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Cluster formation in complex multi-scale systems

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Based on the competition between members of a hierarchy of length scales in complex multi-scale systems, it is shown how clustering of active quantities into concentrated sets, like bubbles in a Swiss cheese, is a generic property that dominates the intermittent structure. The halo-like surfaces of these clusters have scaling exponents lower than that of their kernels, which can be as high as the domain dimension. Possible examples include spots in fluid turbulence and droplets in spin-glasses.

Keywords: clustering; intermittency; fluid turbulence; complex systems; multi-scale; spin glass

1. Introduction

It has long been recognized that active quantities in complex systems of many types are not distributed evenly across a domain but cluster strongly into irregular bubbles, as in a Swiss cheese. The nomenclature, the nature and shape of the bubbles, and the physics in each subject may be substantially different. Some examples are: spottiness in high Reynolds number fluid turbulence (Batchelor & Townsend 1949; Kuo & Corrsin 1971; Meneveau & Sreenivasan 1991; Frisch 1995; Zeff *et al.* 2003) and boundary layers (Emmons 1951); droplet formation in spin-glasses (Fisher & Huse 1986; Bray & Moore 1987; Palassini & Young 2000); clustering behaviour in networks (West *et al.* 1999; Albert & Barabási 2002); the preferential concentration of inertial particles (Eaton & Fessler 1994; Sigurgeirsson & Stuart 2002; Bec 2003; Bec *et al.* 2005; Holm & Putkaradze 2005), with applications to rain initiation by cloud turbulence (Falkovich *et al.* 2002); the clustering of luminous matter (Bak & Chen 2001, 2002; Paczuski & Hughes 2004; Bak & Paczuski 2005) and magnetic bubbles in astrophysics (Zweibel 2002). Clusters display strong features whose typical length scales are much shorter than their averages, thus raising the question of the nature of the interface between them and the surrounding longer scale regions. For instance, in spin glasses Palassini & Young (2000) have shown that the 'surface' of the droplets has a fractal-like structure, whereas the droplets themselves have the full domain

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dimension. In fluid turbulence the concentrated sets on which vorticity accumulates are tubes and sheets, although the fractal nature of these is unclear. These sets dominate the associated Fourier spectra which display a spikiness that is the hallmark of what is usually referred to as intermittency (Batchelor & Townsend 1949; Kuo & Corrsin 1971; Kerr 1985; Meneveau & Sreenivasan 1991; Vincent & Meneguzzi 1994; Frisch 1995; Zeff *et al.* 2003).

While one cannot hope to make a uniform theory for so many disparate examples whose origins, scale and governing equations widely differ, the ubiquity of clustering phenomena suggests the existence of an underlying set of organizing principles. *The function of this paper is primarily mathematical; using simple but broadly applicable ideas, it will demonstrate that a dominant principle behind clustering is the existence of a hierarchy of length scales whose members are in competition.*

2. Competition within a hierarchy of length scales

Consider a d-dimensional system whose smallest characteristic (integral) scale L is such that the system is statistically homogeneous on boxes $\Omega=[0,L]^d$. Moreover, it is endowed with the following two properties. Firstly, at each point $x\in\Omega$, it possesses an ordered set of length scales $\ell_n=\ell_n(x)$ associated with a hierarchy of features labelled by $n\geq 2$

$$L>\ell_1(x)\geq\ell_2(x)\geq\cdots\geq\ell_n(x)\geq\ell_{n+1}(x)\cdots. \tag{2.1}$$

The ℓ_n could be thought of as an ordered set of correlation or coherence lengths; their inverses $\kappa_n(x)=\ell_n^{-1}(x)$ clearly obey $1<L\kappa_n\leq L\kappa_{n+1}$. Secondly, the assumption is that the ensemble averages of the $L\kappa_n(x)$ are bounded above by some ordered, positive parameters of the system satisfying $1<R_n\leq R_{n+1}$,

$$1<L\langle\kappa_n\rangle\leq R_n. \tag{2.2}$$

The ensemble average $\langle\cdot\rangle$ is a spatial average with respect to the Lebesgue measure over Ω. Thus, while the ordering of the $\ell_n(x)$ must be respected at each point, the ℓ_n themselves could be quite rough, e.g. they could consist of a series of step functions.

