \), we have \( dy = dx \). Hence the integral becomes

\[
I = \int_C (2x^2\, dx - 2x^2\, dx) = 0
\]

3b) To evaluate \( \int_C [(x^2 + y^2)\, dx - 2xy\, dy] \) where \( C \) is the curve \( y = x^{1/2} \):
On $C$, which is the curve $y = x^{1/2}$, we have $dy = \frac{1}{2}x^{-1/2}dx$. Hence the integral becomes

$$I = \int_0^1 \left\{ (x^2 + x) \, dx - 2x \, x^{1/2} \left( \frac{1}{2}x^{-1/2}dx \right) \right\}$$

$$= \int_0^1 x^2 \, dx = \frac{1}{3}$$

3c) To evaluate $\int_C \left[ (x^2 + y^2) \, dx - 2xy \, dy \right]$ where $C$ is the curve $y = x^2$:

On $C$, which is the curve $y = x^2$, we have $dy = 2x \, dx$. Hence the integral becomes

$$I = \int_0^1 \left\{ (x^2 + x^4) \, dx - 2x \, x^2 \left( 2x \, dx \right) \right\}$$

$$= \int_0^1 (x^2 - 3x^4) \, dx = -\frac{4}{15}$$

4) To evaluate the line integrals:

$$I_1 = \int_C \left[ y^2 \cos x \, dx + 2y \sin x \, dy \right] \quad I_2 = \int_C \left[ 2y^2 \, dx - x \, dy \right]$$

over a path $C_1$ which is the straight line between $(0,0)$ and $(\pi/2, 1)$: that is, the line $y = 2x/\pi$. For $I_1$ we have $F_1 = y^2 \cos x$; $F_2 = 2y \sin x$ so $F_{1,y} = F_{2,x} = 2y \cos x$. Hence the integral is independent of path and so $I_1$ over $C_1$ must be the same as $I_1$ over $C_2 + C_3$. We calculate $I_1$ over $C_1$ only.

The integral $I_1$ over $C_1$ can be written as

$$I_1 = \int_{C_3} \left[ y^2 \cos x \, dx + 2y \sin x \, dy \right]$$

$$= \int_{C_3} d \left( y^2 \sin x \right) = \left. y^2 \sin x \right|_{(0,0)}^{(\pi/2,1)}$$

$$= 1$$

To evaluate $I_2$, which is not independent of path ($F_{1,y} \neq F_{2,x}$), we integrate first over $C_1$; that is, along the line $y = \frac{2}{\pi}x$

$$\int_{C_1} \left( 2y^2 \, dx - x \, dy \right) = \int_0^{\pi/2} \left\{ 2 \left( \frac{2}{\pi} \right)^2 x^2 - \frac{2}{\pi} x \right\} \, dx = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$
Now we find \( \int_{C_2 + C_3} = \int_{C_2} + \int_{C_3} \) by evaluating first over \( C_2 \) (which is the line \( y = 0 \)) and then over \( C_3 \) (which is the line \( x = \pi/2 \)). On \( C_2 \) we have \( y = 0 \) and so \( dy = 0 \) whereas on \( C_3 \) we have \( x = \pi/2 \) so \( dx = 0 \). Hence
\[
\int_{C_2} (2y^2 dx - xdy) = 0 \quad \int_{C_3} (2y^2 dx - xdy) = -\frac{\pi}{2} \int_0^1 dy = -\frac{\pi}{2}
\]

Hence \( \int_{C_2 + C_3} = -\frac{\pi}{2} \). Note that \( \int_{C_1} \neq \int_{C_2 + C_3} \) because \( I_2 \) is not independent of path.

5) We want to evaluate \( \oint_C (x dy - y dx) \), where \( C \) is the unit circle \( x = \cos t, \ y = \sin t \). For \( C \) to be closed we need \( t : 0 \rightleftharpoons 2\pi \). We have \( dx = -\sin t \ dt, \ dy = \cos t \ dt \) so the integral is
\[
\oint_C (x dy - y dx) = \int_0^{2\pi} (\sin^2 t + \cos^2 t) \ dt = \int_0^{2\pi} \ dt = 2\pi
\]

6) The integral can be written as
\[
\int_C \mathbf{E} \cdot d\mathbf{r} = \int \left[(3x^2 + 6y) \ dx - 14yz \ dy + 20xz^2 \ dz\right]
\]
Along the path \( x = t, \ y = t^2, \ z = t^3 \) we have \( dx = dt, \ dy = 2t \ dt \) and \( dz = 3t^2 \ dt \), in which case the integral becomes
\[
\int_C \mathbf{E} \cdot d\mathbf{r} = \int_0^1 \left[(3t^2 + 6t^2) \ dt - 14t^2 t^3 (2t \ dt) + 20t^6 (3t^2 \ dt)\right]
= \int_0^1 [9t^2 - 28t^6 + 60t^9] \ dt = 3 - 4 + 6 = 5
\]

7) If \( \mathbf{E} = (2xy + z^3) \hat{i} + x^2 \hat{j} + 3xz^2 \hat{k} = -\nabla \phi \), then \( \phi_x = -2xy - z^3; \ \phi_y = -x^2 \) and \( \phi_z = -3xz^2 \). We can check that \( \phi_{xy} = \phi_{yx}, \ \phi_{xz} = \phi_{zx} \) and \( \phi_{yz} = \phi_{zy} \). Integrating the three equations for \( \phi \) we find that \( (c \ is an arbitrary constant)\)
\[
\phi = - (x^2 y + xz^3) + c
\]

Therefore, because \( \mathbf{E} = -\nabla \phi \)
\[
\int_C \mathbf{E} \cdot d\mathbf{r} = - \int_C \nabla \phi \cdot d\mathbf{r} = - \int_C d\phi = [x^2 y + xz^3]^{(3,1,4)}_{(1,-2,1)} = 202
\]