If they become very small near points x^* then they must obey $\ell_n>O(r^{d-\varepsilon})$ ($r=|x-x^*|$ and $\varepsilon>0$) so as not to violate equation (2.2).

Following an idea used by Gibbon & Doering (2003, 2005), consider the real arbitrary parameters $0<\mu<1$ and $0<\alpha<1$ such that $\mu+\alpha=1$. A version of Hölder's inequality, with A and B positive, is

$$\langle AB\rangle\leq\langle A^p\rangle^{1/p}\langle B^q\rangle^{1/q},\quad p^{-1}+q^{-1}=1. \tag{2.3}$$

With a choice of $p=\mu^{-1}$ and $q=\alpha^{-1}$, we have

$$\langle\kappa_n^\alpha\rangle\leq\langle\kappa_{n+1}^\alpha\rangle=\left\langle\left(\frac{\kappa_{n+1}}{\kappa_n}\right)^\alpha\kappa_n^\alpha\right\rangle\leq\left\langle\left(\frac{\kappa_{n+1}}{\kappa_n}\right)^{\alpha/\mu}\right\rangle^\mu\langle\kappa_n\rangle^\alpha. \tag{2.4}$$

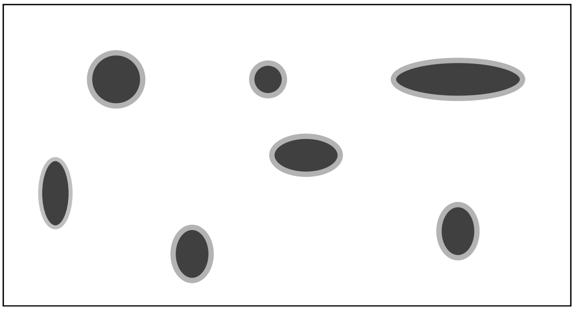

Figure 1. A cartoon-like illustrative slice through Ω for one value of n: the black kernels, surrounded by grey halos may take many shapes depending on the problem. Very small-scale behaviour concentrates on the black and grey regions which constitute the set $\mathcal{A}_n^+(\mathcal{L}_n\kappa_n > 1)$. The halos have scaling exponents lower than those of the black kernels.

Re-arranging and factoring out a term $\langle\kappa_n^\alpha\rangle$ gives

$$\left\langle\left(\frac{\kappa_{n+1}}{\kappa_n}\right)^{\alpha/\mu}\right\rangle \geq \langle\kappa_n^\alpha\rangle\left(\frac{\langle\kappa_n^\alpha\rangle}{\langle\kappa_n\rangle}\right)^{\alpha/\mu}. \tag{2.5}$$

Lower bounds on the ratio $\langle\kappa_n^\alpha\rangle/\langle\kappa_n\rangle$ can be found from equations (2.1) and (2.2) thereby turning equation (2.5) into

$$\left\langle\left(\frac{\kappa_{n+1}}{\kappa_n}\right)^{\alpha/\mu} - [(L\kappa_n)^\mu R_n^{-1}]^{\alpha/\mu}\right\rangle \geq 0. \tag{2.6}$$

While it is possible that the integrand in equation (2.6) could be positive everywhere in Ω, this cannot be assumed; the generic case is that the integrand could take either sign. With the definition $\mathcal{L}_n = LR_n^{-1/\mu}$ we have the pair of inequalities

$$\frac{\kappa_{n+1}}{\kappa_n} \gtrless (\mathcal{L}_n\kappa_n)^\mu, \tag{2.7}$$

for which $\geq$ is valid on regions where the integrand is positive, designated as *good regions*, and negative $(<)$ on *bad regions*. The term $(\mathcal{L}_n\kappa_n)^\mu$ on the right-hand side of equation (2.7) remarkably contains the arbitrary parameter μ which lies in the range $0<\mu<1$. Its existence is important because the ordering in equation (2.1) applied to equation (2.7) makes it clear that everywhere within the bad regions ($<$ in equation (2.7)) there are large lower bounds on κ_n with exponents containing $1/\mu$

$$\mathcal{L}_n\kappa_n > 1 \Rightarrow L\kappa_n > R_n^{1/\mu}. \tag{2.8}$$

Let $\mathcal{A}_n^+$ be the set on which $\mathcal{L}_n\kappa_n > 1$ and $\mathcal{A}_n^-$ the set on which $\mathcal{L}_n\kappa_n \leq 1$. Then all the bad regions ($<$ in equation (2.7)), designated by the clusters of black kernels in figure 1, lie in $\mathcal{A}_n^+$. The grey halos also lie in $\mathcal{A}_n^+$, and correspond to those parts of the good regions ($\geq$ in equation (2.7)), neighbouring the bad, for which $\mathcal{L}_n\kappa_n > 1$; see also figure 2. It is in these halos where the lower bound $(\mathcal{L}_n\kappa_n)^\mu$ becomes operative. The white areas of figure 1 belong to $\mathcal{A}_n^-$ in which the κ_n can

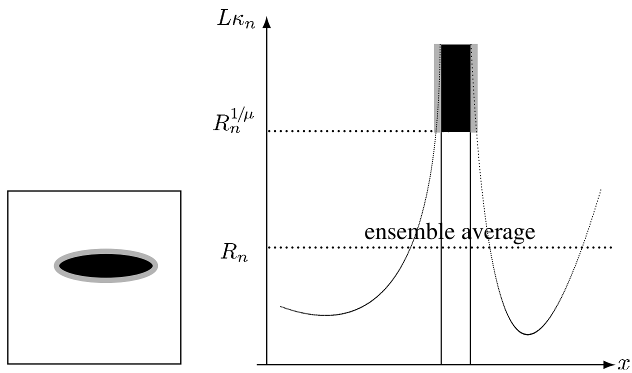

Figure 2. An illustration of a single cluster with its grey halo (left figure) and their corresponding regions (right figure); these lie in the range $L\kappa_n > R_n^{1/\mu}$. For many clusters the corresponding plot of $L\kappa_n$ versus x would have an intermittent structure.

be randomly distributed subject to their ordering in equation (2.1). It is clear from equation (2.6) that the existence and location of the clusters may differ for each n although no information is obtained regarding their distribution.

To show that the volume $\mathcal{V}_n^+$ of $\mathcal{A}_n^+$ comprises a small part of Ω, Chebychev's inequality relates the normalized Lebesgue measure $m(\mathcal{A}_n^+)$ to the integral of $L\kappa_n$ over $\mathcal{A}_n^+$

$$\int_{\mathcal{A}_n^+} L\kappa_n \mathrm{d}m \geq m(\mathcal{A}_n^+) R_n^{1/\mu} = L^{-d} \mathcal{V}_n^+ R_n^{1/\mu}. \tag{2.9}$$

Together with the relation $\int_{\mathcal{A}_n^+} L\kappa_n \mathrm{d}m \leq \langle L\kappa_n \rangle \leq R_n$ we have

$$m(\mathcal{A}_n^+) \leq R_n^{-(1/\mu)+1}. \tag{2.10}$$

Hence $m(\mathcal{A}_n^+)$ is significantly smaller than unity and decreases as R_n increases. Thus $\mathcal{A}_n^+$ can fill, at most, a small fraction of Ω.

3. Scaling exponents

To estimate the fractal or Hausdorff dimensions of the set $\mathcal{A}_n^+$ in a precise set-theoretic sense requires more information than is available. However, something very close to this can be found by estimating scaling exponents; see Hentschel & Procaccia (1983) and Frisch (1995). This entails making a third assumption of self-similarity to estimate the smallest number of balls $\mathcal{N}_n^+$ of radius $\boldsymbol{\lambda}_n^+$ needed to cover $\mathcal{A}_n^+$. Defining $\boldsymbol{\lambda}_n^+$ as

$$(\boldsymbol{\lambda}_n^+)^{-1} \equiv k_n^+ = \langle \kappa_n^p \rangle^{1/p}, \tag{3.1}$$

for some $p > 1$, it is clear that k_n^+ cannot be large enough when $p = 1$ because of equation (2.2). However, any value[1] of $p \gg 1$ will do that makes k_n^+ large enough

[1] As $p \to \infty$, $\langle \kappa_n^p \rangle^{1/p} \to \sup_\Omega \kappa_n$, which certainly lies within $\mathcal{A}_n^+$. The p-labelling of k_n^+ is suppressed.

to be a member of $\mathcal{A}_n^+$. The simplest and worst estimate would be to write

$$\mathcal{N}_n^+ \sim (L/\boldsymbol{\lambda}_n^+)^d. \tag{3.2}$$

Inequality (2.10), however, shows that $\mathcal{A}_n^+$ occupies only a small fraction of Ω. A multiplicative factor of $m(\mathcal{A}_n^+)$ is introduced thus

$$\mathcal{N}_n^+ \sim m(\mathcal{A}_n^+)(L/\boldsymbol{\lambda}_n^+)^d = m(\mathcal{A}_n^+)(Lk_n^+)^d. \tag{3.3}$$

Instead of using equation (2.10) to estimate $m(\mathcal{A}_n^+)$, an assumption of self-similar scaling is introduced that requires that the change in volume of the balls with respect to n should scale as $\mathcal{V}_n^+$ scales to L^d. Thus

$$m(\mathcal{A}_n^+) \sim \frac{\mathcal{V}_n^+}{L^d} \sim \left(\frac{\lambda_{n+1}^+}{\boldsymbol{\lambda}_n^+}\right)^d. \tag{3.4}$$

We observe that the definition of the set $\mathcal{A}_n^+$ in principle involves the length scales L and λ_n^+, but not overtly λ_{n+1}^+. Yet the good and bad sets involve all three scales; L, λ_n^+ and λ_{n+1}^+. The self-similarity assumption (3.4) is an assumption about the nature of the set $\mathcal{A}_n^+$ that relates successive length scales λ_n^+ and λ_{n+1}^+ in an *ad hoc*, yet reasonable fashion. Using equation (3.4) in (3.3) we have

$$\mathcal{N}_n^+ \sim \left(\frac{\lambda_{n+1}^+}{\lambda_n^+}\right)^d \left(\frac{L}{\boldsymbol{\lambda}_n^+}\right)^d = \frac{(\mathcal{L}_n k_n^+)^{2d}}{(\mathcal{L}_n k_{n+1}^+)^d} R_n^{d/\mu}. \tag{3.5}$$

From these, two estimates for $\mathcal{N}_n^+$ emerge, one each for the grey halo and black kernel regions of figure 1, whose scaling exponents[2] are independent of p

$$\mathcal{N}_n^+ \lesssim \begin{cases} (\mathcal{L}_n k_n^+)^{d(1-\mu)} R_n^{d/\mu} & \text{(grey halos)}, \\ (\mathcal{L}_n k_n^+)^d R_n^{d/\mu} & \text{(black kernels)}. \end{cases} \tag{3.6}$$

For the halos, the > direction of the inequality in equation (2.7) has been used together with a simple Hölder inequality

$$\langle \kappa_n^{p(1+\mu)} \rangle^{1/p} \ge \langle \kappa_n^p \rangle^{(1+\mu)/p} = (k_n^+)^{1+\mu}, \tag{3.7}$$

whereas for the kernels $\kappa_n \le \kappa_{n+1}$ has been used. In contrast, without any evidence of contraction of volume, the formula corresponding to equation (3.3) for $\mathcal{N}_n^-$ is

$$\mathcal{N}_n^- \sim (L/\lambda_n^-)^d = (\mathcal{L}_n k_n^-)^d R_n^{d/\mu}, \tag{3.8}$$

where k_n^- satisfies $\mathcal{L}_n \kappa_n^- \le 1$. The uniform scaling exponents in equation (3.6) satisfy (table 1)

$$\mathcal{D}_{n,\text{halo}}^+ \le d(1-\mu) \quad \mathcal{D}_{n,\text{ker}}^+ \le d, \tag{3.9}$$

whereas $\mathcal{D}^- = d$ from equation (3.8). The coefficients $R_n^{d/\mu}$ in equations (3.5)–(3.8) reflect the fact that this effect is taking place only at length scales smaller than $LR_n^{-1/\mu}$ where the better lower bound $(\mathcal{L}_n \kappa_n)^\mu$ becomes effective in equation (2.7). The grey halo clearly plays the role of an interface of small but finite thickness

[2] Since we expect $\mathcal{N}_n^+ \gg 1$, the estimate (3.5) implies that $L\lambda_{n+1}^+ \gg (\lambda_n^+)^2$. This is consistent with $\kappa_n > L^{-1}$ as in equation (2.1) but technically imposes an additional constraint.

Table 1. *Summary regarding the sets* $\mathcal{A}_n^{\pm}$ *and the regions in figure 1*

figure 1	black	grey	white
set	$\mathcal{A}_n^+$	$\mathcal{A}_n^+$	$\mathcal{A}_n^-$
inequality (2.7)	$<$ (bad)	$\geq$ (good)	$\geq$ (good)
exponent	$\leq d$	$\leq d(1-\mu)$	$= d$

between the d-dimensional (white) outer region and the (black) inner kernel whose dimension can be as high as d but could be less; figure 2 illustrates the correspondence between a kernel with its halo and the magnitude of $L\kappa_n$. The full system, with many clusters, would manifest many excursions away from the ensemble average that would dominate the intermittent structure. When $\mathcal{D}_{n,\mathrm{ker}}^+$ saturates its upper bound we have

$$\mathcal{D}_{n,\mathrm{halo}}^+ \leq d(1-\mu) < \mathcal{D}_{n,\mathrm{ker}}^+ = d. \tag{3.10}$$

For any system in question, a numerical experiment would be necessary to estimate the R_n by finding the maximum value of the ensemble average $\langle\kappa_n\rangle$. In principle μ could then be found from numerical estimates of $\ell_n^{\mathrm{crit}} \sim LR_n^{-1/\mu}$ within the black kernels, although if the κ_n take very large values there it might not be possible to achieve resolution. μ itself may have upper and lower bounds that are themselves n-dependent, as in Gibbon & Doering (2003, 2005).

4. Some final remarks

The arguments in this paper have revolved around a statistically steady system with an ensemble average based on a Lebesgue measure. It has been shown that when a system of this type has an ordered hierarchy of length scales, and ensemble averages are finite, then intermittency is generally inevitable. Very short length scales crowd into clusters whose boundaries are fractal-like. This crowding effect can be illustrated in an alternative way: a Schwarz inequality, in combination with equation (2.2), yields a lower bound on the ensemble average of ℓ_n,

$$\langle\ell_n\rangle \geq LR_n^{-1}, \tag{4.1}$$

whereas in the black clusters (2.8) shows that there is a point-wise upper bound,

$$\ell_n < LR_n^{-1/\mu}. \tag{4.2}$$

Comparing the two shows that the length scales within these clusters are *very* much smaller than the lower bound on the averages expressed in equation (4.1).

Historically, the first ideas on clustering came more than half a century ago from Batchelor & Townsend (1949) who observed intermittent behaviour in their high Reynolds number turbulent flow experiments, closely followed by observations in boundary layers by Emmons (1951). Batchelor & Townsend (1949) called this phenomenon ‘spottiness’ and suggested that the energy associated with the small scale components is distributed unevenly in space and roughly confined to regions which concomitantly become smaller with eddy size

(Kuo & Corrsin 1971). Mandelbrot (1974) then suggested that these clustered sets on which energy dissipation is the greatest might be fractal in nature. In measurements of the energy dissipation rate in the atmospheric surface layer, Meneveau & Sreenivasan (1991) interpreted the intermittent nature of their signals in terms of multi-fractals. A newer generation of experiments measuring intense dissipation in turbulent flows have been pursued by Zeff *et al.* (2003).

Standard cascade theories of turbulence (Frisch 1995) are generally statistically stationary, but there are some significant challenges in applying the formulation of this paper to time-evolving three-dimensional Navier–Stokes turbulence. Thin sets of high vorticity and strain, taking on the nature of short-lived quasi-one-dimensional tubes and quasi-two-dimensional sheets, evolve rapidly in time (Kerr 1985; Vincent & Meneguzzi 1994; Frisch 1995). To include time and space in the average $\langle\cdot\rangle$ means a different measure could be necessary because of the semi-infinite nature of the time-axis. With the exception of the paper by Caffarelli *et al.* (1982) on the Navier–Stokes (potentially) singular set, methods of analysis are unfortunately not advanced enough to deal with the full space–time three-dimensional Navier–Stokes equations. Conventional methods use Sobolev norms to L^2-average the velocity field and its derivatives over space while the pressure is removed by projection (Constantin & Foias 1988; Foias *et al.* 2001; Majda & Bertozzi 2002) leaving only time as an independent variable. Using this approach Gibbon & Doering (2003, 2005) have reached a half-way stage in this process by showing that an ordered hierarchy of $\kappa_n(t)$ can be constructed for the three-dimensional Navier–Stokes equations that are comprised of ratios of norms (of derivatives of order n). Because the $\kappa_n(t)$ are functions only of time, $\langle\cdot\rangle$ means time-average. The clusters of figure 1 appear as gaps in the time-axis, as in figure 2, although conclusions about the halos would not be applicable when time is the only variable. Gibbon & Doering (2005) found it necessary to prove that these gaps are finite in width and decreasing with increasing Reynolds number, which involves finding bounds on μ.

Thus for time-evolving intermittent systems such as the Navier–Stokes equations, the challenge is to include both space and time while showing that these equations also possess a space–time hierarchy of scales. The appearance of short-lived clusters in numerical computations of the vorticity field suggests that such hierarchies may exist in local sub-domains of the flow for finite times.

A further example is that of the low-temperature phase of spin glasses (Sherrington & Kirkpatrick 1975; Mézard *et al.* 1987). In this case $\langle\cdot\rangle$ means ensemble average in the sense defined in this paper. The role played by the competition between members of a hierarchy of length scales is consistent with the observation of ultrametricity, a term that is used to denote the presence of a hierarchy of scales (Mézard *et al.* 1987; Parisi & Ricci-Tersenghi 2000; Berthier *et al.* 2004). This has been observed in computations on the low-temperature spin glass phases of the Sherrington–Kirkpatrick (Hed *et al.* 2004) and Edwards–Anderson models (Stariolo 2001), as well as in dynamic phenomena in complexity (Boettcher & Paczuski 1996). The results in this paper, particularly with reference to equation (3.10), are consistent with the droplet theory (Fisher & Huse 1986; Bray & Moore 1987; Palassini & Young 2000) where the kernel of the droplet is of full dimension d but its surface has a scaling exponent $<d$. Palassini & Young (2000) have shown numerically that $\mathcal{D}^{+}_{\text{halo}}=2.58\pm0.02$ when $d=3$ and $\mathcal{D}^{+}_{\text{halo}}=2.77\pm0.02$ when $d=4$.

The physics behind many clustering phenomena is often difficult to express in terms of well-posed boundary value problems that can be subjected to rigorous analysis. Instead, the approach taken in this paper, highlighting the competition between length scales as the dominant mechanism, may be a useful paradigm in explaining the behaviour of multi-scale systems.

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